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On a Variational Formula for Starlike Functions

O pewnym wzorze wariacyjnym dla funkcji gwiaździstych

О некоторой вариационной формуле для звездообразных функций

In this paper a simple derivation of Hummel's variational formula [1] is given. Hummel proved that if the function $w = f(z) = z + a_2 z^2 + \dots$ is starlike w.r.t. the origin, then the function

$$f^*(z) = f(z) + \lambda(1-|z_0|^2)\left\{e^{i\alpha}\left[\frac{zf(z)}{z_0(z-z_0)} - \frac{f(z_0)}{z_0 f'(z_0)}\left(\frac{zf'(z)}{z-z_0} + \frac{f(z)}{z_0}\right)\right] + \right.$$

(1)

$$\left. + e^{-i\alpha}\left[\overline{\left(\frac{f(z_0)}{z_0 f'(z_0)}\right)}\frac{z^2 f'(z)}{1-\bar{z}_0 z} + \frac{zf(z)}{1-\bar{z}_0 z}\right]\right\} + O(\lambda^2)$$

with arbitrary $z_0 (0 < |z_0| < 1)$ and arbitrary real α is univalent and starlike w.r.t. the origin for all sufficiently small $\lambda > 0$. Besides, $f^*(0) = 0$, $f^{*\prime}(0) = 1$. We will need the following theorem due to G. M. Golusin [2]: Let $w = f(z)$, $f(0) = 0$, be a function regular and univalent in the unit circle $C_1 = \{z: |z| < 1\}$ and let $F(z, \lambda)$ be regular and univalent in the annulus $A = \{z: r \leqslant |z| < 1\}$ for all $0 < \lambda < \lambda_0$, besides $F(z, \lambda)$ is supposed to be regular ($z \in A$ being fixed) for every $|\lambda| < \lambda_0$, and to have the form

(2) $$F(z, \lambda) = f(z) + \lambda q(z) + O(\lambda^2)$$

where the estimation of the last term is uniform on compact subsets of A. Let B_λ be a simply connected domain which arises by adjoining to the domain $F(A, \lambda)$ the interior of the map of $|z| = r$ by $F(z, \lambda)$. For λ small enough B_λ will contain the origin and the function $f^*(z)$ mapping C_1 on B_λ ($f^*(0) = 0$) has the form

$$f^*(z) = f(z) + \lambda q(z) + \lambda z f'(z) \overline{S\left(\frac{1}{\bar{z}}\right)} - \lambda z f'(z) S(z) + O(\lambda^2) \tag{3}$$

where $S(z)$ is the sum of terms with negative powers of z in the Laurent's development of $q(z)/zf'(z)$ in A. Consider the function

$$F(z, \lambda) = f(z) + \lambda f(z) R(z)$$

where

$$R(z) = e^{i\alpha} \frac{1 - \bar{z}_0 z}{z - z_0} + e^{-i\alpha} \frac{z - z_0}{1 - \bar{z}_0 z}$$

$R(z)$ clearly real and bounded on $|z| = 1$ so that the boundary of B_λ arises from that of $f(C_1)$ by a suitable shifting along a ray from the origin. It is easy to see that $F(z, \lambda)$ fulfils all the conditions of Golusin's formula for small λ. We have

$$S(z) = (1 - |z_0|^2) e^{i\alpha} \frac{f(z_0)}{z_0 f'(z_0)} \frac{1}{z - z_0}$$

$$\overline{S\left(\frac{1}{\bar{z}}\right)} = (1 - |z_0|^2) e^{-i\alpha} \overline{\left(\frac{f(z_0)}{z_0 f'(z_0)}\right)} \frac{z}{1 - \bar{z}_0 z}$$

so that (3) takes the form

$$f_1^*(z) = f(z) + \lambda f(z) \left[e^{i\alpha} \frac{1 - \bar{z}_0 z}{z - z_0} + e^{-i\alpha} \frac{z - z_0}{1 - \bar{z}_0 z} \right] +$$

$$+ (1 - |z_0|^2 \lambda z f'(z) \left[\overline{\left(\frac{(f(z_0)}{z_0 f'(z_0)}\right)} e^{-i\alpha} \frac{z}{1 - \bar{z}_0 z} - e^{-i\alpha} \frac{f(z_0)}{z_0 f'(z_0)} \frac{1}{z - z_0} \right] + O(\lambda^2)$$

Hence

$$f_1^{*\prime}(0) = 1 + \lambda \left[(1 - |z_0|^2) e^{i\alpha} \frac{f(z_0)}{z^2 f'(z_0)} - e^{i\alpha} \frac{1}{z_0} - z_0 e^{-i\alpha} \right] + O(\lambda^2).$$

By dividing and developing the quotient into the powers of λ we obtain

$$f^*(z) = \frac{f_1^*(z)}{f_1^{*\prime}(0)} = f_1^*(z) - f(z)[f_1^*(0) - 1] + O(\lambda^2)$$

which becomes (1) after a suitable rearranging of terms.

REFERENCES

[1] Hummel, J. A., *A variational method for starlike functions*, Proc. Amer. Math. Soc., **9** (1958), p. 82-87.

[2] Голузин Г. М., *Геометрическая теория функций комплексного переменного*, Москва-Ленинград, 1952.

Streszczenie

Autor posługując się twierdzeniem G. M. Gołuzina i dobierając funkcję $F(z, \lambda)$ otrzymuje wzór wariacyjny dla klasy funkcji gwiaździstych.

Резюме

Автор пользуется теоремой Г. М. Голузима получает вариационную формулу для класса звездообразных функций.