ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN — POLONIA
VOL. XVI, 4 SECTIO A 1962

Z Zakładu Statystyki Matematycznej Wydziału Rolniczego
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Mixed Models $I\times J$ and $I\times 2$ with Interaction in the Case of Non-Orthogonal Data [1]

Mieszane modele $I\times J$ i $I\times 2$ z interakcją w przypadku danych nieortogonalnych

Смешанные модели $I\times J$ и $I\times 2$ с взаимодействием в случае неортогональных данных

1. Review of literature

a. This is a short review of literature concerning the two-way classification $A\times B$ in the case of unequal subclass numbers.

The general form of the mathematical model in which we are interested is as follows:

$$(1)\qquad y_{ijl}=\mu+a_i+\beta_j+\gamma_{ij}+e_{ijl};\quad i=1,2,\ldots,I;\ j=1,2,\ldots,J;\quad l=1,2,\ldots,n_{ij}$$

where y_{ijl} is the l^{th} observation in the (i,j) cell, μ is the general constant, e_{ijl} — random errors which are normally distributed i. e. $N(0,\sigma_e^2)$, and $a_i's$, $\beta_j's$ and $\gamma_{ij}'s$ can be random or fixed effects; a_i is the effect of the i^{th} A class, β_j is the effect of the j^{th} B class and γ_{ij} is the interaction effect of the (i,j) cell.

When all the effects on the right hand of (1) except for e_{ijl} and μ are fixed then (1) is *the fixed model*. In the case when $a_i's$, and $\beta_j's$ and $\gamma_{ij}'s$ are random we have *the random model*. If one of the effects $a_i's$ and $\beta_j's$ is random then γ_{ij} is random and the model is called *mixed*.

The case when the numbers of observations in each cell are proportional ($n_{ij}=p_is_j$) or the same ($n_{ij}=k=\text{constant};\ k\geqslant 1$) is known as *the orthogonal case*.

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[1] This work was carried out while at Statistical Laboratory, Iowa State University, Ames, Iowa, USA, under a Rockefeller Foundation fellowship.

b. **Fixed model.** The literature in the case of non-orthogonal data (n_{ij} are not constant) is extensive, but there are many problems still unsolved. The method of fitting constants devised by R. A. Fisher [9], a special case of which was discussed by A. E. Brandt [2] has been developed and perfected by F. Yates [39]. This method based on the theory of least squares can be applied to the solution of tables of two-way classification corresponding to the model (1). It provides the test of significance for interaction when some of n_{ij} can be equal to zero. F. Yates [40] has presented the theory of the two-way classification under the assumption that the interaction is not existent, and the test of significance for interaction for the case of fixed $\alpha's$ and $\beta's$. Moreover, the same author, had suggested the method of weighted squares of means [39, 40] when the interaction is present and when we assume that the population has equal subclass numbers. He has presented estimates of the main effects only. The effect of the interaction was given for the $I\times 2$ classification. The test of significance for interaction in this case is identical with that given by the method of fitting constants. In the same papers F. Yates has presented the approximate method of unweighted means. The second approximate method called the method of expected subclass numbers has been suggested by G. W. Snedecor and G. M. Cox [30]. One can use it under the assumption that the population has proportional subclass numbers and under the condition that all $n_{ij}\geqslant 1$. The analysis of variance is simple since it is based on a standard procedure used in the orthogonal case. Both approximate methods present estimates of interactions and test of significance of interactions. R. E. Patterson [23] is the author of another approximate method called method of adjusting factors. C. Y. Kramer [18] has presented an approximate method for fixed model in the case of no interaction.

W. L. Stevens [32] has given an arithmetic method useful in working out non-orthogonal data. Other papers by R. O. Johnson and J. Neyman [15], K. R. Nair [21], S. S. Wilks [38], F. Tsao [33, 34], are connected with the problems of working out non-orthogonal data. The problems are discussed in the text-books by G. W. Snedecor [31], O. Kempthorne [16], R. L. Anderson and T. A. Bancroft [1], M. G. Kendall [17].

A general case of weighted restrictions has been presented by J. Norton [22] but his notes were not published. He had used the weights of the form

$$W_i=\left[\sum_j\frac{[p(j/i)]^2}{n_{ij}}\right]^{-1},\quad \text{where}\quad \sum_{j=1}^{J}p(j/i)=1.$$

The special case of the weights when $p(j/i)=w_j$ has been suggested by H. Scheffé [26] in his book. In the article by W. R. Harvey [10] several numerical examples are presented on the use of least squares analysis of data for the case of both factors fixed.

c. Random model. S. L. Crump [6] has given the expected mean squares for the method of expected subclass numbers, while W. T. Federer [8] has presented the expected mean squares for the method of weighted squares of means. C. R. Henderson [11] and W. T. Federer [8] have given them for the method of fitting constants. Moreover, S. L. Crump has developed the sampling variances of the estimates for method of expected subclass numbers and for the method of unweighted squares of means. S. L. Crump [7] states that the method of unweighted squares of means is the simplest computationally.

In the case of uncorrelated and normally distributed random variables S. R. Searle [27] has used Henderson's [13] method 1 to estimate the variance components σ_a^2, σ_b^2, σ_{ab}^2, and σ_c^2. The nature of this method is to equate observed and expected mean squares. Both Henderson's methods 2 and 3 are based on least-squares principles. The variances of estimates of variance components from non-orthogonal data are unknown.

The problem of the estimation of variance components under the method of fitting constants has been discussed by H. L. Lucas [19] and C. R. Henderson [12]. A. Wald [36] has proved that exact confidence limits for the ratio of any variance component to the error component may be obtained in two-way classification with equal numbers. Under sufficiently large number of degrees of freedom it is possible to present approximate confidence limits of any covariance component, I. Bross [3].

Random models in the case of non-orthogonal data are described by O. Kempthorne [16] in his textbook.

H. F. Smith [28] has discussed the random model in the case of proportional numbers of observations in the sub-classes.

Numerical data of examples of random models and corresponding analyses of variance are presented by W. R. Harvey [10].

d. Mixed model. A special type of mixed model with fixed effects α_i, β_j and random interaction γ_{ij} in orthogonal case (n_{ij} = constant) was discussed by S. L. Crump [7]. N. L. Johnson [14] has presented the tests of significance for mixed model when $n_{ij} = k$. Bancroft and Anderson [1] describe a mixed model when β_j and γ_{ij} are random. The mixed model under general assumptions in the orthogonal case is particularly considered by H. Scheffé in his book [26].

The problem of estimation of variance components is presented and illustrated by C. R. Henderson [13]. Henderson's method 3 yields unbiased estimates of variance components in the case of non-orthogonal data. Henderson's method 1 leads to biased estimates in the case of mixed model. He suggests to use method 2 for mixed models when effects are uncorrelated or correlated.

W. R. Harvey [10] states that there is test of significance for the fixed main effects in the mixed model when interaction is significant in the case of non-orthogonal data. The author by using least squares analysis presents expectations of mean squares under standard assumptions. No proofs are given.

e. The three types of the above mentioned models: fixed, random and mixed are special cases of the model suggested by M. B. Wilk and O. Kempthorne [37]. The authors have introduced the concept of *experimental unit* and that of *true response* as well as the use of *randomization* in the design; they have also developed the methods of *finite model analysis* given by O. Kempthorne [16] for orthogonal data.

2. Notation

The symbols used in the text are as follows:

1. $$y_{ijl} = \underbrace{\mu + \alpha_i + \beta_j + \gamma_{ij}}_{\text{fixed}} + e_{ijl}; \quad i = 1, 2, \ldots, I;$$

$$j = 1, 2, \ldots, J; \quad l = 1, 2, \ldots, n_{ij}; \quad \text{fixed model}$$

2. $$y_{ijl} = \mu + \underbrace{a_i + b_j + c_{ij} + e_{ijl}}_{\text{random}}; \quad \text{random model}$$

3. $$y_{ijl} = \mu + \alpha_i + \underbrace{b_j + c_{ij} + e_{ijl}}_{\text{random}}; \quad \text{mixed model}$$

4. $v_i, w_j, p(j/i)$ — weights

5. $$W_i = \left(\sum_{j=1}^{J} \frac{w_j^2}{n_{ij}}\right)^{-1}, \quad V_j = \left(\sum_{i=1}^{I} \frac{v_i^2}{n_{ij}}\right)^{-1}$$

6. $$n_{i.} = \sum_{j=1}^{J} n_{ij}, \quad n_{.j} = \sum_{i=1}^{I} n_{ij}, \quad n = \sum_{j=1}^{J} n_{.j} = \sum_{i=1}^{I} n_{i.}$$

7. $$Y_{ij} = \sum_{l=1}^{n_{ij}} y_{ijl} = n_{ij}\bar{y}_{ij}.$$

8. $$Y_{i.} = \sum_{j=1}^{J} Y_{ij} = n_{i.}\bar{y}_{i..}$$

9. $$Y_{.j} = \sum_{i=1}^{I} Y_{ij} = n_{.j}\bar{y}_{.j.}$$

10. $$\hat{A}_i = \sum_{j=1}^{J} w_j \bar{y}_{ij.}; \quad \bar{A} = \frac{\sum_{i=1}^{I} W_i \hat{A}_i}{\sum_{i=1}^{I} W_i}$$

11. $$\hat{B}_j = \sum_{i=1}^{I} v_i \bar{y}_{ij.}; \quad \bar{B} = \frac{\sum_{j=1}^{J} V_j \hat{B}_j}{\sum_{j=1}^{J} V_j}$$

12. $$Q_{i.} = Y_{i.} - \sum_{j=1}^{J} n_{ij}\bar{y}_{.j.} = Y_{i.} - \sum_{j=1}^{J} \frac{n_{ij} Y_{.j}}{n_{.j}}$$

13. $$Q_{.j} = Y_{.j} - \sum_{i=1}^{I} \frac{n_{ij} Y_{i.}}{n_{i.}} = Y_{.j} - \sum_{i=1}^{I} n_{ij}\bar{y}_{i..}$$

14. $$\bar{y}_{...} = \frac{\sum Y_{i.}}{n} = \frac{\sum Y_{.j}}{n}$$

15. $$e_{ij.} = \frac{1}{n_{ij}} \sum_{l=1}^{n_{ij}} e_{ijl}$$

16. $$e_{.j.} = \frac{1}{n_{.j}} \sum_{i=1}^{I} n_{ij} e_{ij.}$$

17. $$e_{i..} = \frac{1}{n_{i.}} \sum_{j=1}^{J} n_{ij} e_{ij.}$$

18. $$e_{...} = \frac{1}{n} \sum_{i=1}^{I} n_{i.} e_{i..} = \frac{1}{n} \sum_{j=1}^{J} n_{.j} e_{.j}$$

19. $$\tilde{y}_{i..} = \frac{1}{J} \sum_{j=1}^{J} \bar{y}_{ij.}; \quad \tilde{y}_{.j.} = \frac{1}{I} \sum_{i=1}^{I} y_{ij.}$$ (the method of weighted squares of means)

20. $$n_{ij} = p_i s_j; \quad s_. = \sum_{j=1}^{J} s_j; \quad p_. = \sum_{i=1}^{I} p_i$$ (proportional frequencies)

21. $\hat{a}_i, \hat{\beta}_j$ estimates under the fixed model without interaction

$(y_{ijl} = \mu + a_i + \beta_j + e_{ijl})$

22. $l_i = \dfrac{n_{i1} n_{i2}}{n_{i2} + n_{i1}}$ (model $I \times 2$)

23. $z_i = \bar{y}_{i1.} - \bar{y}_{i2.}$

24. $r_{jj'} = \sum_{i=1}^{I} \dfrac{n_{ij} n_{ij'}}{n_{i.}}; \quad j, j' = 1, 2, \ldots, J;$

$$u_{ii'} = \sum_{j=1}^{J} \frac{n_{ij} n_{i'j}}{n_{.j}}$$

25. $\xi_j = n_{.j} - r_{jj}, \quad \eta_i = n_{i.} - u_{ii}$

26. $d_{il} = \sum_{j=1}^{J} n_{ij} n_{lj}; \quad i, l = 1, 2, \ldots, I$

27. $$\mathrm{Var}\left(\sum_{i=1}^{I} x_i\right) = \sum_{i=1}^{I} \mathrm{Var}(x_i) + 2 \sum_{\substack{i<i' \\ i,i'=1,2,\ldots,I}} \mathrm{Cov}(x_i, x_{i'}) =$$

$$= \sum_{i=1}^{I} \mathrm{Var}(x_i) + \sum_{\substack{i \neq i' \\ i,i'=1,2,\ldots,I}} \mathrm{Cov}(x_i, x_{i'})$$

28. $\theta_i = \sum_{j=1}^{J} \bar{y}_{ij.}$ (Table 3)

29. $$L = \frac{2}{I-1}\left\{\left[\sum_{i=1}^{I} l_i \mathrm{Var}(c_i(v)) - \frac{\sum^{I} l_i^2 \mathrm{Var}(c_i(v))}{\sum_i l_i}\right] - \frac{2 \sum_{\substack{i<k \\ i,k=1,2,\ldots,I}} l_i l_k \mathrm{Cov}(c_i(v), c_k(v))}{\sum_i l_i}\right\}$$ (Table 3)

30. $P = \dfrac{(\sum^{I} l_i z_i)^2}{\sum_i l_i}$ (Table 4)

3. Outline of cases considered

H. Scheffé has presented in detail the mixed model with interaction in the case of orthogonal data: $n_{ij} = k =$ constant. The assumptions concerning the model are moderately general.

The purpose of this note is to consider the mixed model with interaction in the case of non-orthogonal data following Scheffé's assumptions [26]. We are especially interested in the case when interaction is significant, so we want to introduce a general form of weights w_j, v_i and discuss the generalization of the method of weighted squares of means. In this case the definitions of the main effects depend on the system of weights. In the case of fixed model under non-orthogonal data the method of weighted squares of means does not give a test or an estimate of interactions except for the case $I \times 2$, so we consider especially the case $I \times 2$ (Table 3).

Particular cases are written down at the bottom of Table 1. They are defined by:

1. the weights w_j and v_i,
2. assumptions concerning the numbers of observations, n_{ij},
3. assumption about correlation among random interactions.

We include Tables 2 and 4 for the cases $I \times J$ and $I \times 2$ respectively when the interaction is insignificant.

The general procedure is to use the analysis of variance for fixed model to obtain expectation of mean squares for both effects A and B and for interaction AB. We would like to see if comparisons of mean squares are fair. Distributional properties of ratios of mean squares were not yet considered.

In order to calculate $E(MS_{AB})$ in general case $I \times J$ we have calculated $E(Q_{.j}^2)$ and $E(Q_{.j}Q_{.k})$ for $j \neq k$; $j, k = 1, 2, \ldots, J$.

4. Assumption under mixed model

We consider the model (1) where the $\{b_j\}$, $\{c_{ij}\}$, $\{e_{ijl}\}$ are jointly normal, the $\{e_{ijl}\}$ are independently $N(0, \sigma_e^2)$ and independent of the $\{b_j\}$ and $\{c_{ij}\}$, which have means $E(b_j) = 0$, $E(c_{ij}) = 0$ for all $i = 1, 2, \ldots, I$ and the following variances and covariances: $\mathrm{Cov}(b_j, b_{j'})$, $\mathrm{Cov}(c_{ij}, c_{i'j'})$ and $\mathrm{Cov}(b_j, c_{ij'})$. In the case $j \neq j'$ these variances and covariances are equal to zero, but when $j = j'$ they are assumed to be non-zero. They can be defined in terms of an $I \times I$ covariance matrix Σ_m with elements $\{\sigma_{ii'}\}$, where $y_{ijl} = m_{ij} + e_{ijl}$. The J vector random variables $(m_{1j}, \ldots, m_{Ij})$ are independently $N(\mu, \Sigma_m)$, where $\mu = (\mu_1, \ldots, \mu_I)$ and are independent of the $\{e_{ijl}\}$.

The unknown parameters are: σ_e^2, the elements $\{\sigma_{ii'}\}$ of the covariance matrix Σ_m, and the means $\{\mu_i\}$, which are written $\{\mu + a_i\}$ (cf. Scheffé's book [26]).

5. Definitions of main effects and interactions effects based on weights v_i

We consider the population presented by the A classification and a random variable v with the population distribution P_v. Let $m(i, v)$ be the „true" value of the individual labeled v on ith level of the A classification. It is necessary to note that i corresponds to a definite and fixed level of A classification in the experiment. We want to generalize Scheffé's definitions of main effects and of interactions effects by using weights v_i.

A vector random variable $m = m(v)$ has the I components $\{m(i, v)\}$. We can represent it as follows:

$$m = m(v) = \big(m(1, v), m(2, v), \ldots, m(I, v)\big)$$

Def. 1. The „true" mean for the ith level of classification A is

$$\mu_i = m(i,.) = E[m(i, v)]$$

where a dot signifies the expected value of $m(i, v)$ has been taken with respect to P_v.

Def. 2. The general mean is defined as

$$\mu = \mu_. = \sum_{i=1}^{I} v_i \mu_i = m(.,.) \quad \text{where} \quad \sum_{i=1}^{I} v_i = 1.$$

Def. 3. The main effect of the ith level of the classification A is defined as

$$a_i = \mu_i - \mu_. = m(i, .) - m(., .)$$

Def. 4. The „true" mean for the individual labeled v is

$$m(., v) = \sum_{i=1}^{I} v_i m(i, v)$$

Def. 5. The main effect of the individual labeled v in the population is

$$b(v) = m(., v) - m(., .).$$

Def. 6. The main effect of the individual labeled v, specific to the ith level of A, is defined as

$$m(i, v) - m(i, .).$$

Def. 7. The interaction of the ith level of A and the individual labeled v in the population is

$$c_i(v) = m(i, v) - m(i, .) - m(., v) + m(., .).$$

Thus we have $m(i, v) = \mu + a_i + b(v) + c_i(v)$.

6. Restrictions

From these definitions it follows that the main effects and interactions in the population satisfy the following weighted restrictions:

$$\sum_{i=1}^{I} v_i a_i = 0, \quad \sum_{i=1}^{I} v_i c_i(v) = 0 \quad \text{for all } v,$$

$$E[b(v)] = 0, \quad E[c_i(v)] = 0 \quad \text{for all } i.$$

Particular cases of weights and restrictions are given at the bottom of Table 1.

7. Relations between Var$[b(v)]$, Cov$[b(v), c_i(v)]$, Cov$[c_i(v), c_{i'}(v)]$, Var$[c_i(v)]$ and $\sigma_{ii'}$

We can express Var$[b(v)]$, Cov$[b(v), c_i(v)]$, Cov$[c_i(v), c_{i'}(v)]$, Var$[c_i(v)]$; $i < i'$; $i, i' = 1, 2, \ldots, I$; in terms of $\sigma_{ii'} = \text{Cov}[m(i, v), m(i', v)]$; $i, i' = 1, 2, \ldots, I$; as follows:

$$\text{Var}[b(v)] = \sum_{i=1}^{I} v_i^2 \sigma_{ii} + 2 \sum_{\substack{i<i' \\ i,i'=1,2,\ldots,I}} v_i v_{i'} \sigma_{ii'},$$

$$\text{Cov}[b(v), c_i(v)] = \sum_{r=1}^{I} v_r \sigma_{ri} - \text{Var}[b(v)],$$

$$\text{Cov}[c_i(v), c_{i'}(v)] = \sigma_{ii'} - \sum_{r=1}^{I} v_r(\sigma_{ri} + \sigma_{ri'}) + \text{Var}[b(v)]$$

where $\sum_{i=1}^{I} v_i = 1$

$$\text{Var}[c_i(v)] = \sigma_{ii} - 2 \sum_{r=1}^{I} v_r \sigma_{ri} + \text{Var}[b(v)].$$

8. Sums of squares under fixed models and expectations of mean squares under mixed models. Tests

Let us examine Tables 1 and 3. Expectations of mean squares for A and B have been obtained under the assumption that interaction is significant. Usually we put $\sum_{i=1}^{I} v_i = \sum_{j=1}^{J} w_j = 1$. In the case of insignificant interaction expectations of mean squares are given in Tables 2 and 4 for the classifications $I\times J$ and $I\times 2$ respectively.

From Table 3 it is clear that comparison of expectations of mean squares given in Table 3 leads to the tests written down at the bottom of this Table.

In the orthogonal case $n_{ij} = k =$ constant we obtain as a particular case the results presented by J. W. Tukey [35], O. Kempthorne [16], M. B. Wilk and O. Kempthorne [37], J. Cornfield and J. W. Tukey [4] and H. Scheffé [24, 25].

Now, let us consider Table 1. In Table 1, sum of squares for interaction, SS_{AB}, is given by least squares analysis. In order to calculate $E(MS_{AB})$ it is necessary to calculate $E\sum_{j=1}^{J}\hat{\beta}_j Q_{.j}$ or $E(Q_{.j}^2)$ and $E(Q_{.j}Q_{.k})$; $j, k = 1, 2, \ldots, J$ which is explained in sections 9 and 10.

From the Table 1 it is seen that by using the ratio $F = MS_B/MS_e$ we can test the null hypothesis H_B that $\mathrm{Var}[b(v)] = 0$.

In the general case the expectation of MS_{AB} is not yet given explicitly. It is done in the $I\times 2$ case, so we can compare it whith the $E(MS_A)$ given in Tables 1 and 2, and suggest a test for A.

Table 4 contains tests for interaction and for effects A and B in the case of mixed model $I\times 2$ when interaction is insignificant. We can use any restrictions: weighted or unweighted.

From Table 2 (the case $I\times J$) we can suggest the test $F = MS_B/MS_e$ under the hypothesis H_B: $\mathrm{Var}[b(v)] = 0$.

Particular cases. We can choose the weights in different ways; this depends on the form of the population.

I. Disproportional frequencies: $n_{ij} \neq p_i s_j \neq$ constant.

1. $w_j = n_{.j}/n$, $v_i = n_{i.}/n$, $\sum^{I} n_{i.} a_i = \sum^{I} n_{i.} c_{ij} = 0$ for all j; $E[b(v)] = E[c_i(v)] = 0$ for all i; $\sum^{I} v_i = \sum^{J} w_j = 1$.

2. $w_j = 1/J$, $v_i = 1/I$, $\sum^{I} a_i = \sum^{I} c_{ij} = 0$ for all j; $E[b(v)] = E[c_i(v)] = 0$ for all i; $\sum v_i = \sum w_j = 1$ (Method of weighted squares of means). Assumption: equal subclass numbers in the classes of the population.

II. Proportionate frequencies: $n_{ij} = p_i s_j$.

1. $w_j = s_j/s_.$, $v_i = p_i/p_.$, $\sum^I p_i a_i = 0$, $\sum^I p_i c_{ij} = 0$ for all j; $E[b(v)] = E[c_i(v)] = 0$; (weighted restrictions); $W_i = s_. p_i$; $V_j = s_j p_.$; $n_{i.} = p_i s_.$, $n_{.j} = p_. s_j$, $n = p_. s_.$.

Under fixed model ($\sum^I p_i a_i = \sum^J s_j \beta_j = \sum^I p_i \gamma_{ij} = \sum^J s_j \gamma_{ij} = 0$ and e_{ijl} are normally and independently distributed, all with mean value 0 and the same unknown variance σ^2) we have the following sums of squares:

$$SS_A = s_. \sum_i^I p_i (\bar{y}_{i..} - \bar{y}_{...})^2; \quad SS_B = p_. \sum_j s_j (\bar{y}_{.j.} - \bar{y}_{...})^2,$$

$$SS_{AB} = \sum_{ij} p_i s_j (\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...})^2, \quad SS_e = \sum_{ijl} (y_{ijl} - \bar{y}_{ij.})^2,$$

where

$$\hat{\mu} = \bar{y}_{...}, \quad \bar{y}_{ij.} = Y_{ij.}/p_i s_j, \quad \bar{y}_{i..} = Y_{i..}/p_i s_., \quad \bar{y}_{.j.} = Y_{.j.}/p_. s_j;$$
$$\hat{a}_i = \bar{y}_{i..} - \bar{y}, \quad \hat{\beta}_j = \bar{y}_{.j.} - \bar{y}, \quad \hat{\gamma}_{ij} = \bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}.$$

For mixed model we obtain:

$$E(MS_A) = \sigma_e^2 + \frac{1}{I-1}\left[s_. \sum^I p_i a_i^2 + \frac{\sum_j s_j^2}{s_.} \sum^I p_i \operatorname{Var}[c_i(v)]\right]$$

(it does not depend on $\operatorname{Cov}(c_i, c_{i'})$ because of the restrictions),

$$E(MS_B) = \sigma_e^2 + \frac{p_.}{J-1}\left(s_. - \frac{\sum^J s_j^2}{s_.}\right) \operatorname{Var}[b(v)],$$

$$E(MS_{AB}) = \sigma_e^2 + \frac{1}{(I-1)(J-1)} \sum^I p_i \left(s_. - \frac{\sum^J s_j^2}{s_.}\right) \operatorname{Var}[c_i(v)],$$

$E(MS_e) = \sigma_e^2$.

Approximate test. When the hypothesis H_0: all $a_i = 0$ is true we have $E(MS_A) \neq E(MS_{AB})$ as it is in the case $I \times 2$. Then both expressions $E(MS_A)$ and $E(MS_{AB})$ though not the same, depend on σ_e^2 and $\operatorname{Var}[c_i(v)]$. Therefore we can use the Satterthwaite's method (1946) to test the hypothesis H_A.

2. $w_j = 1/J$, $v_i = 1/I$, $\sum_i a_i = \sum_i c_{ij} = 0$ for all j; $E[b(v)] = E[c_i(v)] = 0$ for all i (the unweighted restrictions)

$$W_i = p_i J^2 \left(\sum^J \frac{1}{s_j}\right)^{-1}, \quad V_j = s_j I^2 \left(\sum^I \frac{1}{p_i}\right)^{-1}.$$

Table 1.

Mixed model $I \times J$ with unequal subclass numbers (significant interaction)

(General weighted restrictions: $\sum_{i=1}^{I} v_i a_i = \sum_{i=1}^{I} v_i c_{ij} = 0$ for all $j = 1, 2, \ldots, J$; $E[b(v)] = 0$; $E[c_i(v)] = 0$ for all i)

Source of variation	D. F.	Sums of squares under fixed model	$E(MS)$ under mixed model
1. A (fixed)	$I-1$	$SS_A = \sum_{i=1}^{I} W_i(\hat{A}_i - \bar{A})^2$ $= \sum_{i=1}^{I} W_i \hat{A}_i^2 - \frac{(\sum_i W_i \hat{A}_i)^2}{\sum_i W_i}$	$E(MS_A) = \sigma_e^2 + \frac{1}{I-1}\Big\{\Big(\sum_{j=1}^{J} w_j\Big)^2 \sum_{i=1}^{I} W_i\Big(a_i - \frac{\sum W_i a_i}{\sum W_i}\Big)^2 + \Big(\sum_{j=1}^{J} w_j^2\Big)\Big[\sum_{i=1}^{I} W_i \operatorname{Var}[c_i(v)] - \frac{\sum W_i^2 \operatorname{Var}[c_i(v)]}{\sum W_i}\Big] - \frac{2}{\sum W_i} \cdot \sum_{\substack{i<i' \\ i,i'=1,2,\ldots,I}} W_i W_{i'} \operatorname{Cov}[c_i(v), c_{i'}(v)]\Big\}$ When $\sum_j w_j = 1$ and $\operatorname{Var}[c_i(v)] = \operatorname{Cov}[c_i(v), c_{i'}(v)] = 0$ we obtain $E(MS_A) = \sigma_e^2 + \frac{1}{I-1}\sum_{i}^{I} W_i\Big(a_i - \frac{\sum W_i a_i}{\sum W_i}\Big)^2$
2. B (random)	$J-1$	$SS_B = \sum_{j=1}^{J} V_j(\hat{B}_j - \bar{B})^2 =$ $= \sum_{j=1}^{J} V_j \hat{B}_j^2 - \frac{(\sum V_j \hat{B}_j)^2}{\sum_j V_j}$	$E(MS_B) = \sigma_e^2 + \frac{1}{J-1}\Big(\sum_{i=1}^{I} v_i\Big)^2\Big[\sum_{j=1}^{J} V_j - \frac{\sum V_j^2}{\sum V_j}\Big] \operatorname{Var}[b(v)]$

3. AB (random) interaction	$(I-1)(J-1)$	$SS_{AB} = \sum_i^I \sum_j^J n_{ij}(\bar{y}_{ij.} - \bar{y}_{.j.})^2 - \sum_i \hat{a}_i Q_{i.} = \sum_i^I \sum_j^J n_{ij}(\bar{y}_{ij.} - \bar{y}_{i..})^2 - \sum_j \hat{\beta}_j Q_{.j}$ (least-squares analysis)	$E(MS_{AB}) = \frac{J}{J-1}\sigma_e^2 + \frac{1}{(I-1)(J-1)}\Big\{\sum_{i=1}^I \mathrm{Var}[c_i(v)]\eta_i - 2\sum_{\substack{i<i' \\ i,i'=1,2,\ldots,I}} u_{ii'}\mathrm{Cov}[c_i(v), c_{i'}(v)] + \sum_i \sum_j n_{ij}\Big(a_i - \frac{\sum n_{ij} a_i}{n_{.j}}\Big)^2 - E\sum^I \hat{a}_i Q_{i.}\Big\} = \frac{I}{I-1}\sigma_e^2 + \frac{1}{(I-1)(J-1)}\Big\{\Big(n - \sum_i \frac{\sum n_{ij}^2}{n_{i.}}\Big)\mathrm{Var}[b(v)] + \sum_{i=1}^I \Big[n_{i.} - \frac{\sum^J n_{ij}^2}{n_{i.}}\Big]\mathrm{Var}[c_i(v)] + 2\sum_{i=1}^I \mathrm{Cov}[b(v), c_i(v)]\Big[n_{i.} - \frac{\sum^J n_{ij}^2}{n_{i.}}\Big] - E\sum^J \hat{\beta}_j Q_{.j}\Big\}$
4. Error (within subclasses)	$N-IJ$	$SS_e = \sum_i \sum_j \sum_l (y_{ijl} - \bar{y}_{ij.})^2$	$E(MS_e) = \sigma_e^2$

Table 2

Mixed model $I \times J$ with unequal subclass numbers (insignificant interaction)

Source of variation	D. F.	Sums of squares	$E(MS)$ under mixed model
1. A (fixed)	$I-1$	$SS_A = \sum_{i=1}^{I} \hat{a}_i Q_{i.}$ when $\gamma_{ij} = 0$	$y_{ijl} = \mu + a_i + b_j + e_{ijl}$ $E(MS_A) = \sigma_e^2 + \frac{1}{I-1} \sum_{i=1}^{I} \sum_{j=1}^{J} n_{ij} \left(a_i - \frac{\sum_{i=1}^{I} n_{ij} a_i}{n_{.j}} \right)^2$ In the particular case: $n_{ij} = k$ we obtain $E(MS_A) = \sigma_e^2 + \frac{1}{I-1} kJ \sum_{i=1}^{I} a_i^2$
2. B (random)	$J-1$	$SS_B = \sum_j \hat{\beta}_j Q_{.j}$ when $\gamma_{ij} = 0$	$y_{ijl} = \mu + a_i + b_j + e_{ijl}$ $E(MS_B) = \sigma_e^2 + \frac{1}{J-1} \sum_{j=1}^{J} \xi_j \mathrm{Var}[b(v)] =$ $= \sigma_e^2 + \frac{1}{J-1} \left[n - \sum_i \sum_j \frac{n_{ij}^2}{n_{i.}} \right] \mathrm{Var}[b(v)]$ When $n_{ij} = k =$ constant we obtain $E(MS_B) = \sigma_e^2 + Ik \, \mathrm{Var}[b(v)]$

3. AB (random) interaction	$(I-1)(J-1)$	$SS_{AB} = \sum_i \sum_j n_{ij}(\bar{y}_{ij.}-\bar{y}_{.j.})^2 - \sum_i \hat{a}_i Q_{i.}$ $= \sum_i \sum_j n_{ij}(\bar{y}_{ij.}-\bar{y}_{i..})^2 - \sum_j \hat{\beta}_j Q_{.j}$ $(y_{ijl} = \mu+\alpha_i+\beta_j+\gamma_{ij}+e_{ijl})$	$y_{ijl} = \mu+a_i+b_j+c_{ij}+e_{ijl}$ As in Table 1
4. Error (within subclasses)	$N-IJ$	$SS_e = \sum\sum\sum (y_{ijl}-\bar{y}_{ij.})^2$	$E(MS_e) = \sigma_e^2$

Table 3

Mixed model $I\times 2$ with unequal subclass numbers (significant interaction)

Method of weighted squares of means; the unweighted restrictions:

$$\sum_{i=1}^{I} a_i = 0, \quad \sum_{i=1}^{I} c_{ij} 0 \text{ for all } j = 1, 2, \ldots, J;\ E(b_j) = 0,\ E(c_{ij}) = 0 \text{ for all } i.$$

Source of variation	D. F.	Sums of squares	$E(MS)$ under mixed model
1. A (fixed)	$I-1$	$SS_A = \sum_{i=1}^{I} l_i\left(\theta_i - \frac{\sum_i l_i\theta_i}{\sum_i l_i}\right)^2 =$ $= \sum_{i=1}^{I} l_i\theta_i^2 - \frac{\left(\sum_i l_i\theta_i\right)^2}{\sum l_i}$	$E(MS_A) = \sigma_e^2 + \frac{4}{I-1}\sum_i l_i\left(a_i - \frac{\sum^I l_i a_i}{\sum l_i}\right)^2 + L$ When $n_{ij} = k =$ constant we obtain: $E(MS_A) = \sigma_e^2 + \frac{2k}{I-1}\sum^I a_i^2 + \frac{k}{I-1}\sum^I \mathrm{Var}[c_i(v)]$ as it should be (cf. Scheffé's book [26] p. 269 Table 8. 1.1)

Source of variation	D. F.	Sums of squares	$E(MS)$ under mixed model
2. B (random)	1	$SS_B = \frac{(\sum^I z_i)^2}{\sum_{i=1}^{I} \frac{1}{l_i}}$	$E(MS_B) = \sigma_e^2 + \frac{2I^2}{\sum_{i=1}^{I} \frac{1}{l_i}} \operatorname{Var}[b(v)]$ When $n_{ij} = k =$ constant we obtain $E(MS_B) = \sigma_e^2 + kI \operatorname{Var}[b(v)]$ as it should be (cf. ibid.)
3. AB (random) interaction	$I-1$	$SS_{AB} = \sum_{i=1}^{I} l_i \left(z_i - \frac{\sum^I l_i z_i}{\sum_{i=1}^{I} l_i} \right)^2$ $= \sum^I l_i z_i^2 - \frac{(\sum l_i z_i)^2}{\sum_i l_i}$	$E(MS_{AB}) = \sigma_e^2 + L$ When $n_{ij} = k$ we obtain $E(MS_{AB}) = \sigma_e^2 + \frac{k}{I-1} \sum^I \operatorname{Var}[c_i(v)]$ as it should be (cf. ibid.)
2. Error (within subclasses)	$N-2I$	$SS_e = \sum_i \sum_j \sum_l (y_{ijl} - \bar{y}_{ij.})^2$	$E(MS_e) = \sigma_e^2$

Tests. Let us note that except for the component including a_i, $E(MS_A)$ is of the same form as $E(MS_{AB})$. Hence we can test the hypothesis (1) H_A: all $a_i = 0$ using the ratio $F_A = \frac{MS_A}{MS_{AB}}$. We can also present the following hypotheses and corresponding ratios F:

(2) $$H_B\colon \operatorname{Var}[b(v)] = 0;\ F_B = MS_B/MS_e;$$

(3) $$H_{AB}\colon \operatorname{Cov}[c_i(v), c_{i'}(v)] = 0 \text{ for all } i, i' = 1, 2, \ldots, I;$$
$$F_{AB} = MS_{AB}/MS_e.$$

Table 4 *Mixed model $I\times 2$ with unequal subclass numbers (insignificant interaction) (Any restrictions)*

Source of variation	D. F.	Sums of squares	$E(MS)$ under mixed model
1. A (fixed)	$I-1$	$SS_A = \sum_i^I\sum_j^J n_{ij}(\bar{y}_{ij.}-\bar{y}_{.j.})^2 - \sum_{i=1}^I l_i\left(z_i-\frac{\sum^I l_iz_i}{\sum l_i}\right)^2 = \sum_i \hat{a}_iQ_{i.}$ when $\gamma_{ij}=0$	$y_{ijl}=\mu+a_i+b_j+e_{ijl}$; $E(MS_A)=\sigma_e^2+\frac{1}{I-1}\sum_i\sum_j n_{ij}\left(a_i-\frac{\sum_i n_{ij}a_i}{n_{.j}}\right)^2$. When $n_{ij}=k=$ constant we obtain $E(MS_A)=\sigma_e^2+\frac{Jk}{I-1}\sum_i^I a_i^2$ (cf. [26], p. 269)
2. B (random)	1	$SS_B=\frac{(\sum^I l_iz_i)^2}{\sum l_i}=\sum\hat{\beta}_jQ_{.j}=P$ when $\gamma_{ij}=0$	$y_{ijl}=\mu+a_i+b_j+e_{ijl}$; $E(MS_B)=\sigma_e^2+2\left(\sum_{i=1}^I l_i\right)\mathrm{Var}[b(v)]$. When $n_{ij}=k=$ constant we obtain $E(MS_B)=\sigma_e^2+Ik\,\mathrm{Var}\,b(v)$ (cf. ibid.)
3. AB (random) interaction	$I-1$	$SS_{AB}=\sum^I l_i\left(z_i-\frac{\sum l_iz_i}{\sum l_i}\right)^2=\sum_i l_iz_i^2-P$	$y_{ijl}=\mu+a_i+\beta_j+c_{ij}+e_{ijl}$; $E(MS_{AB})=\sigma_e^2+L$ (cf. Table 3)
4. Error (within subclasses)	$N-2I$	$SS_e=\sum_i^I\sum_j^J\sum_l^{n_{ij}}(y_{ijl}-\bar{y}_{ij.})^2$	$E(MS_e)=\sigma_e^2$

1. Test for interaction: H_{AB}: $\mathrm{Cov}[c_i(v), c_{i'}(v)]=0$ for all $i, i'=1, 2, \ldots, I$; $F_{AB}=MS_{AB}/MS_e$
2. H_A: all $a_i=0$; $F_A=MS_A/MS_e$
3. H_B: $\mathrm{Var}[b(v)]=0$; $F_B=MS_B/MS_e$

The method of weighted squares of means does not produce the same results as the standard method given in II. 1. But it is interesting that under the fixed model we can reparametrize the model with weighted restrictions into the model with unweighted restrictions (cf. H. B. Mann [20]). Then using normal equations we obtain the same sums of squares as in II.1. Thus in application it is advisable to use the standard method.

III. Orthogonal case: $n_{ij} = k = \text{constant}$; $w_j = 1/J$, $v_i = 1/I$, $W_i = kJ$, $V_j = kI$; $\sum^{I} a_i = \sum^{I} c_{ij} = 0$ for all j; $E[b(v)] = E[c_i(v)] = 0$ for all i.

We obtain:

$$E(MS_A) = \sigma_e^2 + \frac{1}{I-1}\left\{kJ\sum_{i=1}^{I} a_i^2 + k\sum^{I}\mathrm{Var}[c_i(v)]\right\},$$

$$E(MS_B) = \sigma_e^2 + kI\,\mathrm{Var}[b(v)] \quad \text{and} \quad E(MS_{AB}) = \sigma_e^2 + \frac{k}{I-1}\sum_{i=1}^{I}\mathrm{Var}[c_i(v)]$$

as it should be (cf. [26], p. 269, Table 8.8.1).

IV. $\mathrm{Cov}[c_i(v), c_{i'}(v)] = 0$, $\mathrm{Var}[c_i(v)] = \sigma_{AB}^2 = \text{constant}$ for all $i = 1, 2, \ldots, I$; $\mathrm{Cov}[c_i(v), c_i(v)] = 0$. We can consider all particular cases given above in I, II and III.

Remark. In the case of fixed model (non-orthogonal data) the method of weighted squares of means does not give a test or an estimate of interaction except for the case $I\times 2$ (cf. G. W. Snedecor and G. M. Cox [30]).

9. Calculation of $E(Q_{.j}Q_{.k}), j \neq k; j,k = 1,2,\ldots,J$ in the case of mixed model $I\times J$ with interaction in the case of non-orthogonal data

From Table 1 it is evident that in order to find $E(MS_{AB})$ it is necessary to calculate $E(\sum_{i=1}^{I}\hat{a}_iQ_{i.})$ and $E(\sum_{j=1}^{J}\hat{\beta}_jQ_{.j})$, where $\sum_i Q_{i.} = \sum_j Q_{.j} = 0$. The $\hat{a}_i$'s and the $\hat{\beta}_j$'s are estimates under the model without interaction:

$$y_{ijl} = \mu + a_i + \beta_j + e_{ijl}$$

In order to calculate $E(\sum_{j=1}^{J}\hat{\beta}_jQ_{.j})$ we can calculate $E(Q_{.j}^2)$ and $E(Q_{.j}Q_{.k})$, $j \neq k$; $j,k = 1,2,\ldots,J$, because we can express $\hat{\beta}_j$ as function of $Q_{.j}$, $Q_{.k}$ and y_{ijl}. Then because of $Q_{.j} = Y_{.j} - \sum^{I} n_{ij}\bar{y}_{i..}$ we can express $\sum_j \hat{\beta}_jQ_{.j}$ in terms of y_{ijl} and then as a function of μ, a_i, b_j, c_{ij}, and e_{ijl}.

Under the assumptions: $\sum_i a_i = \sum_i c_{ij} = 0$ for all $j = 1, 2, \ldots, J$; $E(b_j) = E[c_i(v)] = 0$ and $\sum_j b_j \neq 0$, $\sum_j c_{ij} \neq 0$ we obtain:

$$Q_{.j} = \left(n_{.j}b_j - \sum_{i=1}^{I} \frac{n_{ij}\sum_j n_{ij}b_j}{n_{i.}}\right) + \sum_{i=1}^{I} n_{ij}\left(c_{ij} - \frac{\sum_j n_{ij}c_{ij}}{n_{i.}}\right) + \left(n_{.j}e_{.j.} - \sum_{i=1}^{I} n_{ij}e_{i..}\right)$$

and the following expressions:

1. $$a_1 = E\left(n_{.j}b_j - \sum^{I} \frac{n_{ij}\sum_j n_{ij}b_j}{n_{i.}}\right)\left(n_{.k}b_k - \sum_{i=1}^{I} \frac{n_{ik}\sum^{J} n_{ij}b_j}{n_{i.}}\right) =$$
$$= \left[\sum_{l=1}^{J} r_{jl}r_{kl} - (n_{.j} + n_{.k})r_{jk}\right]\operatorname{Var}[b(v)].$$

2. $$a_2 = E\left(n_{.j}b_j - \sum^{I} \frac{n_{ij}\sum_j n_{ij}b_j}{n_{i.}}\right)\left[\sum_i n_{ik}\left(c_{ik} - \frac{\sum_j n_{ij}c_{ij}}{n_{i.}}\right)\right] =$$
$$= -n_{.j}\sum_{i=1}^{I} \frac{n_{ik}n_{ij}}{n_{i.}}\operatorname{Cov}[b(v), c_i(v)] - r_{jk}\sum_{i=1}^{I} n_{ik}\operatorname{Cov}[b(v), c_i(v)] +$$
$$+ \sum_{i=1}^{I} \operatorname{Cov}[b(v), c_i(v)]\frac{n_{ik}}{n_{i.}}\sum_{l=1}^{I} \frac{n_{lj}d_{jl}}{n_{l.}}.$$

3. $$a_3 = E\left(n_{.k}b_k - \sum^{I} \frac{n_{ik}\sum^{J} n_{ij}b_j}{n_{i.}}\right)\left[\sum_{i=1}^{I} n_{ij}\left(c_{ij} - \frac{\sum^{J} n_{ij}c_{ij}}{n_{i.}}\right)\right] =$$
$$= -n_{.k}\sum_{i=1}^{I} \frac{n_{ik}n_{ij}}{n_{i.}}\operatorname{Cov}[b(v), c_i(v)] - r_{jk}\sum_{i=1}^{I} n_{ij}\operatorname{Cov}[b(v), c_i(v)] +$$
$$+ \sum_{i=1}^{I}\left(\operatorname{Cov}[b(v), c_i(v)]\frac{n_{ij}}{n_{i.}}\sum_{l=1}^{I} \frac{n_{lk}d_{il}}{n_{l.}}\right).$$

4. $$a_4 = E\left[\sum_{i=1}^{I} n_{ij}\left(c_{ij} - \frac{\sum_j n_{ij}c_{ij}}{n_{i.}}\right)\right]\left[\sum^{I} n_{ik}\left(c_{ik} - \frac{\sum_j n_{ij}c_{ij}}{n_{i.}}\right)\right] =$$
$$= \sum_{i=1}^{I}\left\{\frac{n_{ij}n_{ik}}{n_{i.}}\left[\frac{\sum_j n_{ij}^2}{n_{i.}} - n_{ij} - n_{ik}\right]\right\}\operatorname{Var}[c_i(v)] +$$
$$+ \sum_{\substack{i<i' \\ i,i'=1,2,\ldots,I}}\left\{\frac{n_{ij}n_{i'k} + n_{ik}n_{i'j}}{n_{i.}n_{i'.}}d_{ii'} - \frac{n_{ij}n_{ik}(n_{i'j} + n_{i'k})}{n_{i.}} - \frac{n_{i'j}n_{i'k}(n_{ij} + n_{ik})}{n_{i'.}}\right\}\times$$
$$\times \operatorname{Cov}[c_i(v), c_{i'}(v)].$$

5. $a_5 = E\left(n_{.j}e_{.j.} - \sum_{i=1}^{I} n_{ij}e_{i..}\right)\left(n_{.k}e_{.k.} - \sum_{i=1}^{I} n_{ik}e_{i..}\right) = -\sigma_e^2 r_{jk}.$

6. $E(Q_{.j}Q_{.k}) = \mathrm{Cov}(Q_{.j}, Q_{.k}) = a_1 + a_2 + a_3 + a_4 + a_5;$
$j \neq k;\ j, k = 1, 2, \ldots, J.$

7. For fixed model, i. e. when $\mathrm{Cov}[c_i(v), c_{i'}(v)] = 0 = \mathrm{Var}[b(v)] = \mathrm{Var}[c_i(v)]$ we obtain $E(Q_{.j}Q_{.k}) = \mathrm{Cov}(Q_{.j}, Q_{.k}) = -\sigma_e^2 \cdot r_{jk}$, as it should be (cf. [21]).

8. In the orthogonal case: $n_{ij} = k =$ constant the $a_2 = 0$, $a_3 = 0$ and $a_4 = 0$ because of the restrictions $\sum_{i=1}^{I} c_{ij} = 0$ for all $j = 1, 2, \ldots, J$. $Q_{.j}$ depends only on b_j and e_{ijl}:

$$Q_{.j} = Ik\left[\left(b_j - \frac{\sum_j b_j}{J}\right) + (e_{.j.} - e_{...})\right]$$

under the restrictions $\sum_i a_i = \sum_i c_{ij} = 0$ for all j.

Then

$$E(Q_{.j}Q_{.k}) = \mathrm{Cov}(Q_{.j}, Q_{.k}) = -\frac{I^2k^2}{J}\mathrm{Var}[b(v)] - \frac{Ik}{J}\sigma_e^2.$$

Let us calculate $E(Q_{.j}^2)$.

10. The calculation of $E(Q_{.j}^2)$; $j = 1, 2, \ldots, J$

1. $a_{\mathrm{I}} = \mathrm{Var}\left(n_{.j}b_j - \sum_{i=1}^{I} \frac{n_{ij}\sum_{j}^{J} n_{ij}b_j}{n_{i.}}\right) =$

$$= \left(n_{.j}^2 - 2n_{.j}r_{jj} + \sum_{l=1}^{J} r_{jl}^2\right)\mathrm{Var}[b(v)].$$

2. $a_{\mathrm{II}} = \mathrm{Var}\left(\sum_{i=1}^{I} n_{ij}c_{ij} - \sum_{i=1}^{I} \frac{n_{ij}\sum_{j}^{J} n_{ij}c_{ij}}{n_{i.}}\right) =$

$$= \sum_{i=1}^{I}\left[n_{ij}^2\left(1 - \frac{2n_{ij}}{n_{i.}} + \frac{\sum_j n_{ij}^2}{n_{i.}^2}\right)\mathrm{Var}[c_i(v)] + \right.$$

$$\left. + \sum_{\substack{i \neq i' \\ i, i' = 1, 2, \ldots, I}} \left[n_{ij}n_{i'j}\left(1 - \frac{n_{ij}}{n_{i.}} - \frac{n_{i'j}}{n_{i'.}} + \frac{\sum_j n_{ij}n_{i'j}}{n_{i.}n_{i'.}}\right)\mathrm{Cov}[c_i(v), c_{i'}(v)]\right]\right].$$

3. $$a_{III} = 2\mathrm{Cov}\left(n_{.j}b_j - \sum_{i=1}^{I}\frac{n_{ij}\sum_j n_{ij}b_j}{n_{i.}},\ \sum_{i=1}^{I} n_{ij}c_{ij} - \sum_{i=1}^{I}\frac{n_{ij}\sum_j n_{ij}c_{ij}}{n_{i.}}\right) =$$

$$= 2\Bigg[n_{.j}\sum_{i=1}^{I} n_{ij}\mathrm{Cov}[b(v), c_i(v)] - n_{.j}\sum_{i=1}^{I}\frac{n_{ij}^2}{n_{i.}}\mathrm{Cov}[b(v), c_i(v)] -$$

$$- r_{jj}\sum_{i=1}^{I} n_{ij}\mathrm{Cov}[b(v), c_i(v)] + \sum_{i=1}^{I}\sum_{i'=1}^{I}\frac{n_{ij}n_{i'j}}{n_{i.}n_{i'.}}d_{ii'}\mathrm{Cov}[b(v), c_i(v)]\Bigg].$$

4. $$a_{IV} = \mathrm{Var}\left(n_{.j}e_{.j.} - \sum_{i=1}^{I} n_{ij}e_{i..}\right) = \sigma_e^2\xi_j.$$

5. $E(Q_{.j}) = 0$.

6. $E(Q_{.j}^2) = \mathrm{Var}(Q_{.j}) = a_I + a_{II} + a_{III} + a_{IV}$.

7. For fixed model we have $\mathrm{Cov}[c_i(v), c_{i'}(v)] = 0 = \mathrm{Var}[c_i(v)] = \mathrm{Var}[b(v)]$ and we obtain $E(Q_{.j}^2) = \sigma_e^2\xi_j = \sigma_e^2\left(n_{.j} - \sum_i \frac{n_{ij}^2}{n_{i.}}\right)$ as it should be (cf. [21]).

8. In the orthogonal case: $n_{ij} = k =$ constant the $a_{II} = 0$ and the $a_{III} = 0$ because of the restrictions that $\sum_{i=1}^{I} c_{ij} = 0$ for all $j = 1, 2, \ldots, J$. Then $E(Q_{.j}^2) = \mathrm{Var}(Q_{.j}) = a_I + a_{IV} = \frac{(J-1)I^2k^2}{J}\mathrm{Var}[b(v)] + \frac{Ik(J-1)}{J}\sigma_e^2$ as it should be.

Acknowledgement

The author is indebted to Professor O. Kempthorne for his advice and suggestions.

The author is grateful to Professor T. A. Bancroft, Director and Head of Statistical Laboratory and Department of Statistics, Iowa State University, Ames, Iowa, for making it possible to carry out the present research at his Statistical Laboratory, under a Rockefeller Foundation fellowship.

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Streszczenie

Celem niniejszej pracy jest rozpatrzenie modeli mieszanych $I \times J$ i $I \times 2$ z interakcją w przypadku danych nieortogonalnych (por. 3 pozycja § 2) przy założeniach H. Scheffé'go [26] dla danych ortogonalnych. Przedstawiono zarówno definicje głównych efektów i interakcyjnych, korzystając z ogólnej formy wag v_i jak i odpowiednie restrykcje ważone. W tablicach 1, 2, 3 i 4 podano sumy kwadratów dla efektów głównych, interakcyjnych i dla błędu przy modelu ze stałymi parametrami (fixed model) oraz wartości oczekiwane odpowiednich średnich kwadratów dla rozważanego mieszanego modelu. Znaleziono wyraźną postać wartości oczekiwanej średniego kwadratu dla interakcji AB w przypadku $I \times 2$ (Tablica 3). Odpowiednia wartość w przypadku ogólnym $I \times J$ (Tablica 1) nie została explicite obliczona. W tablicach 2 i 4 przedstawiono wymienione sumy kwadratów i wartości oczekiwane w przypadku nieistotnej interakcji.

Dla ortogonalnych danych $n_{ij} = \text{constans} = k$ otrzymano jako szczególny przypadek analizowanego modelu wyniki znane w literaturze (J.W. Tukey, O. Kempthorne, H. Scheffé).

W celu obliczenia wartości oczekiwanej średniego kwadratu dla interakcji w przypadku ogólnym $I \times J$ obliczono $E(Q_{.j}^2)$ i $E(Q_{.j}Q_{.k})$; $j, k = 1, 2, \ldots, J$.

Wartości oczekiwane w Tablicach 1, 2, 3 i 4 sugerują testy nieprzypadkowości F dla zweryfikowania hipotez odnośnie efektów głównych i interakcyjnych.

Na str. 62, 63 i 70 rozpatrzono szczególne przypadki modeli określone bądź rodzajem wag w_j i v_i, bądź założeniami dotyczącymi liczby obserwacji n_{ij}, bądź też założeniami o korelacjach między losowymi interakcjami.

Резюме

Целью этой работы является рассмотрение смешанных моделей $I \times J$ и $I \times 2$ с взаимодействием в случае неортогональных данных (ср. 3 поз. 2) при предложениях Х. Шеффе [26] для ортогональных данных. Представлено определения главных эффектов и эффектов взаимодействия, используя общее представление весов w_j к v_j, а также соответствующие взвешенные ограничения. В таблицах 1, 2, 3 и 4 дано, при модели с постоянными параметрами (fixed model), суммы квадратов для главных эффектов, эффектов взаимодействия и для ошибки, а в случае смешанной модели представлено математическое ожидание соответствующих средних квадратов. Найдено четкий вид математического ожидания среднего квадрата для взаимодействия AB в случае $I \times 2$ (Таблица 3). Соответствующее значение в общем случае $I \times J$ (Таблица 1) не вычислено в явном виде. В таблицах 2 и 4 представлено указанные суммы квадратов и математические ожидания в случае несущественного взаимодействия.

Для ортогональных данных $n_{ij} = \text{constans} = к$ получено как частный случай рассматриваемой модели известные в литературе результаты (И. В. Тукей, О. Кемптгорн, Х. Шеффе).

Целью вычисления математического ожидания среднего квадрата для взаимодействия в общем случае $I \times J$ вычислено $E(Q_{.j}^2)$ и $E(Q_{.j}Q_{.k})$; $j, k = 1, 2, \ldots, J$.

Математическое ожидание в таблицах 1, 2, 3 и 4 внушают критерии значимости F для проверки гипотез относительно эффектов главных и эффектов взаимодействия.

На стр. 62, 63, и 70 рассмотрено частные случаи моделей определенных либо родом весов w_j и v_i, либо предположениями относительно числа наблюдений n_{ij}, либо предположениями о корреляциях между случайными взаимодействиями.