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On Univalent Functions with Two Preassigned Values

O funkcjach jednolistnych z dwiema zadanymi wartościami

Об однолистных функциях с двумя заданными значениями

1. Introduction

The class (S) of regular functions F(z) univalent in the unit circle K and subject to conditions F(0) = 0, F'(0) = 1 has been investigated very widely. In particular the both above conditions impose well known restrictions on |F(z)| at $z \in K$, whereas the single condition F(0) = 0 does not. In [6] p. 66, P. Montel suggested to find the precise bounds of $|\varphi(z)|$ for $\varphi(z)$ being a function regular and univalent in K and such that

(1.1)
$$\varphi(0) = 0, \ \varphi(z_0) = 1, \ 0 < |z_0| < 1.$$

If e.g. F(0) = 0, $F(z_0) = A$, then $\varphi(z) = A^{-1}F(z)$ satisfies (1.1) so that the value of $\varphi(z)$ at z_0 is irrelevant.

Some extremal problems connected with functions $\varphi(z)$ satisfying (1.1) i. e. $\varphi \epsilon(Sz_0)$ have been investigated by several authors. V. Singh [8] and Z. Lewandowski [5] obtained precise bounds for $|\varphi(z)|$ with some further restrictions concerning φ , the former assumed φ to be typically real whereas $z_0 = 1$, the latter assumed φ to be starlike with respect to the origin.

On the other hand the well known double inequality

(1.2)
$$\frac{|z|}{(1+|z|)^2} \leqslant |F(z)| \leqslant \frac{|z|}{(1-|z|)^2}, \ F \epsilon(S),$$

may be replaced by the following single inequality

(1.3)
$$\frac{F(z_1)}{F(z_2)} \leqslant \left| \frac{z_1}{z_2} \left| \left(\frac{1+|z_2|}{1-|z_1|} \right)^2 \right| \right|$$

The equality in (1.3) is attained only when F(z) is the Koebe function $z(1-e^{-i\theta}z)^{-2}$ and $z_1 = |z_1|e^{i\theta}$, $z_2 = -|z_2|e^{i\theta}$. Hence, if z_1 and z_2 are not situated on opposite radii, the precise bound in (1.3) can be lowered. We are thus led to the following problem:

(i) Find the precise upper bound $k(z_1, z_2)$ of $|F(z_1)/F(z_2)|$

for fixed $z_1, z_2 \in K$, $z_1 \neq z_2$, $z_1 \neq 0 \neq z_2$ and varying $F \in (S)$.

The problem (i) is a particular case of a more general one:

(ii) Find the region of variability of $F(z_1)/F(z_2)$ for fixed $z_1, z_2 \in K$ when F is varying within the class (S).

It is obvious that the assumption F'(0) = 1 in (i) and (ii) can be dropped since $F(z_1)/F(z_2)$ does not depend on F'(0). Still it is advisable sometimes to assume F'(0) = 1 in order to achieve compactness.

Z. Lewandowski [5] solved both problems for the subclass of functions starlike w.r.t. the origin.

Obviously the solution of (i) provides at once the solution of Montel's problem mentioned above. In that case $F(z_2) = 1$, so that $|\varphi(z)| \leq \langle k(z, z_0)$. Again $|F(z_2)/F(z_1)| \leq k(z_2, z_1)$, or $k^{-1}(z_2, z_1) \leq |F(z_1)/F(z_2)|$ and this means $k^{-1}(z_0, z) \leq |\varphi(z)|$, both bounds being precise and being attained, since the class (S) is compact.

It is easy to see that the problem (ii) is equivalent to the general interpolation problem for univalent functions (n = 2) as considered in [7]. Although this problem has been solved by H. L. Royden for n = 2 in terms of inequalities between periods of certain elliptic and hyperelliptic integrals, no explicite determination of the domain of variability of $\varphi(z_1)$ (z_1 fixed, $\varphi \in (Sz_0)$ varying) is given in [7] so that an immediate derivation of the solution of (i) from Royden's inequalities does not seem to be possible.

In this paper which is a part of research done by the author as a Research Assistant at the Imperial College of Science and Technology in London, we will use variational methods to obtain the solution of the problem (i). The author is very much indebted to Prof. W. K. Hayman for his encouragement and his helpful criticism.

2. Statement of results

Let z_1, z_2 be two points of the unit circle K different from 0 and from each other and let

(2.1)
$$\eta = \frac{\bar{z}_1 z_2 - z_1 \bar{z}_2 - |z_1 - z_2| |1 - z_1 \bar{z}_2|}{\bar{z}_1 (1 + |z_2|^2) - \bar{z}_2 (1 + |z_1|^2)} = e^{i_a} (a \text{ is real}).$$

If λ_j is a closed Jordan curve situated in K containing $0, z_j$, but not z_k in its interior and

(2.2)
$$\Omega_j = \int_{\lambda_j} \sqrt{Q(\zeta)} d\zeta, \quad j, k = 1, 2.$$

(2.3)
$$Q(z) = \frac{\eta(z-\eta)^2}{z(z-z_1)(z-z_2)(1-\bar{z}_1z)(1-\bar{z}_2z)}$$

then Ω_1 is purely real, Ω_2 is purely imaginary.

If \wp is Weierstrass's elliptic function with periods Ω_1, Ω_2 , and

$$e_k = \wp\left(rac{1}{2} \, arOmega_k
ight), k = 1, 2,$$

we have

(2.4)
$$k(z_1, z_2) = \sup_{(F)} \left| \frac{F(z_1)}{F(z_2)} \right| = \left| \frac{f(z_1)}{f(z_2)} \right| = \left| \frac{2e_2 + e_1}{2e_1 + e_2} \right|$$

with the extremal function

(2.5)
$$f(z) = \wp \left(\int_{0}^{z} \left[Q(\zeta) \right]^{1/2} d\zeta + \frac{1}{2} \left(\Omega_{1} + \Omega_{2} \right) \right) + e_{1} + e_{2}$$

which is unique apart from a constant factor. F(z) is supposed to be a function regular and univalent in K and F(0) = 0.

3. A differential equation for the extremal function

Let f(z) be a function for which the functional $|F(z_1)/F(z_2)|(z_1, z_2 \neq 0, z_1, z_2 \in K, z_1 \neq z_2$ are supposed to be fixed) attains its upper bound $k(z_1, z_2)$ within the class (S). We assume that $F \in (S)$ in order to obtain a compact class of functions although the condition F'(0) = 1 is not necessary, as pointed out above.

We will need Schiffer's variational formula (see e.g. [2], p. 302). Let $f(z) = z + a_2 z^2 + ... = w$ map K conformally onto a domain G with the boundary γ . If $w_0 \neq 0$ is a fixed interior point of G then the functions

(3.1)
$$w^* = w + \varrho^2 e^{2i\varphi} / (w - w_0)$$

are univalent for $|w - w_0| > \varrho$ and $0 \le \varphi < \pi$ and for all $\varrho > 0$ sufficiently small they will transform γ into γ^* being the boundary of a new simply connected domain G^* containing w = 0 as an interior point. G and G^* differ very little if ϱ is small enough. According to Schiffer's formula the univalent function $f^*(z)$ mapping K onto G^* such that $f^*(0) = 0$ and $f^{*'}(0) > 0$, has the form:

(3.2)
$$f^{*}(z) = f(z) + \varrho^{2} e^{2i\varphi} \left[\frac{zf'(z)}{u [f'(u)]^{2}(u-z)} + \frac{1}{f(z) - f(u)} + \frac{f'(z)}{f(u)} \right] + \\ + \varrho^{2} e^{-2i\varphi} \left[\frac{z^{2}f'(z)}{\overline{u} [\overline{f'(u)}]^{2}(1 - \overline{u}z)} - \frac{z^{2}f'(z)}{\overline{f(u)}} \right] + O(\varrho^{3}),$$

where $f(u) = w_0$ and the estimation of the term $O(\varrho^3)$ is uniform on compact subsets of K. In general $f^* \notin (S)$, this is however irrelevant for the ratio $f^*(z_1)/f^*(z_2)$ which does not depend on $f^*'(0)$. (3.2) implies

(3.3)
$$\log \frac{f^{*}(z)}{f(z)} = \varrho^{2} e^{2i\varphi} \left[\frac{zf'(z)}{f(z)} \frac{1}{u[f'(u)]^{2}(u-z)} + \frac{1}{f(z)[f(z)-f(u)]} + \frac{f'(z)}{f(z)f(u)} \right] + \varrho^{2} e^{-2i\varphi} \left[\frac{z^{2}f'(z)}{f(z)\overline{u}[f'(u)]^{2}(1-\overline{u}z)} - \frac{z^{2}f'(z)}{f(z)\overline{f(u)}} \right] + O(\varrho^{3}).$$

Thus for the extremal function giving the maximal value to $|F(z_1)/F(z_2)|$, and also to $\log |F(z_1)/F(z_2)|$ we have, using (3.3)

$$\begin{split} \delta \log \left| \frac{f(z_1)}{f(z_2)} \right| &= \delta \Re \left\{ \log \frac{f(z_1)}{f(z_2)} \right\} = \Re \delta \log \frac{f(z_1)}{f(z_2)} = \\ &= \Re \left\{ \log \frac{f^*(z_1)}{f^*(z_2)} - \log \frac{f(z_1)}{f(z_2)} \right\} + O(\varrho^3) = \\ &= \Re \left\{ \log \frac{f^*(z_1)}{f(z_1)} - \log \frac{f^*(z_2)}{f(z_2)} \right\} + O(\varrho^3) = \\ .4) &= \Re \left\{ \varrho^2 e^{2i\varphi} \left[\frac{1}{u[f'(u)]^2} \left(\frac{z_1 f'(z_1)}{f(z_1)} \frac{1}{u - z_1} - \frac{z_2 f'(z_2)}{f(z_2)} \frac{1}{u - z_2} \right) \right. \\ &+ \frac{1}{f(z_1)(f(z_1) - f(u))} - \frac{1}{f(z_2)(f(z_2) - f(u))} + \frac{1}{f(u)} \left(\frac{f'(z_1)}{f(z_1)} - \frac{f'(z_2)}{f(z_2)} \right) \right] \\ &+ \varrho^2 e^{-2i\varphi} \left[\frac{1}{\overline{u}[f'(u)]^2} \left(\frac{z_1^2 f'(z_1)}{f(z_1)} \frac{1}{1 - \overline{u}z_1} - \frac{z_2^2 f'(z_2)}{f(z_2)} \frac{1}{1 - \overline{u}z_2} \right) \right. \\ &- \frac{1}{\overline{f(u)}} \left(\frac{z_1^2 f'(z_1)}{f(z_1)} - \frac{z_2^2 f'(z_2)}{f(z_2)} \right) \right] \right\} \leqslant 0 \,. \end{split}$$

Putting

(3

(3.5)
$$\frac{z_1 f'(z_1)}{f(z_1)} = A, \quad \frac{z_2 f'(z_2)}{f(z_2)} = B$$

and using the fact that $\Re\{a+\beta\} = \Re\{a+\overline{\beta}\}$ we see that (3.4) takes the form

$$\begin{aligned} \Re \Big\{ \varrho^2 e^{2i\varphi} \Big[\frac{1}{u \left[f'(u) \right]^2} \Big(\frac{A}{u-z_1} - \frac{B}{u-z_2} + \frac{\bar{A}\bar{z}_1}{1-u\bar{z}_1} - \frac{\bar{B}\bar{z}_2}{1-u\bar{z}_2} \Big) \\ (3.6) &+ \frac{1}{f(u)} \Big(\frac{A}{z_1} - \bar{A}\bar{z}_1 - \frac{B}{z_2} + \bar{B}\bar{z}_2 \Big) + \frac{1}{f(z_1)(f(z_1) - f(u))} \\ &- \frac{1}{f(z_2)(f(z_2) - f(u))} \Big] \Big\} \leqslant 0 \,. \end{aligned}$$

Hence since φ is arbitrary we obtain writing z instead of u the following differential equation for f(z):

(3.7)

$$egin{aligned} \overline{z(f'(z))^2} \left(\overline{z-z_1} + \overline{1-zar{z}_1} - \overline{z-z_2} - \overline{1-zar{z}_2}
ight) \ &+ rac{1}{f(z)} \left(rac{A}{z_1} - ar{A}ar{z}_1 - rac{B}{z_2} + ar{B}ar{z}_2
ight) + rac{1}{f(z_1)(f(z_1)-f(z))} \ &- rac{-1}{f(z_2)(f(z_2)-f(z))} = 0 \,. \end{aligned}$$

 $1 \mid A \quad \overline{A}\overline{z}_1 \quad B \quad \overline{B}\overline{z}_2 \setminus$

We now show that

(3.8)
$$A/z_1 - \bar{A}\bar{z}_1 - B/z_2 + \bar{B}\bar{z}_2 = 1/f(z_1) - 1/f(z_2),$$

for the extremal function.

The transformation $z^* = (z + \varepsilon)/(1 + \overline{\varepsilon} z)$ with complex ε , $|\varepsilon| < 1$, carries the unit circle K into itself. Put

$$f^*(z) = f\left(rac{z+arepsilon}{1+ar arepsilon z}
ight) - f(arepsilon) \,.$$

Clearly $f^{\bullet}(0) = 0$. Since $(z+\varepsilon)/(1+\varepsilon z) = (z+\varepsilon)(1-\varepsilon z+\varepsilon^2 z^2-\ldots) = z+\varepsilon-\varepsilon z^2+O(\varepsilon^2)$ we have by Taylor's formula, in view of $f(\varepsilon) = \varepsilon f'(0)+O(\varepsilon^2)$,

$$f^{*}(z) = f(z) + (\varepsilon - \overline{\varepsilon} z^{2})f'(z) - \varepsilon f'(0) + O(\varepsilon^{2}),$$

or

$$f^{*}(z)/f(z) = 1 + (\varepsilon - \overline{\varepsilon} z^{2}) f'(z)/f(z) - \varepsilon f'(0)/f(z) + \mathcal{O}(\varepsilon^{2})$$

and finally

$$\log(f^*(z)/f(z)) = (\varepsilon - \varepsilon z^2)f'(z)/f(z) - \varepsilon f'(0)/f(z) + O(\varepsilon^2)$$

Hence

$$\begin{split} \delta \log \left| \frac{f(z_1)}{f(z_2)} \right| &= \delta \Re \log \frac{f(z_1)}{f(z_2)} = \Re \left\{ \log \frac{f^*(z_1)}{f^*(z_2)} - \log \frac{f(z_1)}{f(z_2)} \right\} + O(\varepsilon^3) = \\ &= \Re \left\{ \log \frac{f^*(z_1)}{f(z_1)} - \log \frac{f^*(z_2)}{f(z_2)} \right\} + O(\varepsilon^2) = \\ &= \Re \left\{ \varepsilon \left[\frac{f'(z_1)}{f(z_1)} - \frac{f'(0)}{f(z_1)} - \frac{f'(z_2)}{f(z_2)} + \frac{f'(0)}{f(z_2)} \right] + \overline{\varepsilon} \left[z_2^2 \frac{f'(z_2)}{f(z_2)} - z_1^2 \frac{f'(z_1)}{f(z_2)} \right] \right\} = \\ &= \Re \left\{ \varepsilon \left[\frac{f'(z_1)}{f(z_1)} - \frac{f'(0)}{f(z_1)} - \frac{f'(z_2)}{f(z_2)} + \frac{f'(0)}{f(z_2)} + \frac{f'(0)}{f(z_2)} + \frac{f'(0)}{f(z_2)} + \frac{f'(z_2)}{f(z_2)} + \frac{f'(z_2)}{f(z_2)}$$

for any small complex ε . This implies

$$\frac{f'(z_1)}{f(z_1)} - \frac{f'(z_2)}{f(z_2)} + \bar{z}_2^2 \frac{f'(z_2)}{\bar{f}(z_2)} - \bar{z}_1^2 \frac{f'(z_1)}{\bar{f}(z_1)} = \frac{f'(0)}{f(z_1)} - \frac{f'(0)}{\bar{f}(z_2)}$$

and in view of f'(0) = 1 it implies (3.8). Putting w = f(z), $w_1 = f(z_1)$, $w_2 = f(z_2)$, we see that (3.7) takes, in view of (3.8), the form

$$\frac{P(z)}{zf'(z)^3} + \frac{1}{w} \left(\frac{1}{w_1} - \frac{1}{w_2} \right) + \frac{1}{w_1(w_1 - w)} - \frac{1}{w_2(w_2 - w)} = 0,$$

or

(3.9)
$$zP(z)\left(\frac{dz}{z}\right)^2 = \frac{(w_1 - w_2)dw^2}{w(w_1 - w)(w_2 - w)}$$

where

(3.10)
$$P(z) = \frac{A}{z-z_1} + \frac{A\bar{z}_1}{1-z\bar{z}_1} - \frac{B}{z-z_2} - \frac{B\bar{z}_2}{1-z\bar{z}_2}$$

the constants A, B being defined by (3.5).

4. The form of P(z)

In this chapter we will prove that

(4.1)
$$P(z) \equiv \frac{-Ce^{-ia}(z-e^{ia})^2}{(z-z_1)(z-z_2)(1-z\overline{z}_1)(1-z\overline{z}_2)}$$

C > 0 and a being real constants. Besides, we will show in sect. 5 that

(4.2)
$$e^{ia} = \frac{\bar{z}_1 z_2 - z_1 \bar{z}_2 - |z_1 - z_2| |1 - z_1 \bar{z}_2|}{\bar{z}_1 (1 + |z_2|^2) - \bar{z}_2 (1 + |z_1|^2)}.$$

We now prove that, if f(z) is the extremal function for which

$$|f(z_1)/f(z_2)| = k(z_1, z_2),$$

then

(4.3)
$$\Im \frac{z_1 f'(z_1)}{f(z_1)} = \Im \frac{z_2 f'(z_2)}{f(z_2)}$$

resp. with the notation of (3.5)

(4.4)
$$(A - \overline{A}) - (B - \overline{B}) = 0.$$

Let us rotate z_1 and z_2 about the origin by the same small angle θ so that $z^* = ze^{i\theta}$ and $z^* - z = \Delta z = iz\theta + O(\theta^2)$. Then $f^*(z) = f(z^*) = f(z) + \Delta z f'(z) + O(\theta^2)$ and

$$f^*(z)/f(z) = 1 + [izf'(z)/f(z)]\theta + \mathrm{O}(\theta^2),$$

or

(4.5)
$$\log \frac{f^*(z)}{f(z)} = \frac{izf'(z)}{f(z)} \theta + \mathcal{O}(\theta^2).$$

Hence

$$egin{aligned} \delta \log \left| rac{f(z_1)}{f(z_2)}
ight| &= \mathfrak{R} \left\{ \log rac{f^*(z_1)}{f(z_1)} - \log rac{f^*(z_2)}{f(z_2)}
ight\} + \mathrm{O}\left(heta^2
ight) = \ &= \mathfrak{R} \left\{ i heta \left[rac{z_1 f'(z_1)}{f(z_1)} - rac{z_2 f'(z_2)}{f(z_2)}
ight\} \leqslant 0 \end{aligned}$$

for the extremal case. Since θ can be either positive, or negative, we see that $\Re\{i(A-B)\}=0$, $\Re(iA)=\Re(iB)$, or $\Im A=\Im B$ which gives (4.4).

Now we will prove that

$$(4.6) \qquad \qquad \overline{P(1/\overline{z})} \equiv z^2 P(z).$$

We have

$$P\left(\frac{1}{\bar{z}}\right) = \frac{A\bar{z}}{1-z_1\bar{z}} + \frac{\bar{A}\bar{z}_1\bar{z}}{\bar{z}-\bar{z}_1} - \frac{B\bar{z}}{1-z_2\bar{z}} - \frac{\bar{B}\bar{z}_2\bar{z}}{\bar{z}-\bar{z}_2}$$

and hence

$$\overline{P(1/\bar{z})} = \frac{\bar{A}z}{1-z\bar{z}_1} + \frac{Az_1z}{z-z_1} - \frac{\bar{B}z}{1-z\bar{z}_2} - \frac{Bz_2z}{z-z_2} = z \Big[\frac{\bar{A} - \bar{A}z\bar{z}_1 + \bar{A}z\bar{z}_1}{1-z\bar{z}_1} + \frac{Az_1 - Az + Az}{z-z_1} - \frac{\bar{B} - \bar{B}z\bar{z}_2 + \bar{B}z\bar{z}_2}{1-z\bar{z}_3} - \frac{Bz_2 - Bz + Bz}{z-z_2} \Big] = z \big[(\bar{A} - A) - (\bar{B} - B) + zP(z) \big] = z^2 P(z)$$

in view of (4.4).

For $z = e^{i\theta}(\theta \text{ real})$ (4.6) takes the form

$$\overline{e^{i\theta}P(e^{i\theta})} = e^{i\theta}P(e^{i\theta})$$

which means that zP(z) is real on |z| = 1.

Next we apply Julia's variational formula in order to prove that also $zP(z) \ge 0$ on |z| = 1. At the same time we will also show that the map of K by the extremal function is a slit domain, the boundary of f(K) being a single analytic arc.

Julia's variational formula is an analogue of Hadamard's formula, the variation of Green's function being replaced by that of the mapping function (see e.g. [1], or [4]).

Given a piecewise analytic curve γ being the boundary of a simply connected domain G containing w = 0 inside, let $z = \varphi(w)$ be the function mapping G conformally on K so that $\varphi(0) = 0$, $\varphi'(0) > 0$. Let $n(s) = \varrho p(s)$ be the normal displacement of the point w on γ , where p(s) > 0 for the outward normal, p(s) being a continuous function of the arc length s on γ , vanishing in the neighbourhood of corner points of γ . If $f(z) = \varphi^{-1}(z)$ and if $f^*(z)$ maps K on the new domain G^* with the boundary γ^* resulting from normal displacements p(s) of points of γ , and if $f^*(0) = 0$, $f^{*'}(0) > 0$, then

$$f^{*}(z) - f(z) = \frac{1}{2\pi} \int \frac{zf'(z)(\zeta + z)}{\zeta - z} |\varphi'(w)|^{2} \varrho p(s) ds_{v} + O(\varrho^{2})$$

where ζ is the point on |z| = 1 corresponding to the point w on γ . Hence

$$\log rac{f^*(z)}{f(z)} = rac{1}{2\pi} \int rac{z f'(z)}{f(z)} \, rac{\zeta+z}{\zeta-z} \, |arphi'(w)|^2 arrho p\left(s
ight) ds_w + \mathrm{O}\left(arrho^2
ight).$$

As before, with the notation of (3.5),

$$\delta \log \left| \frac{f(z_1)}{f(z_2)} \right| = \Re \left\{ \frac{1}{2\pi} \int\limits_{\gamma} \left(A \frac{\zeta + z_1}{\zeta - z_1} - B \frac{\zeta + z_2}{\zeta - z_2} \right) |\varphi'(w)|^2 \varrho p(s) ds_w \right\} = 0$$

(4.7)

$$=\frac{1}{2\pi}\int\limits_{\gamma}\Re\left\{A\frac{\zeta+z_1}{\zeta-z_1}-B\frac{\zeta+z_2}{\zeta-z_2}\right\}|\varphi'(w)|^2\varrho p(s)\,ds_w\leqslant 0$$

This implies that

(4.8)
$$\Re \left\{ A \frac{e^{i\theta} + z_1}{e^{i\theta} - z_1} - B \frac{e^{i\theta} + z_2}{e^{i\theta} - z_2} \right\} \ge 0$$

for any real θ . In fact, if for certain θ_0 the l.h.s. in (4.8) were negative, then it would be negative in a certain neighbourhood of θ_0 corresponding to a subarc γ_0 of γ . Taking now p(s) < 0 on γ_0 and $p(s) \equiv 0$ otherwise (this corresponds to a shrinking of G which is always admissible) we would obtain by (4.7) $\delta \log |f(z_1)/f(z_2)| > 0$ (since $\varphi'(w)$ vanishes at isolated points in view of (3.9)) and this is impossible for the extremal case. So the inequality (4.8) is proved. Besides, it is readily seen from (4.8) that no expansion of the domain in the extremal case is possible in view of the fact that the l.h.s. of (4.8) vanishes only at isolated values of θ . This means that CG has no interior points. On the other hand (3.9) shows that γ is piecewise analytic. So we have shown that G is a slit domain, the slit being piecewise analytic.

We have still to show that $zP(z) \ge 0$ on |z| = 1. We have for $z = e^{i\theta}$

$$\begin{split} \Re\left\{\frac{Az}{z-z_1} + \frac{\bar{A}\bar{z}_1z}{1-z\bar{z}_1}\right\} &= \Re\left\{\frac{A}{1-z_1e^{-i\theta}} + \frac{Az_1e^{-i\theta}}{1-z_1e^{-i\theta}}\right\} = \\ &= \Re\left\{A\frac{1+z_1e^{-i\theta}}{1-z_1e^{-i\theta}}\right\} = \Re\left\{A\frac{e^{i\theta}+z_1e^{-i\theta}}{e^{i\theta}+z_1e^{-i\theta}}\right\} = \Re\left\{A\frac{e^{i\theta}+z_1e^{-i\theta}}{e^{i\theta}+z_1e^{-i\theta}}\right\} = \\ &= \Re\left\{A\frac{1+z_1e^{-i\theta}}{1-z_1e^{-i\theta}}\right\} = \Re\left\{A\frac{e^{i\theta}+z_1e^{-i\theta}}{e^{i\theta}+z_1e^{-i\theta}}\right\} = \Re\left\{A\frac{e^{i\theta}+z_1e^{-i\theta}}{e^{i\theta}+z_1e^{-i\theta}}\right\} = \\ &= \Re\left\{A\frac{1+z_1e^{-i\theta}}{1-z_1e^{-i\theta}}\right\} = \Re\left\{A\frac{e^{i\theta}+z_1e^{-i\theta}}{e^{i\theta}+z_1e^{-i\theta}}\right\} = \Re\left\{A\frac{e^{i\theta}+z_1e^{-i\theta}}{e^{i\theta}$$

and hence

$$\Re\{e^{i\theta}P(e^{i\theta})\} = \Re\left\{A\frac{e^{i\theta}+z_1}{e^{i\theta}-z_1} - B\frac{e^{i\theta}+z_2}{e^{i\theta}-z_2}\right\} \ge 0$$

by (4.8). Thus $zP(z) \ge 0$ on |z| = 1. Besides, we have

$$P(z) = \frac{(\bar{A}\bar{z}_1 - A\bar{z}_1)z + \text{const.}}{(z - z_1)(1 - z\bar{z}_1)} - \frac{(\bar{B}\bar{z}_2 - \bar{B}\bar{z}_2)z + \text{const.}}{(z - z_2)(1 - z\bar{z}_2)} = \\ = \frac{[(A - \bar{A}) - (B - \bar{B})]\bar{z}_1\bar{z}_2z^3 + (\text{lower powers of } z)}{(z - z_1)(1 - z\bar{z}_2)(z - z_2)(1 - z\bar{z}_1)}$$

and in view of (4.4) the leading coefficient in the numerator vanishes. Hence P(z) has at most two roots different from ∞ . However, by (4.6)

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and this means that if z_0 is a finite root of P(z), $z_0 \neq 0$, so $1/\overline{z}_0$ is also a root of P(z). Thus P(z) having at most two roots has necessarily a pair of roots symmetric with respect to |z| = 1. Now the r.h.s. in (3.9) does not vanish for any $w \epsilon f(K)$ so that also P(z) cannot vanish inside K. This means that both roots of P(z) lie on |z| = 1. However, zP(z) is real and positive on |z| = 1 and having two roots on |z| = 1, it has necessarily one double root $e^{i\alpha}$ on |z| = 1. Hence

(4.10)
$$P(z) = \frac{\mu (z - e^{ia})^2}{(z - z_1)(1 - \overline{z}_1 z)(z - z_2)(1 - \overline{z}_2 z)}, \quad a \text{ real, } \mu = \text{const.}$$

Now the denominator in (4.10) for |z| = 1 has the form $z^2 |z - z_1|^2 |z - z_2|^2$ and this means that

$$zP(z) = rac{\mu (z - e^{ia})^2}{z \, |z - z_1|^2 \, |z - z_2|^2} \geqslant 0 \,, \quad ext{ or } rac{\mu (z - e^{ia})^2}{z} = \lambda \geqslant 0 \,, \, |z| = 1$$

Hence $\mu = \lambda(\theta)e^{i\theta}(e^{i\theta} - e^{ia})^{-2} = \lambda(\theta)e^{-ia}[2i\sin(\frac{1}{2}(\theta - a))]^{-2}$, or $\mu e^{ia} = -\lambda(\theta)/4\sin^2[\frac{1}{2}(\theta - a)] = -C$. Ultimately $\mu = -Ce^{-ia}$ or

(4.11)
$$zP(z) \equiv \frac{-Cze^{-ia}(z-e^{ia})^2}{(z-z_1)(1-z\overline{z}_1)(z-z_2)(1-z\overline{z}_2)}, \quad C > 0, a \text{ real},$$

and hence (3.9) takes the form

(4.12)
$$\frac{Ce^{-ia}(z-e^{ia})^2 dz^2}{z(z-z_1)(1-z\overline{z}_1)(z-z_2)(1-z\overline{z}_2)} = \frac{(w_2-w_1)dw^2}{w(w_1-w)(w_2-w)}, \quad C > 0, \text{ a real.}$$

Making z tend to z_1 in (4.12) we have

(4.13)
$$A = \frac{z_1 f'(z_1)}{f(z_1)} = \frac{C e^{-ia} (z_1 - e^{ia})^2}{(1 - |z_1|^2) (z_2 - z_1) (1 - z_1 \overline{z}_2)}$$

and similarly

(4.14)
$$B = \frac{z_2 f'(z_2)}{f(z_2)} = \frac{C e^{-ia} (z_2 - e^{ia})^2}{(1 - |z_2|^2) (z_1 - z_2) (1 - \overline{z}_1 z_2)}$$

(4.12) means that f'(z) is finite and different from 0 for all points of |z| = 1 but two. For $z = e^{ia}$, $f'(e^{ia}) = 0$, for $z = \tau$ such that $f(\tau) = \infty$ we have $f'(\tau) = \infty$. This means that both arcs of |z| = 1 with end points e^{ia} , τ

are mapped by f(z) onto two analytic arcs with the same endpoints $f(e^{ia}), f(\tau) = \infty$ which by the univalency and by the absence of exterior points necessarily coincide.

5. Determination of a

Putting $\eta = e^{ia}$ and using (4.13), (4.14) we see that (4.4) takes the form

(5.1)
$$\bar{\eta}(z_1 - \eta)^2 G - \eta (\bar{z}_1 - \bar{\eta})^2 \bar{G} = \bar{\eta} (z_2 - \eta)^2 H - \eta (\bar{z}_2 - \bar{\eta})^2 \bar{H}$$

where

(5.2)
$$G = \frac{1}{(1-|z_1|^2)(z_1-z_2)(1-z_1\bar{z}_2)}, \quad H = \frac{1}{(1-|z_2|^2)(z_2-z_1)(1-\bar{z}_1z_2)}$$

Hence $(z_1 - \eta)^2 G - (1 - \eta \bar{z}_1)^2 \bar{G} = (z_2 - \eta)^2 H - (1 - \eta \bar{z}_2)^2 \bar{H}$ which is a quadratic equation with respect to η , of the form

(5.3)
$$\eta^{2} [(G - \bar{z}_{1}^{2}\bar{G}) - (H - \bar{z}_{2}^{2}\bar{H})] - 2\eta [(z_{1}G - \bar{z}_{1}\bar{G}) - (z_{2}H - \bar{z}_{2}\bar{H})] + [(z_{1}^{2}G - \bar{G}) - (z_{2}^{2}H - \bar{H})] = 0.$$

However, (5.2) implies

(5.41)
$$(G - \bar{z}_1^2 \bar{G}) - (H - \bar{z}_2^2 \bar{H}) = \frac{2 \left[\bar{z}_1 (1 + |z_2|^2) - \bar{z}_2 (1 + |z_1|^2) \right]}{|z_1 - z_2|^2 |1 - z_1 \bar{z}_2|^2},$$

(5.42)
$$(z_1G - \bar{z}_1\bar{G}) - (z_2H - \bar{z}_2\bar{H}) = \frac{2(\bar{z}_1z_2 - z_1\bar{z}_2)}{|z_1 - z_2|^2 |1 - z_1\bar{z}_2|^2},$$

(5.43)
$$(z_1^2 G - \bar{G}) - (z_2^2 H - \bar{H}) = -\frac{2 \left[z_1 (1 + |z_2|^2) - z_2 (1 + |z_1|^2) \right]}{|z_1 - z_2|^2 \left| 1 - z_1 \bar{z}_2 \right|^2},$$

so that (5.3) takes the form

(5.5)
$$\eta^{2} [\bar{z}_{1}(1+|z_{2}|^{2})-\bar{z}_{2}(1+|z_{1}|^{2})] - 2\eta [\bar{z}_{1}z_{2}-z_{1}\bar{z}_{2}] - [z_{1}(1+|z_{2}|^{2})-z_{2}(1+|z_{1}|^{2})] = 0.$$

Since $|z_1(1+|z_2|^2) - z_2(1+|z_1|^2)|^2 = |z_1-z_2|^2|1-z_1\overline{z}_2|^2 - (z_1\overline{z}_2-\overline{z}_1z_2)^2$, we have for the discriminant Δ of (5.5)

$$\sqrt{\Delta} = |z_1 - z_2| |1 - z_1 \overline{z}_2|,$$

and so the equation (5.3) has for any z_1, z_2 different from each other and from the origin and situated in the unit circle precisely two different roots η_1, η_2 with $|\eta_1| = |\eta_2| = 1$:

(5.6)
$$\eta_{1,2} = \frac{\bar{z}_1 z_2 - z_1 \bar{z}_2 \mp |z_1 - z_2| |1 - z_1 \bar{z}_2|}{\bar{z}_1 (1 + |z_2|^2) - \bar{z}_2 (1 + |z_1|^2)}$$

which interchange if we interchange z_1 and z_2 .

Now, both numbers η_1 , η_2 have a simple geometrical meaning. They correspond to the intersection points of the circle |z| = 1 and the circle through z_1, z_2, \bar{z}_1^{-1} (and hence through \bar{z}_2^{-1}) orthogonal to |z| = 1. In fact, the equation of the circle through z_1, z_2, \bar{z}_1^{-1} has the form

and after expanding w.r.t. the first row and dropping the factor $(1 - |z_1|)^2$ we obtain

(5.7)
$$|z|^{2}(z_{1}\bar{z}_{2}-\bar{z}_{1}z_{2})+z[\bar{z}_{1}(1+|z_{2}|^{2})-\bar{z}_{2}(1+|z_{1}|^{2})]- - \bar{z}[z_{1}(1+|z_{2}|^{2})-z_{2}(1+|z_{1}|^{2})]+(z_{1}\bar{z}_{2}-\bar{z}_{1}z_{2})=0.$$

If δ is the intersection point of the circle (5.7) and the unit circle then $\bar{\delta} = \delta^{-1}$ and using this we see that (5.7) takes the form

$$\begin{split} \delta^{\mathfrak{s}}[\bar{z}_{1}(1+|z_{2}|^{\mathfrak{s}})-\bar{z}_{\mathfrak{s}}(1+|z_{1}|^{\mathfrak{s}})] &= 2\,\delta(\bar{z}_{1}z_{\mathfrak{s}}-z_{1}\bar{z}_{\mathfrak{s}}) - \\ &- [z_{1}(1+|z_{\mathfrak{s}}|^{\mathfrak{s}})-z_{\mathfrak{s}}(1+|z_{1}|)^{\mathfrak{s}}] = 0 \end{split}$$

which is identical with (5.5).

(S) is a compact family and this implies that $\sup |F(z_1)/F(z_2)|$ must be always attained. Also $k(z_1, z_2)$ varies continuously with z_1 and z_2 . For $-1 < z_2 < 0 < z_1 < 1$ the extremal function is unique and f(z) = $= z(1-z)^{-2}$. Besides f'(z) = 0 for z = -1 $(f'(z) = (1+z) \cdot (1-z)^{-3})$ so that in this case

(5.8)
$$e^{i_{\alpha}} = \eta = \frac{\bar{z}_1 z_2 - z_1 \bar{z}_2 - |z_1 - z_2| |1 - z_1 \bar{z}_2|}{\bar{z}_1 (1 + |z_2|^2) - \bar{z}_2 (1 + |z_1|^2)}$$

is the only point at which f'(z) = 0. Now changing both z_1 and z_2 from their starting position on the real axis we see that f and also η will change

continuously so that the same formula (5.7) must be valid. Both values η_1, η_2 cannot interchange since $|z_1 - z_2| |1 - z_1 \overline{z}_2| \neq 0$ and so is the denominator. Hence z_2 lies on the arc of the circle (5.7) between η and z_1 . At the same time interchanging z_1, z_2 gives us another value of η which thus necessarily corresponds to the function providing $\inf |F(z_1)/F(z_2)|$, since similar considerations for the case of minimum hold. We have thus proved the

Theorem 1. If z_1, z_2 are two fixed points of the unit circle different from the origin and from each other, then the function w = f(z) realising $\sup_{F \in (S)} |F(z_1)/F(z_2)|$ satisfies the following differential equation

(5.9)
$$\frac{C\eta(z-\eta)^2}{z(z-z_1)(z-z_2)(1-\bar{z}_1z)(1-\bar{z}_2z)} = \frac{w_2-w_1}{w(w_1-w)(w_2-w)} \left(\frac{dw}{dz}\right)^2$$

where $\eta = e^{ia}$ is the point of the unit circle which is the end point of a circular arc (z_1, η) orthogonal to |z| = 1 and containing z_2 inside, further η is determined by (5.8), C is a real positive constant and $w_1 = f(z_1), w_2 = f(z_2)$. The extremal function maps |z| = 1 onto a slit domain G, CG being a single analytic arc extending from $f(\eta)$ to ∞ .

6. Qualitative investigation of the extremal function

So far we have proved that the extremal function satisfies the differential equation (5.9) or (3.9) which may be written in the form

(6.1)
$$zP(z)\left[-\left(\frac{dz}{z}\right)^{2}\right] = \frac{(w_{2}-w_{1})dw^{2}}{w(w_{1}-w)(w_{2}-w)}$$

with

(6.2)
$$zP(z) = \frac{-Cze^{-i\alpha}(z-e^{i\alpha})^2}{(z-z_1)(z-z_2)(1-z\overline{z}_1)(1-z\overline{z}_2)} \quad (C > 0)$$

zP(z) being non-negative on |z| = 1. Thus our problem is associated, in accordance with Teichmüller's general principle, with the positive quadratic differential $zP(z)[-(dz/z)^2] = CQ(z)dz^2$, $|z| \leq 1$, having three simple poles inside K and a double zero on the boundary. At the same time inner points of K have as a local uniformizing parameter the variable w = f(z) and so the r.h.s. of (6.1) is also a positive quadratic differential in the domain f(K). Since the shape of the trajectories can be determined separately for the l.h.s. and r.h.s. and w = f(z) carries the trajectories on K into those on f(K), so a qualitative investigation of the mapping function can be done in this way. We begin by investigating the trajectories of $C^{-1}zP(z)[-(dz/z)^2] \equiv Q(z)dz^2$.

In view of (4.9) $Q(z)dz^2$ may be continued into the whole sphere and the trajectories lying outside |z| = 1 arise by reflecting of those situated inside |z| = 1 w.r.t. |z| = 1. In fact, on the trajectories $zP(z)[-(dz/z)^2] > 0$, |z| < 1. If $\zeta = \overline{z}^{-1}$, then $dz/z = -(\overline{d\zeta/\zeta})$, and so $zP(z)[-(dz/z)^2] \equiv \overline{\zeta P(\zeta)}[-(\overline{d\zeta/\zeta})^2] = \zeta P(\zeta)[-(\overline{d\zeta/\zeta})^2] = \zeta P(\zeta)[-(d\zeta/\zeta)^2]$.

So we can consider $Q(z)dz^2$ as a positive quadratic differential on the sphere. $Q(z)dz^2$ has six simple poles $z_1, z_2, \overline{z_1}^{-1}, \overline{z_2}^{-1}, 0, \infty$ and one double zero $z = \eta$ (in accordance with the general formula, see e.g. [3] p. 36).

Let Φ denote the union of all trajectories which have a limiting point at a point of the set C of zeros and simple poles. Then according to The Basic Structure Theorem ([3], p. 37) in the absence of poles of order at least 2, the system of curves Φ divides the sphere into a finite number of ring domains ([3], p. 37) which are swept out by not intersecting trajectories and each of these is a closed Jordan curve. We now determine the shape of the trajectories for $-1 < z_2 < 0 < z_1 < 1$. In this case $e^{ia} = e^{-ia} = -1$ and

$$Q(z)dz^2 = rac{-(z+1)^2 dz^2}{z(z+r_2)(z-r_1)(1-r_1z)(1+r_2z)}, \hspace{1em} r_i = |z_i|\,.$$

Hence the set Φ consists of the unit circumference |z| = 1 and the segments $-r_2^{-1} \leq x \leq -r_2$, $0 \leq x \leq r_1$, $r_1^{-1} \leq x < +\infty$ of the real axis. Thus we obtain two ring domains, the remaining trajectories being closed curves filling those completely without intersecting each other. Now the quadratic differential $Q(z)dz^2$ depends continuously on z_1 and z_2 and this implies that continuous changing of z_1 and z_2 does not change the topological properties of the trajectories.

Hence the set Φ for $Q(z)dz^2$ consists of |z| = 1, two non-intersecting arcs: γ_1 joining 0 to z_1 and γ_2 joining η to z_2 and of the reflections of γ_1, γ_2 w.r.t. |z| = 1. The set Φ divides the sphere \Re into two ring domains.

Since the function w = f(z), in view of (6.1), carries trajectories into trajectories, the corresponding set Φ_w in the w-plane consists of a simple analytic arc joining $f(\eta)$ and ∞ which is the map of |z| = 1 by f(z), of an arc $\Gamma_2 = f(\gamma_2)$ joining w_2 and $f(\eta)$ and an arc $\Gamma_1 = f(\gamma_1)$ joining 0 and w_1 . The complement of Φ_w w.r.t. the sphere is thus a ring domain. By considering the map of K by the Koebe function for $-1 < z_2 <$ $< 0 < z_1 < 1$ and a subsequent continuous change of z_1 and z_2 resp. w_1 and w_2 we see that the orthogonal trajectories of $Q_1(w)dw^2$ which is the r.h.s. of (6.1) have the same topological structure as the trajectories of $Q_1(w)dw^2$: there are two arcs, one $\tilde{\Gamma}_2$ joining 0 and w_2 , another $\tilde{\Gamma}_1$ joining w_1 and ∞ , the remaining orthogonal trajectories being closed curves surrounding $\tilde{\Gamma}_2$. The map of $\tilde{\Gamma}_1$ by $f^{-1}(w)$ is also an orthogonal trajectory $\tilde{\gamma}_1$ of $Q(z)dz^2$ joining z_1 to $\tau = e^{i\beta}$, where $f(\tau) = \infty$. The value of β can be found as follows:

There are two arcs l_1 , l_2 on |z| = 1 with common end points η , τ which are carried by f(z) into both edges of the slit. If $\zeta_1 \epsilon l_1$, $\zeta_2 \epsilon l_2$ are points corresponding to the same point w on the slit l, then

$$\int_{l_1} \sqrt{Q(\zeta_1)} \, d\zeta_1 = \int_{l_2} \sqrt{Q(\zeta_2)} \, d\zeta_2 = C^{-1/2} \int_l \sqrt{Q_1(w)} \, dw$$

For $z = e^{i\theta}$, however, $\sqrt{Q(z)} dz = \sin \left[\frac{1}{2}(\theta - \alpha)\right] \cdot |e^{i\theta} - z_1|^{-1} |e^{i\theta} - z_2|^{-1} d\theta$ so that $\tau = e^{i\theta}$ is uniquely determined by the equation

$$\int_{a}^{b} \sin\left[\frac{1}{2}(\theta-a)\right] \cdot |e^{i\theta}-z_{1}|^{-1} \cdot |e^{i\theta}-z_{2}|^{-1} d\theta =$$

=
$$\int_{b}^{2\pi+a} \sin\left[\frac{1}{2}(\theta-a)\right] \cdot |e^{i\theta}-z_{1}|^{-1} \cdot |e^{i\theta}-z_{2}|^{-1} d\theta.$$

Later on we will show that τ lies on the smaller arc of |z| = 1 with end points $-\eta$, η_2 , $(\eta_2$ is defined by (5.6) when we take the sign "+").

On the other hand the ring domain slit along an orthogonal trajectory is mapped by $\int V\overline{Q(z)} dz$ 1:1 conformally onto a rectangle. Let us slit the ring domain $K - (\gamma_1 \cup \gamma_2)$ along the orthogonal trajectory $\tilde{\gamma}_1$ joining z_1 and τ . Taking the integral $\int_{\tau}^{s} V\overline{Q(\zeta)} d\zeta = v(z)$ we see that $K - (\gamma_1 \cup \gamma_2 \cup \tilde{\gamma}_1)$ is mapped 1:1 conformally by

(6.3)
$$v(z) = \int \frac{e^{-ia/2} (\zeta - e^{ia}) d\zeta}{[\zeta(\zeta - z_1)(\zeta - z_2)(1 - \overline{z}_1 \zeta)(1 - \overline{z}_2 \zeta)]^{1/2}}$$

on a rectangle in the v-plane with corners corresponding to the points τ and z_1 . The trajectories of $Q(z)dz^2$ (on which $\sqrt{Q(z)}dz$ is positive) are carried into segments parallel to the real axis, whereas the arcs of orthogonal trajectories on which $\sqrt{Q(z)}dz$ is purely imaginary are carried into segments parallel to the imaginary axis. Both edges of the slit $\tilde{\gamma}_1$ correspond to one pair of sides, the third side corresponds to l_1 , both edges of γ_2 , and l_2 , the forth side is the map of both edges of γ_1 . Similarly

(6.4)
$$v(w) = C^{-1/2} \int_{\infty}^{w} \left[\frac{w_2 - w_1}{\omega (w_1 - \omega) (w_2 - \omega)} \right]^{1/2} d\alpha$$

maps $f(K) - (\Gamma_1 \cup \Gamma_2 \cup \tilde{\Gamma}_1)$ on the same rectangle. Both edges of the slit $\tilde{\Gamma}_1$ correspond to one pair of sides parallel to the imaginary axis, whereas the third side parallel to the real axis is the map by (6.4) of the slit l and both edges of Γ_2 and the forth side (parallel to the real axis) is the map by (6.4) of both edges of Γ_1 . The identity of both rectangles means the equality of suitably chosen periods of both integrals (6.3) and (6.4). At the same time w(z) being defined implicitly by (6.3) and (6.4) is univalent.

7. Determination of the extremal function

Put now

(7.1)
$$\Omega_1 = \int_{\lambda_1} \frac{e^{-ia/2} (\zeta - e^{ia}) d\zeta}{[\zeta(\zeta - z_1)(\zeta - z_2)(1 - \overline{z}_1 \zeta)(1 - \overline{z}_2 \zeta)]^{1/2}}$$

where λ_1 is a closed Jordan curve surrounding 0 and z_1 but leaving z_2 outside. At the same time λ_1 is a closed curve on the Riemann surface of $[z(z-z_1)(z-z_2)(1-\bar{z}_1z)(1-\bar{z}_2z)]^{1/2}$. Of course we can replace λ_1 by an arbitrary trajectory of the ring domain $K-\gamma_1$ and hence we see that $\Omega_1 > 0$ $(\Omega_1 = \int_{\lambda_1} \sqrt{Q(\zeta)} d\zeta$ and $\sqrt{Q(\zeta)} d\zeta > 0$ on trajectory after a

suitable orientation and a suitable choice of the branch of VQ(z).

Next we put

(7.2)
$$\Omega_2 = \int_{\lambda_2} \frac{e^{-ia/2} (\zeta - e^{ia}) d\zeta}{[\zeta(\zeta - z_1)(\zeta - z_2)(1 - \bar{z}_1\zeta)(1 - \bar{z}_2\zeta)]^{1/2}}$$

where λ_2 is a Jordan curve surrounding 0 and z_2 but leaving z_1 outside. After replacing λ_1 by a closed orthogonal trajectory lying inside K (some of orthogonal trajectories are open, their end points being points on l_1 resp. l_2 corresponding to the same point of the slit l) we see that Ω_2 is purely imaginary. A suitable orientation of λ_2 gives us $\Re \Omega_2 = 0$, $\Im \Omega_2 > 0$ since $Q(\zeta)d\zeta^2 < 0$ on the orthogonal trajectory.

Put $v = C^{1/2}u$ and $w = 4(w_2 - w_1) W + \frac{1}{3}(w_1 + w_2)$. Then (6.4) after differentiating takes the form

$$\left(\frac{du}{dW}\right)^2 = \frac{1}{4W^3 - g_3W - g_3}$$

where

$$g_2 = \frac{1}{12} \frac{w_1^2 - w_1 w_2 + w_2^2}{(w_2 - w_1)^2} = \frac{1}{12} \frac{1 - x + x^2}{(1 - x)^2}$$
$$g_3 = \frac{1}{8 \cdot 27} \frac{(x + 1)(x - 2)(x - \frac{1}{2})}{(x - 1)^3}, \quad w_j = f(z_j), \ j = 1, 2, x = \frac{w_1}{w_2}.$$

Since $\delta = g_2^3 - 27g_3^2 = x^2/4^4(x-1)^4 \neq 0$, we have necessarily $W = = \wp(u|\omega_1, \omega_2)$ because $W = \infty$ for u = 0. $\wp(u|\omega_1, \omega_2)$ denotes Weierstrass's elliptic function defined by

$$\wp(u|\omega_1, \omega_2) = \frac{1}{u^2} + \sum_{m,n'} \left[\frac{1}{(u - m\omega_1 - n\omega_2)^2} - \frac{1}{(m\omega_1 + n\omega_2)^2} \right].$$

Hence

(7.3)
$$w = 4(w_2 - w_1)\wp(u|\omega_1, \omega_2) + \frac{w_1 + w_2}{3}$$

Now $\mathscr{O}(u|\omega_1, \omega_2) = C^{-1} \mathscr{O}(\sqrt{C}u|\sqrt{C}\omega_1, \sqrt{C}\omega_2) = C^{-1} \mathscr{O}(v|\sqrt{C}\omega_1, \sqrt{C}\omega_2),$ and so (7.3) becomes

(7.4)
$$w = C_1 \wp (v | \sqrt{C} \omega_1, \sqrt{C} \omega_2) + C_2$$

 C_1 , C_2 being constants. When z describes a closed trajectory (resp. an orthogonal trajectory) then v increases by Ω_1 (resp. by Ω_2) and at the same time w attains its initial value. At the same time w could not attain the same value for two different points z on the trajectory. This proves that Ω_1 , Ω_2 are periods in (7.4) and cannot be replaced by Ω_1/k , Ω_2/m (k, m being natural numbers greater than 1). Hence (7.4) takes the form

$$w = C_1 \wp(v | \Omega_1, \Omega_2) + C_2$$

where v, Ω_1, Ω_2 are defined by (6.3), (7.1), (7.2) respectively. Now w = 0for z = 0 and this gives $-C_1 \wp (\int_0^0 \sqrt{Q(\zeta)} d\zeta |\Omega_1, \Omega_2) = C_2$. Finally

(7.5)
$$w = \wp \left(\int_{0}^{3} \sqrt{Q(\zeta)} d\zeta + \frac{1}{2} \left(\Omega_{1} + \Omega_{2} \right) \right) + e_{1} + e_{2}$$

since the constant C_1 can be dropped and

(7.6)
$$k(z_1, z_2) = \left| \frac{2e_2 + e_1}{2e_1 + e_2} \right|, \quad e_k = \wp\left(\frac{1}{2}\Omega_k\right), \quad k = 1, 2.$$

 $\wp(v)$ is clearly a single valued function of $z \in K$. If z describes a closed curve on the Riemann surface of VQ(z) lying entirely over K, v changes by $m_1 \Omega_1 + m_2 \Omega_2$ which does not change $\wp(v)$. If, however, the end points of the two paths of integration lie on different sheets but over the same point z the corresponding values of v have the sum equal 0 mod. (Ω_1, Ω_2) . $\wp(v)$ being even, we again obtain the same value of $\wp(v)$. Both functions (6.3) and (6.4) are univalent and this implies also the univalency of (7.4).

We have proved

Theorem II. If z_1, z_2 are two fixed points of the unit circle different from the origin and from each other, and F(z) is a function regular and univalent in the unit circle vanishing at the origin, then

$$\sup_{(F)} |F(z_1)/F(z_2)| = |f(z_1)/f(z_2)| = k(z_1, z_2)$$

where $k(z_1, z_2)$ is defined by (7.6).

The extremal function f(z) is defined by (7.5), (2.1)-(2.4) and is unique apart from a constant factor.

8. The case of the Koebe function

We will now prove that for $z_1, z_2 \neq 0$ lying on the same diameter of the unit circle, and in this case only, the extremal function is the Koebe function.

Suppose first that both points A_1 , A_2 (corresponding to the complex numbers $z_1, z_2 \neq 0$) lie on the arc BB' of the circle orthogonal to |z| = 1, $B \leftrightarrow \eta$, $B' \leftrightarrow \eta_2$, whose centre will be denoted by C. Let P, Q be the points of intersection of |z| = 1 and a ray emanating from C. If φ is the angle between OC and CPQ and CB' = CB = r, then

$$CP = \cos\varphi (\sqrt{1 + r^2} - \sqrt{1 - r^2 \tan^2 \varphi})$$

$$CQ = \cos\varphi (\sqrt{1 + r^2} + \sqrt{1 - r^2 \tan^2 \varphi})$$

$$CP \cdot CQ = r^2$$

Putting $\theta_P = \langle COP, \theta_Q = \langle COQ \rangle$ we have $OC - \cos\theta_Q = CQ\cos\varphi = \cos^2\varphi (\sqrt{1+r^2} + \sqrt{1-r^2}\tan^2\varphi)$, and hence

$$d heta_Q = - \left(1 + \sqrt{rac{1+r^2}{1-r^2 an^2arphi}}
ight) darphi = -rac{CQ\cdot darphi}{\cosarphi\sqrt{1-r^2 an^2arphi}},$$

and similarly

$$d\theta_P = \left(\sqrt{\frac{1+r^2}{1-r^2\tan^2\varphi}} - 1\right)d\varphi = \frac{CP \cdot d\varphi}{\cos\varphi \sqrt{1-r^2\tan^2\varphi}}$$

We have

$$PB = r \cdot QB/QC = QB \cdot PC/r,$$

$$A_1P = r \cdot A_1Q/QC = A_1Q \cdot PC/r,$$

$$A_2P = r \cdot A_2Q/QC = A_2Q \cdot PC/r.$$

Hence

$$\begin{split} I_2 &= \int_{\beta_2}^{a+2\pi} \frac{\sin\frac{\theta-a}{2}d\theta}{|e^{i\theta}-z_1||e^{i\theta}-z_2|} = \int_{B^\prime B} \frac{PB \cdot d\theta_P}{A_1P \cdot A_2P} = \\ &= \int_{-\varphi_0}^{\varphi_0} \frac{QB \cdot PC \cdot r^{-1}}{A_1Q \cdot A_2Q \cdot PC^2 \cdot r^{-2}} \cdot \frac{PC \cdot d\varphi}{\cos\varphi \sqrt{1-r^2 \tan^2\varphi}} = \\ &= \int_{-\varphi_0}^{\varphi_0} \frac{QB}{A_1Q \cdot A_2Q} \cdot \frac{rd\varphi}{\cos\varphi \sqrt{1-r^2 \tan^2\varphi}}, \quad e^{i\theta_2} = \eta_2. \end{split}$$

and similarly

$$I_1 = \int_{BB'} \frac{QB \cdot d\Theta_Q}{A_1 Q \cdot A_2 Q} = \int_{-\varphi_0}^{\varphi_0} \frac{QB}{A_1 Q \cdot A_2 Q} \frac{QC \cdot d\varphi}{\cos\varphi \sqrt{1 - r^2 \tan^2\varphi}}$$

Since QC > r for $C \neq \infty$, $I_1 > I_2$ for r finite. Again, it is easy to see that, if the end point B' is replaced by $B'' \leftrightarrow -\eta$, then $I_1 < I_2$ in case $r < +\infty$. This means that the point τ (for which both integrals are the same) corresponding to $w = \infty$ lies in this case on the open arc (B', B''). For the Koebe function, however, $\tau = -\eta$ and this is impossible, as we have seen now, for $C \neq \infty$.

When z_1 and z_2 are on opposite radii, the extremal function is unique and is a Koebe function which is known from the elementary theory.

Let us now suppose that both points z_1, z_2 lie on the same radius, e. g. $0 < z_2 < z_1 < 1$.

Then $\eta = -1$ according to (5.7) and $\beta = a + \pi$, $\tau = -\eta = 1$ in this case. We see that the function $\overline{f(\overline{z})}$ also would be an extremal function. The uniqueness of the extremal function implies $f(z) \equiv \overline{f(\overline{z})}$. However,

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the only function with this property mapping K onto the plane slit along a single analytic arc is the Koebe function $Az(1 \mp z)^{-2}$ with real A. In our case $(\tau = 1, f'(0) = 1) f(z) = z(1-z)^{-2}$.

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Streszczenie

W pracy tej posługując się metodami wariacyjnymi znajduję $\sup_{F} |F(z_1)/F(z_2)| = k(z_1, z_2)$ przy ustalonych z_1, z_2 , przy czym F(z) jest funkcją regularną i jednolistną w kole jednostkowym, zerującą się w zerze. Funkcją ekstremalną jest funkcja Koebego jedynie dla z_1, z_2 leżących na jednej średnicy koła jednostkowego, a w pozostałych przypadkach funkcją ekstremalną jest superpozycja pewnej całki hipereliptycznej i funkcji \wp Weierstrassa o tych samych okresach, co dana całka hipereliptyczna.

Znając $k(z_1, z_2)$ można już łatwo znaleźć dokładne oszacowanie $|\varphi(z)|$, gdy $\varphi(z)$ jest funkcją regularną i jednolistną w kole jednostkowym taką, że $\varphi(0) = 0$, $\varphi(z_0) = 1$ ($0 < |z_0| < 1$). Wynik ten jest rozwiązaniem problemu postawionego przez P. Montela.

Резюме

В этой работе, пользуясь вариационными методами, я нахожу $\sup |F(z_1)/F(z_2)| = k(z_1, z_2)$ при установленных z_1, z_2 , причём F(z) есть регулярная и однолистная функция в единичном круге, равная

нулю при z = 0. Екстремальной функцией является функция Кёбе исключительно для z_1 и z_2 , лежащих на одном диаметре единичного круга, а в остальных случаях экстремальной функцией является наложение некоторого гиперэллиптического интеграла и функции Вейерштрасса \wp с такими же периодами, как данный гирерэллиптический интеграл.

Зная $k(z_1, z_2)$, можно уже легко найти точную оценку $|\varphi(z)|$, когда $\varphi(z)$ функция регулярная и однолистная в единичном круге такая, что $\varphi(0) = 0$, $\varphi(z_0) = 1$ ($0 < |z_0| < 1$). Этот результат является рещением проблемы, поставленной П. Монтелем.