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On the Speed of Convergence of Sums and Differences

O szybkości zbieżności sum i różnic

О скорости сходимости сумм и разностей

For practical purposes it is often desired to know whether one of two considered sequences of functions of random variables converges quicker to the limit variable than the other one. In the present work we investigate two sequences: $\{Z_n\}$ of sums and $\{Y_n\}$ of differences, defined as follows:

Let $\{X_j\}$ and $\{X'_j\}$ be two independently obtained infinite sequences of independent random variables such that for each value of j X_j and X'_j have a common distribution with all finite moments (with means μ_j and variances σ_j^2). Let us further assume that Liapounoff's condition is fulfilled, i. e.

$$\lim_{n\to\infty} \frac{\sqrt[3]{\sum_{j=1}^{n}\beta_j}}{\sqrt{\sum_{j=1}^{n}\sigma_j^2}} = 0, \tag{1}$$

where β_j is the absolute central moment of third order of random variable X_j.

We define cummulative sums S_n and S'_n by relations

$$S_n = \sum_{j=1}^{n} X_j, \tag{2}$$

$$S'_n = \sum_{j=1}^{n} X'_j. \tag{3}$$

In view of our assumptions the sequences $\{S_n\}$ and $\{S'_n\}$ tend assymptotically to normal variables according to the central limit theorem.

Let us form the sum Z_n and the difference Y_n of S_n and S'_n:

(4) $$Z_n = S_n + S'_n,$$

(4') $$Y_n = S_n - S'_n.$$

Clearly, the sequences $\{Z_n\}$ and $\{Y_n\}$ are also convergent to normal variables.

We raise now a question: does one of these two sequences converge quicker to its limit variable than the other one, and if so, which of the two?

Let $C_j(t) = C_{X_j}(t)$ be the characteristic function of X_j (and for that matter of X'_j). The characteristic function for Z_n will be thus

(5) $$C_{Z_n}(t) = \prod_{j=1}^{n} [C_j(t)]^2$$

and for Y_n

(5') $$C_{Y_n}(t) = \prod_{j=1}^{n} C_j(t) \prod_{j=1}^{n} C_j(-t).$$

It will be more convenient for us to use the logarithms of characteristic functions (l. c. f. s). Denoting

(6) $$L_X(t) = \log_e C_X(t),$$

we have

(7) $$L_{Z_n}(t) = 2\sum_{j=1}^{n} L_j(t)$$

and

(7') $$L_{Y_n}(t) = \sum_{j=1}^{n} L_j(t) + \sum_{j=1}^{n} L_j(-t).$$

Now let us consider new random variables

(8) $$U_n = \frac{Z_n - E(Z_n)}{\sigma_{Z_n}},$$

(8') $$V_n = \frac{Y_n - E(Y)}{\sigma_{Y_n}},$$

where $E(Z_n)$ and $E(Y_n)$ are expectations and σ_{Z_n}, σ_{Y_n} standard deviations of Z_n and Y_n respectively. L. c. f. s of U_n and V_n are

$$L_{U_n}(t) = L_{Z_n}\left(\frac{t}{\sigma_{Z_n}}\right) - \frac{itE(Z_n)}{\sigma_{Z_n}}, \tag{9}$$

$$L_{V_n}(t) = L_{Y_n}\left(\frac{t}{\sigma_{Y_n}}\right) - \frac{itE(Y_n)}{\sigma_{Y_n}}. \tag{9'}$$

Expectations and variances of Z_n and Y_n may be expressed in terms of

$$E(Z_n) = 2m_n, \quad \text{where} \quad m_n = \sum_{j=1}^{n} \mu_j, \tag{10}$$

$$E(Y_n) = 0, \tag{10'}$$

$$\sigma_{Z_n}^2 = \sigma_{Y_n}^2 = 2S_n^2, \quad \text{where} \quad S_n^2 = \sum_{j=1}^{n} \sigma_j^2. \tag{11}$$

Making use of (7), (7') and (10), (10'), (11), we may write

$$L_{U_n}(t) = 2\sum_{j=1}^{n} L_j\left(\frac{t}{\sqrt{2S_n^2}}\right) - \frac{it2m_n}{\sqrt{2S_n^2}}, \tag{12}$$

$$L_{V_n}(t) = \sum_{j=1}^{n} L_j\left(\frac{t}{\sqrt{2S_n^2}}\right) + \sum_{j=1}^{n} L_j\left(\frac{-t}{\sqrt{2S_n^2}}\right). \tag{12'}$$

L. c. f. s may be developed in power series with cumulants as coefficients of powers of t, since random variables X_j, X_j' have finite moments of all orders.

$$L_j(t) = \mu_j it + \frac{\sigma_j^2 (it)^2}{2!} + \sum_{\nu=3}^{\infty} \frac{\varkappa_j^{(\nu)} (it)^\nu}{\nu!}, \tag{13}$$

where $\varkappa_j^{(\nu)}$ is the cumulant (semi-invariant) of order ν of random variable X_j.

Applying (13) we obtain

$$L_{U_n}(t) = 2\sum_{j=1}^{n}\left[\frac{\mu_j (it)}{\sqrt{2S_n^2}} + \frac{\sigma_j^2 (it)^2}{2!2S_n^2} + \sum_{\nu=3}^{\infty} \frac{\varkappa_j^{(\nu)} (it)^\nu}{\nu!(2S_n^2)^{\nu/2}}\right] - \frac{2itm_n}{\sqrt{2S_n^2}} = \tag{14}$$

$$= -\frac{t^2}{2} + 2\sum_{j=1}^{n}\sum_{\nu=3}^{\infty} \frac{\varkappa_j^{(\nu)} (it)^\nu}{\nu!(2S_n^2)^{\nu/2}}$$

and similarily

$$(14')\qquad L_{V_n} = \sum_{j=1}^{n}\left[\frac{\mu_j it}{\sqrt{2S_n^2}} + \frac{\sigma_j^2(it)^2}{2!2S_n^2} + \sum_{\nu=3}^{\infty}\frac{\varkappa_j^{(\nu)}(it)^\nu}{\nu!(2S_n^2)^{\nu/2}} - \frac{\mu_j(it)}{\sqrt{2S_n^2}} + \right.$$

$$\left. + \frac{\sigma_j^2(it)^2}{2!2S_n^2} + \sum_{\nu=3}^{\infty}\frac{(-1)^\nu\varkappa_j^{(\nu)}(it)^\nu}{\nu!(2S_n^2)^{\nu/2}}\right] = -\frac{t^2}{2} + 2\sum_{j=1}^{n}\sum_{\nu=2}^{\infty}\frac{\varkappa_j^{(2\nu)}(it)^{2\nu}}{(2\nu)!(2S_n^2)^\nu}.$$

Comparing the above expressions for $L_{U_n}(t)$ and $L_{V_n}(t)$ we see that the terms in the latter are all of even powers of t, while the former contains also terms with odd powers of t. Thus we are led to a conclusion that the sequence $\{L_{U_n}(t)\}$ may not converge quicker than the sequence $\{L_{V_n}(t)\}$.

Let us however make more precise the meaning of „quicker convergence”. Let us form absolute differences between $L_{U_n}(t)$ and its limit and between $L_{V_n}(t)$ and its limit $-t^2/2$. The difference D between these two absolute differences is

$$(15)\qquad D = \left|L_{U_n}(t) + \frac{t^2}{2}\right| - \left|L_{V_n}(t) + \frac{t^2}{2}\right|.$$

If $D > 0$ for all $t \neq 0$, then the sequence $L_{V_n}(t)$ will be said to converge quicker to $-t^2/2$ than the sequence $L_{U_n}(t)$. If $D = 0$ for all $t \neq 0$, then the sequence $L_{V_n}(t)$ will be said to converge as quickly to $-t^2/2$ as the sequence $L_{U_n}(t)$.

The quicker convergence of $L_{V_n}(t)$ would imply the quicker convergence of random variables V_n which correspond to these l. c. f. s, and in consequence of the sequence Y_n to its limit than of the sequence Z_n to its limit. Now we may formulate

Theorem 1.

The sequence of differences $\{Y_n\}$ defined in (4′) where each pair of independent variables X_j and X_j' (belonging to two independently obtained sequences $\{X_j\}$ and $\{X_j'\}$) has a common distribution with all finite moments and Liapounoff's condition (1) fulfilled, converges to its limit normal variable at least as quickly as sequence of sums $\{Z_n\}$ defined in (4).

Proof. We denote that with real t the real parts of $L_{U_n}(t)+t^2/2$ and $L_{V_n}(t)+t^2/2$ are equal (being composed of identical terms with even powers of it. Since with real t $L_{V_n}(t)+t^2/2$ is pure real, while $L_{U_n}(t)+t^2/2$ may have an imaginary part (composed of terms with odd powers of it), we may put

$$(16)\qquad L_{U_n}(t) + t^2/2 = R(t) + I(t)\cdot i,$$

$$(16')\qquad L_{V_n}(t) + \frac{t^2}{2} = R(t),$$

where $R(t)$ and $I(t)$ are both real. then we obtain at once

$$D = |R(t)+I(t)i| - |R(t)| = \sqrt{R^2(t)+I^2(t)} - |R(t)| \geqslant 0, \tag{17}$$

for all $t \neq 0$, which proves the theorem.

Now, the equality in (17) obtains, if and only if $I(t) = 0$ for all t, i. e. if all cumulants of odd orders vanish, which implies the symmetry of all distributions of X_j (and of X'_j). On the other hand the sharp inequality in (17) obtains, if and only if $I(t) \neq 0$ for at least one value of t, i. e. if at least one cumulant of odd order is not equal to zero, which implies assymmetry for at least one distribution of X_j (and X'_j). Thus we have

Theorem 2.

In the case of two independently obtained sequences $\{X_j\}$ and $\{X'_j\}$ of independent random variables X_j and independent random variables X'_j having for each value of j a common distribution with all finite moments and Liapounoff's condition (1) fulfilled, the following holds:

a) If all distributions of X_j (and A'_j) are symmetrical, then the sequence of differences $\{Y_n\}$ defined in (4′) converges to its limit normal variable as quickly as the sequence of sums $\{Z_n\}$ defined in (4) converges to its limit normal variable;

b) If at least one distribution of X_j (and of X'_j) is skew, then the sequence of differences $\{Y_n\}$ converges quicker to its normal variable than the sequence of sums $\{Z_n\}$ converges to its limit normal variable.

For an example we may take the differences of independent variables χ_n^2 and $\chi_n^{2'}$ with $n = 1, 2, \ldots$ degrees of freedom as compared to the sums of the said Chi-squares. χ_n^2 and $\chi_n^{2'}$ are here considered as sums of n independent random variables χ_1^2 and $\chi_1^{2'}$ respectively (i. e. as sums of independent Chi-squares with one degree of freedom each). In the case of differences we have symmetrical Bessel function distributions ([2], [3]), and in the case of sums we have skew χ_{2n}^2 distributions with doubled degees of freedom, and therefore the sequences of differences converges quicker to its limit normal variable than the sequence of sums.

In a paper by M. Fisz „The limiting distribution of the difference of two independent Poisson random variables" ([1]) two such variables with generally different parameters λ_1 and λ_2 were considered. If we put $\lambda_1 = \lambda_2 = n\lambda_0$, we shall have another special case of our theorem 2. The author of the cited paper, whose main aim was to prove the convergence of the difference of two independent Poisson random variables to normal distribution, has noticed that the distribution of the difference is closer to normal distribution than the distribution of sums.

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Streszczenie

W pracy dowodzi się, że spośród dwóch ciągów sum i różnic określonych przez (4) i (4′) odpowiednio, gdzie $\{X_j\}$ i $\{X_j'\}$ są dwoma niezależnymi ciągami zmiennych losowych niezależnych o wspólnych rozkładach, ciąg różnic jest nie wolniej zbieżny niż ciąg sum, przypadek jednakowej szybkości zbieżności zachodzi, gdy rozkłady zmiennych losowych X_j i X_j' są symetryczne.

Резюме

О скорости сходимости сумм и разностей

Доказывается, что из двух последовательностей сумм и разностей определенных через (4) и (4′), где $\{x_j\}$ и $\{x_j'\|$ являются независимыми последовательностями одинаково распределенных независимых величин, последовательность разностей сходится немедленней чем последовательность сумм, причем случай одинаковой скорости сходимости имеет место, когда распределения случайных величин x_j и x_j' симметричны.