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# Brownian Sheets with Values in a Banach Space

Powierzchnie brownowskie o wartościach w przestrzeni Banacha

**1. Introduction.** Let $(B, \|\cdot\|)$ be a real separable infinite dimensional Banach space and let $p_t$ be the Wiener measure with mean zero and variance parameter $t > 0$ defined on the Borel $\sigma$– field $\mathcal{B}$ of subsets of $B$. In other words, we assume that there exists a real separable infinite dimensional Hilbert space $H \subset B$ with a centered at zero cylindrical Gauss measure $\mu_t$ having variance parameter $t$, such that $\|\cdot\|$ is a $\mu_t$–measurable norm on $H$, $B$ is the completion of $H$ with respect to $\|\cdot\|$ and $p_t$ is the unique $\sigma$–additive extension of a measure $\tilde{\mu}_t$ associated with $\mu_t$ by equality on cylinders in $B$ and $H$. This is possible because any seminorm in $H$ is always measurable or not with respect to all $\mu_t$ simultaneously, furthermore $H$ is determined uniquely by $B$ and $p_t$ for a fixed $t > 0$. It is well known that an arbitrary real separable Banach space $B$ can be used in the described above context, and since a measurable norm is weaker than the original norm $|\cdot| = \sqrt{\langle\cdot,\cdot\rangle}$ generated by the inner product of $H$, $(H, \|\cdot\|)$ is not complete unless it is finite dimensional. Construction and further properties of the Wiener measure in a Banach space were given by Gross [9] (cf. also Kuo [15]).

Let $p_0$ denote the measure assigning the unit mass to the origin $0 \in B$. Then the family of measures $\{p_t,\ t \geq 0\}$ forms a strongly continuous contraction semigroup acting in the Banach space of bounded uniformly continuous (real or complex valued) functions on $B$, in particular $p_t * p_s = p_{t+s}$ for $t, s \geq 0$, where $*$ denotes the convolution. Consequently a one parameter $B$–valued Wiener process $\{\xi_t,\ t \geq 0\}$ with independent $p_{t-s}$–distributed increments $\xi_t - \xi_s$, $t > s \geq 0$ and continuous paths can be constructed (see Gross [8,9] and Kuo [15]). In the presented article we describe a simple construction and basic properties of a multiparameter Wiener process called Brownian sheet with values in a real separable infinite dimensional Banach space $B$.

The notion of a Banach space valued Brownian sheet is not entirely new, because such a process was introduced e.g. by Morrow [16] for the purpose of approximation of rectangular sums of $B$–valued random elements. Moreover, by analogy to the fact observed by Kuelbs [14] for real Brownian sheets we may define $B$–valued Brownian sheet on the cube $\langle 0, t_0\rangle^r$, $r \geq 2$, identifying it with $\{\xi_t,\ t \in \langle 0, t_0\rangle\}$, where the last Wiener process takes values in the Banach space $C(\langle 0, t_0\rangle^{r-1}, B)$ of continuous functions from $\langle 0, t_0\rangle^{r-1}$ into $B$. We do not want to display these considerations, but we will present here some other method based on random series in tensor products of Banach spaces leading quickly to the same results.

Throughout the paper $B^*$ denotes the topological dual of $B$ ($H^*$ resp. for $H$) and the bracket $(\cdot,\cdot)$ means the natural pairing between $B^*$ and $B$. Since the norm $\|\cdot\|$ is weaker than $|\cdot|$, the restriction of any $y^* \in B^*$ to $H$ is a continuous linear functional on $H$, so that $B^* \subset H^*$. In view of the Riesz representation theorem $H^*$ is isometrically isomorphic to $H$. Denote by $\hat{}$ the following isomorphism: $0^* \xrightarrow{\wedge} 0$ and if $0^* \neq y^* \in H^*$, let $\hat{y} = y_1(y^*, y_1)$, where $y_1 \in H$ is the unique vector characterized by the properties: $y_1 \in \{x : (y^*, x) = 0\}^{\perp}$, $|y_1| = 1$ and $(y^*, y_1) > 0$. Thus we have defined an embedding $B^* \subset H^* \xrightarrow{\wedge} H \subset B$, so that for $y^* \in B^*$ and $x \in H$, the scalar product $\langle \hat{y}, x\rangle$ is well-defined and $\langle \hat{y}, x\rangle = (y^*, x)$.

**2. Construction of the process.** Let $T = \{t = (t_1, \ldots, t_r) \in R^r : t_i \in R_+ = = \langle 0, \infty), 1 \leq i \leq r\}$ and $\partial T = \{t \in T : t_i = 0$ for some $i = 1, 2, \ldots, r\}$. In the sequel for $s, t \in T$ we will use the notation: $s \wedge t = (\min(s_1, t_1), \ldots, \min(s_r, t_r))$ and by analogy $s \vee t$ with max, furthermore $s \pm t = (s_1 \pm t_1, \ldots, s_r \pm t_r)$ and $|t| = \prod_{i=1}^{r} t_i$. Let $C(T, B)$ denote the space of continuous functions $x : T \to B$ such that $x|_{\partial T} = 0$. We shall prove that there exist a probability space $(\Omega, \mathcal{F}, P)$ and a stochastic process $X = \{X(t),\ t \in T\}$ defined on it with values in $B$, satisfying the following conditions:

(2.1) for an arbitrary $t \in T$, $X(t) : \Omega \to B$ is a random element in $(B, \mathcal{B})$,

(2.2) the process $X$ has independent increments

$$\Delta X(V) = \sum_{\{1 \leq i \leq r : t_i = a_i \vee t_i = b_i\}} (-1)^{\sum_{j=1}^{r} \chi(t_j = a_j)} X(t_1, t_2, \ldots, t_r)$$

on disjoint rectangles $V = \langle a, b) = \{t \in T : a_i \leq t_i < b_i,\ i \leq r\}$,

(2.3) $\Delta X(V)$ has distribution $p_{\mathrm{vol}\, V}$ for $V = \langle a, b) \subset T$, where $\mathrm{vol}\, V = |b - a|$;

hence $X(t) = 0$ with probability 1 iff $t \in \partial T$ and $X(t)$ is $p_{|t|}$ – distributed whenever $t \in T$.

Moreover,

(2.4) realizations of the process $X$ are a.s. continuous, i.e. belong to $C(T, B)$.

We are going now to describe briefly construction of $X$. Let $T_m = \{t \in T : 0 \leq t_i \leq m_i,\ 1 \leq i \leq r\}$, where $m_i \in N = \{1, 2, \ldots\}$. Suppose $\{y_n,\ n \geq 1\}$ is a CONS in $H$ and $\{f_j,\ j \geq 1\}$ is a CONS in $C'(T_m)$, where $C'(T_m)$ is the Hilbert space generating Wiener measure in $C(T_m)$ — the space of continuous functions from $T_m$ into $R$ vanishing on $\partial T \cap T_m$. It is easy to see that for any sequence of i.i.d. standard normal random variables $\{g_n\}$, defined on the same probability space, $\sum_n g_n y_n$ converges a.s. in $(B, \|\cdot\|)$, and similarly $\sum_j g_j f_j$ is convergent with probability 1 in the usual sup norm in $C(T_m)$. Hence if $g_{nj}$ are independent standard normal random variables defined on a common probability space, on account of the result given by

Chevet [3],

$$\sum_{j,n} g_{nj}\, f_j \otimes y_n \tag{2.5}$$

converges a.s. in $C(T_m) \otimes_\epsilon B$, where $\epsilon$ is the least reasonable cross–norm. It is very well known that the space $C(T_m) \otimes_\epsilon B$ is equivalent to $C(T_m, B)$ — the real separable Banach space of continuous functions $x : T_m \to B$ with norm $\|x\|_m = \sup_{t \in T_m} \|x(t)\|$ such that $x|_{\partial T} = 0$. Thus if we identify the tensor product $\otimes$ with multiplication, the above series (2.5) defines a stochastic process $X_m = \{X_m(t),\ t \in T_m\}$ with realizations in $C(T_m, B)$.

Let $W_m$ be the distribution of $X_m$ in $\big(C(T_m, B)\ ,\ \mathcal{B}(C(T_m, B))\big)$ and let $\mathcal{H}_m = C'(T_m) \otimes_2 H$. Then $\{f_j y_n,\ j, n \geq 1\}$ is a CONS in $\mathcal{H}_m$, which implies that $\mathcal{H}_m$ is the Hilbert space generating Wiener measure $W_m$ in $C(T_m, B)$.

All what we have to prove is that $X_m$ satisfies (2.1)–(2.3). Consider the probability space $\big(C(T_m, B), \mathcal{B}(C(T_m, B)), W_m\big)$. Obviously $X_m(t),\ t \in T_m$, are random elements with values in $B$. Observe now that the functional $G_{y^*,V} \in C^*(T_m, B)$ given by $G_{y^*,V}(x) = \big(y^*, \Delta x(V)\big)$ after embedding into $\mathcal{H}_m$ is equal to $\Delta|t \wedge \cdot|(V)\hat{y}$. Indeed, for each $f_j$ and $y_n$ we have $\langle \Delta|t \wedge \cdot|(V)\hat{y}, f_j y_n\rangle_{\mathcal{H}_m} = \langle \Delta|t \wedge \cdot|(V), f_j\rangle_{C'(T_m)} \langle \hat{y}, y_n\rangle =$ $= \Delta f_j(V)(y^*, y_n) = \big(y^*, \Delta(f_j y_n)(V)\big)$, so $\hat{G}_{y^*,V} = \Delta|t \wedge \cdot|(V)\hat{y}$. It follows that $\Delta X_m(V)$ has distribution $p_{\mathrm{vol}\, V}$, because for each $y^* \in B^*$, $\big(y^*, \Delta X_m(V)\big)$ is distributed normally with mean zero and variance $\mathrm{vol}\, V \cdot |\hat{y}|^2$ (cf. Kuo [15] p. 78).

Furthermore, since $W_m$ is generated by the cylindrical Gauss measure in $\mathcal{H}_m$, if $F \perp G$, $F, G \in \mathcal{H}_m$, then $\langle F, X_m\rangle_{\mathcal{H}_m}$ and $\langle G, X_m\rangle_{\mathcal{H}_m}$ are independent. Suppose $V_1, V_2 \subset T_m$ are disjoint rectangles. Let $u_1^*, \ldots, u_k^*\ ,\ z_1^*, \ldots, z_n^* \in B^*$. Without loss of generality we assume that $\hat{u}_i$ are orthogonal in $H$ and similarly $\hat{z}_j$ (for we can always form a basis in $\mathrm{Lin}(u_1^*, \ldots, u_k^*)$ consisted of orthogonal vectors). However $\hat{G}_{u_i^*,V_1}$ and $\hat{G}_{z_j^*,V_2}$ are then all mutually orthogonal in $\mathcal{H}_m$, so that joint distribution of the random vector
$\big\{\big(u_1^*, \Delta X_m(V_1)\big), \ldots, \big(u_k^*, \Delta X_m(V_1)\big)\ ,\ \big(z_1^*, \Delta X_m(V_2)\big), \ldots, \big(z_n^*, \Delta X_m(V_2)\big)\big\}$
is Gaussian and random vectors $\big\{\big(u_1^*, \Delta X_m(V_1)\big), \ldots, \big(u_k^*, \Delta X_m(V_1)\big)\big\}$ and $\big(z_1^*, \Delta X_m(V_2)\big), \ldots, \big(z_n^*, \Delta X_m(V_2)\big)\big\}$ are independent. Consequently $\Delta X_m(V_1)$ and $\Delta X_m(V_2)$ are independent random elements in $B$.

Finally $X_m$ is the process with continuous realizations on $T_m$ satisfying (2.1)–(2.3). Note also that the measure $W_m$ does not depend on the choice of $f_j$ in $C'(T_m)$ and $y_n$ in $H$ — any other CONS in $C'(T_m)$ as well as in $H$ will lead to the same distribution $W_m$.

Let $C(T, B)$ be viewed with a family of seminorms $\|\cdot\|_m$, $m \in N^r$. Then $C(T, B)$ is a real separable $B_0$–space. Denote by $\pi_m : C(T, B) \to C(T_m, B)$ projections obtained by restriction of the domain of functions $x \in C(T, B)$ to $T_m$ and set $\mathcal{U}_m = {\pi_m}^{-1}\big(\mathcal{B}(C(T_m, B))\big)$, and $W(U) = W_m(A)$ provided $A \in \mathcal{B}(C(T_m, B))$ and $U = {\pi_m}^{-1}(A)$. Then $W$ is well–defined and is a cylindrical measure on the field $\bigcup_{m \in N^r} \mathcal{U}_m$ in $C(T, B)$, thus $W$ is countably additive (see, e.g. Daleckii and Fomin [4], Th. 1.3 p. 25, where $\mathcal{K} = \bigcup_{m \in N^r} \mathcal{K}_m$, $\mathcal{K}_m = {\pi_m}^{-1}(\mathcal{M}_m)$ and $\mathcal{M}_m$ is, for example, the class of compact sets in $C(T_m, B)$ ). Moreover, the $\sigma$–field generated by $\bigcup_{m\in N^r} \mathcal{U}_m$ is equal to $\mathcal{B}(C(T,B))$, therefore $W$ has the unique $\sigma$–additive extension to $\mathcal{B}(C(T,B))$ denoted still by $W$. Defining $X$ on $(C(T,B),\mathcal{B}(C(T,B)),W)$ by

$$X(t,x)=x(t)\ ,\quad x\in C(T,B)\ ,$$

we see that $X$ is the process satisfying (2.1)–(2.4). To check these conditions it suffices to restrict ourselves to $T_m, X_m$ and $W_m$ with an appropriately chosen $m\in N^r$.

Let $\Omega = B^T$ and let $\sigma\mathcal{C}(B^T)$ denote the $\sigma$-field of subsets of $B^T$ induced by the mappings $x\to x(t)$ , $t\in T$. Then $\sigma\mathcal{C}(B^T)\cap C(T,B)=\mathcal{B}(C(T,B))$, so that we can define a measure $P$ on $\sigma\mathcal{C}(B^T)$ by the formula $P[A]=W[A\cap C(T,B)]$ for $A\in\sigma\mathcal{C}(B^T)$. Assume now that $Y$ is a stochastic process on $(\Omega,\sigma\mathcal{C}(B^T))$ satisfying (2.1)–(2.3) obtained on the basis of Kolmogorov's extension theorem. Since $P$ on cylindrical sets coincides with finite dimensional distributions of $Y$, $P$ is precisely the same probability measure as that being the distribution of $Y$ on $(\Omega,\sigma\mathcal{C}(B^T))$ in Kolmogorov's representation. Denoting by $\mathcal{F}$ the completion of $\sigma\mathcal{C}(B^T)$ under $P$ we see that the process $Y$ considered on $(\Omega,\mathcal{F},P)$ possesses the continuous modification $X$, hence separable. However the existence of a separable modification for $Y$ does not follow from a general version of Doob's theorem because infinite dimensional separable Banach space is neither compact nor locally compact (compare Gihman and Skorohod [7], Ch. III). Though $Y$ need not be continuous or separable, it is stochastically continuous (and also in $L^p$ , $0<p<\infty$, uniformly on each set $T_m$) because $X$ is so. Stochastic continuity of $X$ implies in turn that an arbitrary dense subset of $T$ may serve as a set of separability for $X$ (see Gihman and Skorohod [7]).

Conditions (2.1)–(2.4) imply the following properties:

$$\bigwedge_{y^*\in B^*}\ \bigwedge_{t\in T} E(y^*,X(t))=\int_\Omega (y^*,X(t))\,dP=\int_B (y^*,x)\,dp_{|t|}(x)=0 \tag{2.6}$$

and

$$\bigwedge_{y^*,z^*\in B^*}\ \bigwedge_{t,s\in T} E(y^*,X(t))(z^*,X(s))=\int_\Omega (y^*,X(t))(z^*,X(s))\,dP=\langle\hat y,\hat z\rangle|t\wedge s| \,. \tag{2.7}$$

The first formula follows easily from the above construction and arguments. We shall prove (2.7). Since increments of $X$ on disjoint rectangles are independent, we have

$$E(y^*,X(t))(z^*,X(s))=E(y^*,X(t\wedge s))(z^*,X(t\wedge s))\ .$$

If $y^*=0^*$, then (2.7) is obvious. Suppose $y^*\neq 0^*$. Then $\hat z=\langle\hat y/|\hat y|,\hat z\rangle\,\hat y/|\hat y|+\hat v$, where $\langle\hat v,\hat y\rangle=0$. Hence we infer that $\langle\hat y,X(t\wedge s)\rangle$ and $\langle\hat v,X(t\wedge s)\rangle$ are independent random variables with distributions $N(0,|\hat y|^2\,|t\wedge s|)$ and $N(0,|\hat v|^2\,|t\wedge s|)$, thus

$$E(y^*,X(t\wedge s))(z^*,X(t\wedge s))=E\langle\hat y,\hat z\rangle\langle\hat y,X(t\wedge s)\rangle^2\,|\hat y|^{-2}+E\langle\hat y,X(t\wedge s)\rangle\langle\hat v,X(t\wedge s)\rangle=$$
$$=\langle\hat y,\hat z\rangle\ |t\wedge s|\ .$$

**3. Strong Markov property.** Let $Z=\{Z(t),\ t\in T\}$ be a stochastic process defined on a probability space $(\Omega,\mathcal{F},P)$ taking values in a Hausdorff topological group

$E$ with its Baire $\sigma$-field $e$. We say that $Z$ is right continuous, if for each $t \in T$ and $\omega \in \Omega$,

$$Z(s,\omega) \longrightarrow Z(t,\omega) \qquad \text{as } s \geq t\,,\ s \to t\,.$$

Let $\mathcal{F}_t = \sigma\big(Z(s),\ s \in \langle 0,t\rangle\big)$ and let $\mathcal{B}(S)$ denote the Borel $\sigma$-field of subsets $A \subseteq S \subseteq T$. It can be easily seen that under our assumptions the process $Z$ is progressively measurable. For the proof of this fact it suffices to consider a sequence

$$Z_n(s) = \sum_{k \in N^r} Z\big((k-1)2^{-n}\big)\chi_{\langle (k-1)2^{-n},\, k2^{-n})}(s)$$

convergent in $E$ to $Z(s)$ for all $s \in T$ and $\omega \in \Omega$, and observe that the mapping $(s,\omega) \to Z_n(s,\omega)$ of $\langle 0,t\rangle \times \Omega$ into $(E,e)$ is $\mathcal{B}(\langle 0,t\rangle) \times \mathcal{F}_t$ — measurable for a fixed $t \in T$.

As an obvious corollary we conclude that the Brownian sheet $X$ in a Banach space is progressively measurable, and consequently measurable.

Let $\tau : (\Omega, \mathcal{F}, P) \to (T, \mathcal{B}(T))$ be a stopping time. Then it can be noted that $Z(t+\tau)$ is a random element with values in $E$. Recall that a random vector $\tau$ is called a stopping time with respect to the filtration $\{\mathcal{F}_t,\ t \in T\}$ if for every $t \in T$, $\{\tau \leq t\} \in \mathcal{F}_t$. Let $\mathcal{F}_\tau = \{D \in \mathcal{F} : D \cap \{\tau \leq t\} \in \mathcal{F}_t \text{ for each } t \in T\}$ and $\mathcal{G}_\tau = \{D \in \mathcal{F} : D \cap \{\tau \leq t\} \in \sigma\big(Z(s),\ s \in T \setminus (t,\infty)\big) \text{ for each } t \in T\}$. Note that $\mathcal{F}_\tau$ and $\mathcal{G}_\tau$ are $\sigma$-fields and $\mathcal{F}_\tau \subseteq \mathcal{G}_\tau$.

We are now in a position to establish a kind of the strong Markov property for Brownian sheets in $B$. We are able to prove even a somewhat stronger result that implies easily strong Markov property for $X$.

**Proposition 3.1.** *Let $Z$ be a right continuous process with stationary independent increments vanishing at the boundary $Z|_{\partial T} = 0$ taking values in a (Hausdorff) Abelian topological group $E$ such that operations $+,-$ are $(e \times e, e)$ - measurable and let $\tau$ be a stopping time with respect to the filtration $\{\mathcal{F}_t,\ t \in T\}$. Denote $Z_0(t) = \Delta Z\big(\langle \tau, \tau + t\rangle\big)$ , $t \in T$. Then the processes $Z$ and $Z_0$ are stochastically equivalent in the wide sense and the $\sigma$-field $\sigma\big(Z_0(t),\ t \in T\big)$ is independent of $\mathcal{G}_\tau$ ( and $\mathcal{F}_\tau$ ).*

**Proof.** The proof can be obtained by a modification of Breiman's [2] arguments, but details will be given elsewhere.

**Corollary 3.2.** *The Brownian sheet $X$ in $B$ satisfies the strong Markov property formulated in Proposition* 3.1 .

The last conclusion is a consequence of the fact that in a metric space the Baire and Borel $\sigma$-fields coincide.

**4. Vector integrals.** Since in the sequel we make use of integrals of Banach space valued continuous functions $x \in C(T_m, B)$ integrated with respect to vector measures taking values in the conjugate space $B^*$, for convenience of the reader we describe here briefly construction of such integrals.

Let $S$ be a compact (Hausdorff) topological space and let $C(S,B)$ be the space of continuous functions defined on $S$ with values in a real Banach space $(B, \|\cdot\|)$.

The space $C(S,B)$ equipped with the norm $\|x\|_S = \sup_{s\in S}\|x(s)\|$ is then a real Banach space.

A function $e : S \to B$ is said to be simple if it may be represented as a linear combination

$$e(s) = \sum_{i=1}^{n} x_i\, \chi_{E_i}(s) \tag{4.1}$$

of some vectors $x_i \in B$ multiplied by indicators $\chi_{E_i}$, where $E_i$, $1 \le i \le n$, are arbitrary pairwise disjoint Borel subsets of $S$, i.e. $E_i \in \mathcal{B}(S)$. The reader may readily verify that for each continuous function $x \in C(S,B)$ there exists a sequence $\{e_n\}$ of simple functions convergent uniformly on $S$ to $x$ in the norm $\|\cdot\|$ of $B$, so that

$$\|x - e_n\|_S \longrightarrow 0 \qquad \text{as } n \to \infty\,. \tag{4.2}$$

Let $\lambda : \mathcal{B}(S) \to B^*$ be an additive set function, for brevity called vector measure. The variation of $\lambda$ is the extended nonnegative function Var $\lambda$, whose value on a set $E \in \mathcal{B}(S)$ is determined by the formula

$$\operatorname{Var}\lambda(E) = \sup_{\kappa} \sum_{E_i \in \kappa} \|\lambda(E_i)\|_{B^*}\,,$$

where the supremum is extended over all partitions $\kappa = \{E_i,\ 1 \le i \le n_\kappa\}$ of $E$ into a finite number of disjoint Borel sets $E_i \subseteq S$. To simplify the notation we write $\operatorname{Var}\lambda(S) = \operatorname{Var}\lambda$ and assume that $\lambda$ is of bounded variation $\operatorname{Var}\lambda < \infty$.

If $e$ is a simple function given by (4.1), the integral of $e$ over $S$ with respect to $\lambda$ is defined to be

$$\int_S e(s)\, d\lambda(s) = \sum_{i=1}^{n} (\lambda(E_i), x_i)\,.$$

Basing on the inequality

$$\Big|\int_S e(s)\, d\lambda(s)\Big| \le \|e\|_S \ \operatorname{Var}\lambda$$

one can easily demonstrate that for every sequence of simple functions $\{e_n\}$ satisfying (4.2) with a fixed function $x \in C(S,B)$ there exists the unique limit

$$\int_S x(s)\, d\lambda(s) = \lim_n \int_S e_n(s)\, d\lambda(s)\,,$$

which is by definition taken as the integral of $x$ with respect to $\lambda$ over $S$. The obtained integral is a special case of the general Bartle [1] integral, constructed for a larger class of $\lambda$–integrable functions on an arbitrary measurable space $(S,\sigma)$ with a field $\sigma$.

A vector measure $\lambda : \mathcal{B}(S) \to B^*$ is countably additive if and only if for every sequence $\{E_n\}$ of pairwise disjoint Borel subsets of $S$ the series $\sum_n \lambda(E_n)$ converges in the norm of $B^*$ and

$$\sum_n \lambda(E_n) = \lambda\big(\bigcup_n E_n\big)\,.$$

Let us observe that the series $\sum_n \lambda(E_n)$ is then unconditionally convergent, i.e. for each subsequence $\{n_r\}$ the subseries $\sum_r \lambda(E_{n_r})$ converges strongly in $B^*$ to $\lambda(\bigcup_r E_{n_r})$.

A vector measure $\lambda : \mathcal{B}(S) \to B^*$ is called regular, if for each $x \in B$ the real set function $(\lambda(\cdot), x) : \mathcal{B}(S) \to R$ is regular, so that for arbitrary $\varepsilon > 0$ and $A \in \mathcal{B}(S)$ we can select an open set $G_\varepsilon \supseteq A$ and a compact set $K_\varepsilon \subseteq A$, such that

$$\left|(\lambda(A), x) - (\lambda(A'), x)\right| < \varepsilon$$

whenever $A' \in \mathcal{B}(S)$ and $K_\varepsilon \subseteq A' \subseteq G_\varepsilon$ .

Singer [18] proved that the topological dual space $C^*(S, B)$ conjugated to $C(S, B)$ is isometrically isomorphic to the space of countably additive regular vector measures $\lambda : \mathcal{B}(S) \to B^*$ of bounded variation with the norm Var $\lambda$, and every continuous linear functional $L^* \in C^*(S, B)$ possesses the integral representation

$$(L^*, x) = \int_S x(s)\, d\lambda(s) ,$$

where $\lambda \leftrightarrow L$ is the mentioned isomorphism.

We are going to describe besides a special kind of the double integral that will appear in our further considerations, namely

$$\int_S \int_S g(t, s) \langle d\hat{\lambda}(t), d\hat{\mu}(s) \rangle , \tag{4.3}$$

where $S$ is, as before, a compact (Hausdorff) topological space and $\lambda, \mu : \mathcal{B}(S) \to B^*$ are vector measures embedded by the isomorphism $\hat{}$ into the Hilbert space $H \subset B$, being the generator of the Wiener measure in a separable Banach space $B$. The last integral can be defined (at least) for all bounded completely measurable functions $g : S \times S \to R$, i.e. functions which are uniform limits of sequences of simple functions.

A function $f : S \times S \to R$ is called now simple, if it may be represented in the form

$$f(t, s) = \sum_{i=1}^{p} \sum_{j=1}^{k} b_{ij}\, \chi_{D_i}(t)\, \chi_{E_j}(s) , \tag{4.4}$$

where $\{D_i,\ 1 \le i \le p\}$ and $\{E_j,\ 1 \le j \le k\}$ are arbitrary finite partitions of $S$ into disjoint Borel sets and $b_{ij} \in R$. The double integral (4.3) of any simple function (4.4) we define by the formula

$$\int_S \int_S f(t, s)\ \langle d\hat{\lambda}(t), d\hat{\mu}(s) \rangle = \sum_{i=1}^{p} \sum_{j=1}^{k} b_{ij}\ \langle \hat{\lambda}(D_i), \hat{\mu}(E_j) \rangle .$$

The above integral satisfies then the inequality

$$\left| \int_S \int_S f(t, s)\ \langle d\hat{\lambda}(t), d\hat{\mu}(s) \rangle \right| \le C^2 \sup_{t,s \in S} |f(t, s)|\ \text{Var}\, \lambda\ \ \text{Var}\, \mu \tag{4.5}$$

with a positive constant $C$ such that $\|x\| \leq C|x|$ for $x \in H$, and consequently $|\hat{y}| = \sup\{(y^*, x) : |x| \leq 1,\ x \in H\} \leq C\|y^*\|_{B^*}$ provided $y^* \in B^*$. On the basis of (4.5) we infer immediately that for every bounded completely measurable function $g : S \times S \to R$, there exists the limit

$$\int_S \int_S g(t,s)\ \langle d\hat{\lambda}(t), d\hat{\mu}(s)\rangle = \lim_n \int_S \int_S f_n(t,s)\ \langle d\hat{\lambda}(t), d\hat{\mu}(s)\rangle,$$

and is unique for all sequences of simple functions $\{f_n\}$ convergent uniformly to $g$. Therefore the above equality will be treated as the definition of the integral (4.3). One can easily observe that every real continuous function $f : S \times S \to R$ is the uniform limit of a sequence of simple functions $\{f_n\}$, thus it can be used as the integrand in (4.3). Obviously, the described integral is well-defined too for $\mu = \lambda$.

**5. Covariance operators of Brownian sheets.** The covariance operator of a second order in the weak sense random element with expectation zero in a Banach space $\mathfrak{X}$ is in general a mapping $\mathfrak{X}^* \to \mathfrak{X}^{**}$, but it is well-known that covariance operators of Gaussian distributions map $\mathfrak{X}^*$ into $\mathfrak{X} \subset \mathfrak{X}^{**}$ (c.f. Vahania [20], Ch.4). Obviously all the measures $W_m$ are Gaussian as Wiener measures.

**Theorem 5.1.** *For each $m \in N^r$ the covariance operator $\Gamma_m : C^*(T_m, B) \to C(T_m, B)$ of $W_m$ satisfies the formula*

$$(5.1)\quad (\Gamma_m L^*, M^*) = \int_{C(T_m,B)} (L^*, x)\ (M^*, x)\ dW_m(x) = \int_{T_m} \int_{T_m} |t \wedge s|\ \langle d\hat{\lambda}(t), d\hat{\mu}(s)\rangle\ ,$$

*where $L^*, M^* \in C^*(T_m, B)$ and $\lambda, \mu : \mathcal{B}(T_m) \to B^*$ are countably additive regular vector measures with bounded variation associated with $L^*, M^*$ and embedded into $H$ by the isometric isomorphism $\hat{\ } : B^* \subset H^* \to H$.*

**Proof.** To simplify the notation let us put $Q_m(n) = \{k2^{-n} \in T_m : k \in N^r\}$, $n = 1, 2, \ldots$. It can be easily seen that

$$(5.2)\quad \Big| \sum_{k,j \in Q_m(n)} (\lambda(V_k), x(k))\ (\mu(V_j), x(j)) \Big| \leq$$
$$\leq \|x\|_m{}^2 \sum_{k,j \in Q_m(n)} \|\lambda(V_k)\|_{B^*}\ \|\mu(V_j)\|_{B^*} \leq \|x\|_m{}^2\ \mathrm{Var}\,\lambda\ \ \mathrm{Var}\,\mu\ ,$$

and by Fernique's [6] theorem,

$$(5.3)\quad E\|X_m\|_m{}^2 = \int_{C(T_m,B)} \|x\|_m{}^2\ dW_m(x) < \infty\ ,$$

because $W_m$ is a Gaussian measure in $C(T_m, B)$. Moreover, with probability 1,

$$(5.4)\quad \sum_{k,j \in Q_m(n)} (\lambda(V_k), X_m(k))\ (\mu(V_j), X_m(j)) \to \int_{T_m} X_m(t)\, d\lambda(t) \int_{T_m} X_m(s)\, d\mu(s).$$

Hence, on account of (5.2)–(5.4), the Lebesgue dominated convergence theorem and (2.7) we conclude that

$$(5.5)\qquad E\int_{T_m} X_m(t)\,d\lambda(t)\int_{T_m} X_m(s)\,d\mu(s) =$$
$$= E\lim_n \sum_{k,j\in Q_m(n)} (\lambda(V_k), X_m(k))\,(\mu(V_j), X_m(j)) =$$
$$= \lim_n \sum_{k,j\in Q_m(n)} |k\wedge j|\,\langle\hat\lambda(V_k),\hat\mu(V_j)\rangle = \int_{T_m}\int_{T_m} |t\wedge s|\,\langle d\hat\lambda(t), d\hat\mu(s)\rangle\,.$$

**Corollary 5.2.** *The measure $W$ is Gaussian with mean zero and covariance operator $\Gamma : C^*(T,B)\to C(T,B)$ determined by the equation*

$$(5.6)\qquad (\Gamma L^*, M^*) = \int_{C(T,B)} (L^*,x)\,(M^*,x)\,dW(x) = \int_T\int_T |t\wedge s|\,\langle d\hat\lambda(t), d\hat\mu(s)\rangle\,,$$

*where the last integral reduces to the integral over the product $T_m\times T_m$ with $m = (\mathrm{rank}\,L^*)\vee(\mathrm{rank}\,M^*)$ for $L^*, M^*\in C^*(T,B)$.*

**Proof.** Since $C(T,B)$ is a $B_0$–space, each continuous linear functional $L^*\in C^*(T,B)$ has some rank $m\in N^r$, i.e. there exists a constant $C$, $0<C<\infty$, such that for all $x\in C(T,B)$

$$(5.7)\qquad |(L^*,x)|\le C\,\|x\|_m$$

and (5.7) is no longer true if $m'\le m$, $m'\ne m$, $m'\in N^r$. Then it can be proved that there can be found a countably additive regular vector measure $\lambda : \mathcal{B}(T)\to B^*$ with bounded variation having support contained in the set $T_m$, such that

$$(5.8)\qquad (L^*,x) = \int_T x(t)\,d\lambda(t)\qquad \text{for all } x\in C(T,B)\,.$$

Indeed, if $\pi_m x = \pi_m y$ for some $x,y\in C(T,B)$, then $|(L^*,x-y)|\le C\,\|x-y\|_m = 0$ and hence $(L^*,x) = (L^*,y)$. Therefore the restriction $L^*_m = L^*\circ\pi_m$ of $L^*$ to $C(T_m,B)$ determines completely $L^*$ in the unique manner and $L^*_m\in C^*(T_m,B)$., Applying again Singer's result [18] we see that there exists a countably additive regular vector measure $\lambda_1 : \mathcal{B}(T_m)\to B^*$ with bounded variation such that

$$(L^*_m, z) = \int_{T_m} z(t)\,d\lambda_1(t)\qquad \text{for all } z\in C(T_m,B)\,.$$

Let $\lambda : \mathcal{B}(T)\to B^*$ be an extension of $\lambda_1$ defined as follows: $\lambda(G) = \lambda_1(G\cap T_m)$ if $G\in\mathcal{B}(T)$, so that $\lambda(G) = 0$ provided $G\subseteq T\setminus T_m$ and $G\in\mathcal{B}(T)$. Then we have

$$(L^*,x) = (L^*_m,\pi_m x) = \int_{T_m}\pi_m x(t)\,d\lambda_1(t) = \int_T x(t)\,d\lambda(t)\,.$$

Moreover, we observe that for an arbitrary number $a \in R$,

$$W[x \in C(T,B) : (L^*,x) < a] = W[x \in C(T,B) : (L^*_m, \pi_m x) < a] = \\ = W_m[z \in C(T_m,B) : (L^*_m, z) < a] = \Phi(a;0,\sigma),$$

where

$$\sigma^2 = \int_{T_m}\int_{T_m} |t \wedge s| \langle d\hat{\lambda}_1(t), d\hat{\lambda}_1(s)\rangle \qquad \text{(cf. (5.5))}.$$

Thus $W$ is a Gaussian measure. Finally, by analogy to (5.5) we obtain

$$E(L^*,X)\,(M^*,X) = E(L^*_m,\pi_m X)\,(M^*_m,\pi_m X) = E(L^*_m,X_m)\,(M^*_m,X_m) = \\ = \int_{T_m}\int_{T_m} |t \wedge s| \langle d\hat{\lambda}_1(t), d\hat{\mu}_1(s)\rangle = \int_T\int_T |t \wedge s| \langle d\hat{\lambda}(t), d\hat{\mu}(s)\rangle,$$

where $M^*_m, \mu_1$ and $\mu$ are defined similarly as $L^*_m, \lambda_1$ and $\lambda$.

**6. Expansion of Brownian sheets in $B$ into a series of real processes.** Suppose $\gamma$ is a Gaussian measure in a real separable (infinite dimensional) Banach space $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$. From Theorem 3.1 given by Kuelbs [12] we know that there exists then a real separable Hilbert space $\mathcal{H} \subset \mathcal{X}$ such that $\gamma(\overline{\mathcal{H}}) = 1$, where $\overline{\mathcal{H}}$ denotes the closure of $\mathcal{H}$ in $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$, and for an arbitrary CONS $\{\alpha_n\} \subset \mathcal{H}$ for $\gamma$–a.e. $x \in \mathcal{X}$, we have

$$\lim_N \Big\| x - \sum_{k=1}^{N} \langle x, \alpha_k\rangle_{\mathcal{H}}\, \alpha_k \Big\|_{\mathcal{X}} = 0 .$$

Note that according to the definition of functions $\langle \cdot, \alpha_k\rangle_{\mathcal{H}} : \mathcal{X} \to R$ they are independent standard normal random variables on $(\mathcal{X}, \mathcal{B}(\mathcal{X}), \gamma)$ (cf. also Kuo [15]). This observation can be formulated in other words as follows: the measure $\gamma$ on cylindrical subsets of $\mathcal{X}$, and hence on the whole $\sigma$–field $\mathcal{B}(\mathcal{X})$ is determined by the canonical Gauss measure $\gamma_1$ in $\mathcal{H}$ with mean zero and variance parameter 1. Moreover, on the basis of Theorems 2 and 3 given by Dudley, Feldman and LeCam [5], $\|\cdot\|_{\mathcal{X}}$ is a measurable norm with respect to $\gamma_1$ in the sense of Gross [8], thus $\gamma$ is the Wiener measure.

Jain and Kallianpur [10] employed to the same problem the well–known Banach–Mazur theorem asserting that each real separable Banach space $\mathcal{X}$ is isometrically isomorphic (congruent) to some closed subspace $C_0$ of the space $C(0,1)$ with the usual supremum norm. Investigating next Gaussian measure on $C_0$ they obtained some other description of $\mathcal{H}$. Kallianpur [11] has shown besides that $\overline{\mathcal{H}}$ is the topological support of $\gamma$, that is $\gamma(\overline{\mathcal{H}}) = 1$ and for any open set $G$ such that $G \cap \overline{\mathcal{H}} \neq \emptyset$, the inequality $\gamma(G \cap \overline{\mathcal{H}}) > 0$ holds. The approach proposed by Jain and Kallianpur possesses rather theoretical meaning.

Perhaps the most natural and simple characterization of the Hilbert space $\mathcal{H}$ being the generator of a Gaussian measure $\gamma$ in a Banach space $\mathcal{X}$ was found by LePage [17]. Assume for a moment that the space $\mathcal{X}$ consists of real functions on a parameter set $A$, such that distinct elements of $\mathcal{X}$ are distinct functions (it is always possible to take $A = K^*$ or $A = \mathcal{X}^*$ and to define $x(\alpha) = (\alpha, x)$ for $\alpha \in A \subset \mathcal{X}^*$). In addition, suppose that all the projections $\pi_\alpha : \mathcal{X} \to R$, $\alpha \in A$, are continuous in the norm of $\mathcal{X}$, so that $\pi_\alpha(x_n) = x_n(\alpha) \to x(\alpha) = \pi_\alpha(x)$ whenever $\|x_n - x\|_{\mathcal{X}} \to 0$. Let $\mathcal{L}$ be the smallest closed subspace of the space $L^2(\mathcal{X}, \mathcal{B}(\mathcal{X}), \gamma)$ containing the family of projections $\{\pi_\alpha,\ \alpha \in A\}$. Then the space $\mathcal{L}$ is isometrically isomorphic to $\mathcal{H}$ and a congruence between these spaces is given by the Bochner integral

$$\check{y} = \int_{\mathcal{X}} x\,(y, x)\,d\gamma(x) \qquad \text{(convergent strongly in } \mathcal{X}\text{)}$$

where $y \in \mathcal{L}$ and $\check{y} \in \mathcal{H}$. Scalar products in both spaces are connected by the equality

$$\langle \check{z}, \check{y} \rangle_{\mathcal{H}} = \int_{\mathcal{X}} (z, x)\,(y, x)\,d\gamma(x) = \langle z, y \rangle_{\mathcal{L}} ,$$

and the closure $\bar{\mathcal{H}}$ of $\mathcal{H}$ in $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ is the topological support of $\gamma$. Moreover, for an arbitrary CONS $\{y_k,\ k \geq 1\}$ in $\mathcal{L}$ the functions $y_k : \mathcal{X} \to R$ are independent standard normal random variables such that

$$\Big\| x - \sum_{k=1}^{n} (y_k, x)\,\check{y}_k \Big\|_{\mathcal{X}} \longrightarrow 0 \qquad \text{for } \gamma - \text{a.e. } x \in \mathcal{X} ,$$

and for each $p > 0$ ,

$$\int_{\mathcal{X}} \Big\| x - \sum_{k=1}^{n} (y_k, x)\,\check{y}_k \Big\|_{\mathcal{X}}^{p} d\gamma(x) \longrightarrow 0 \qquad \text{as } n \to \infty .$$

From our construction of Brownian sheets $X_m$ it follows directly that all the above results are true for $W_m$, only the last statement may be regarded as a corollary to LePage theorem. We denote by $\mathcal{H}_m$ the Hilbert space generating $W_m$.

**Remark** . It is worth to mention that if we treat $\hat{y}$, obtained from $y^* \in \mathcal{X}^*$, as elements of $\mathcal{X}$, then for an arbitrary $y^* \in \mathcal{X}^*$ we have $\hat{y} = \check{y}$. In fact, since the scalar product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ is normalized so that the canonical Gauss distribution in $\mathcal{H}$ generates $\gamma$, for each $z^* \in \mathcal{X}^*$ we get

$$(z^*, \check{y}) = \int_{\mathcal{X}} (z^*, x)\,(y^*, x)\,d\gamma(x) = \langle \hat{z}, \hat{y} \rangle_{\mathcal{H}} = (z^*, \hat{y}) ,$$

and this gives the desired conclusion.

The space $\mathcal{H}_m$ may be described in a more detailed way by means of the Hilbert space $H \subset B$ and the space $C'(T_m) \subset C(T_m)$. The space $C'(T_m)$ being the generator of the Wiener measure in $C(T_m)$ consists of such functions $f \in C(T_m)$ which are absolutely continuous with respect to the Lebesgue measure on $T_m$ and satisfy the condition

$$\int_{T_m} \{\Delta' f(t)\}^2\,dt < \infty ,$$

where we have put $\Delta' f(t) = \lim_{\substack{\mathrm{vol}(K)\to 0 \\ K\ni t}} \dfrac{\Delta f(K)}{\mathrm{vol}(K)}$, $K = r$–dimensional closed cubs.

The next theorem is a generalization of Lemma 4 given by Kuelbs and LePage [14].

**Theorem 6.1.** *Let $L : T_m \to B$ be an arbitrary function and let $\{\breve{y}_j,\ j \geq 1\}$ form a CONS in $H$. Then $L \in \mathcal{H}_m$ if and only if $L(t) \in H$ for each $t \in T_m$, $L|_{T_m \cap \partial T} = 0$, all the mappings $\langle \breve{y}_j, L(\cdot)\rangle$, $j \geq 1$, belong to the space $C'(T_m)$ and*

$$\sum_j \int_{T_m} \{\Delta' \langle \breve{y}_j\ ,\ L(t)\rangle\}^2\, dt < \infty\ .$$

*The scalar product in $\mathcal{H}_m$ is given by the formula*

$$\langle L, M\rangle_{\mathcal{H}_m} = \sum_j \int_{T_m} \{\Delta' \langle \breve{y}_j, L(t)\rangle\}\ \{\Delta' \langle \breve{y}_j, M(t)\rangle\}\, dt\ .$$

*Moreover,*

a) $\mathcal{H}_m = \overline{\mathrm{Lin}}\ \{|t \wedge \cdot|\breve{y} : t \in T_m\ ,\ y^* \in B^*\}$ ,
*where the closure is taken in the norm induced by the scalar product in $\mathcal{H}_m$. In addition, for all $\breve{f}, \breve{g} \in C'(T_m)$ and $\breve{y}, \breve{v} \in H$, we have*

$$\langle \breve{f}\ \breve{y}\ ,\ \breve{g}\ \breve{v}\rangle_{\mathcal{H}_m} = \langle \breve{f}\ ,\ \breve{g}\rangle_{C'(T_m)}\ \langle \breve{y}\ ,\ \breve{v}\rangle\ ,$$

*in particular $\langle |t \wedge \cdot|\ \breve{y}\ ,\ |s \wedge \cdot|\ \breve{v}\rangle_{\mathcal{H}_m} = |t \wedge s|\ \langle \breve{y}\ ,\ \breve{v}\rangle$ .*

b) *For each $\breve{L} \in \mathcal{H}_m$ , $\breve{y} \in H$ and $t \in T_m$ , $\breve{L}(t) \in H$*

$$\langle \breve{y}\ ,\ \breve{L}(t)\rangle = \langle |t \wedge \cdot|\ \breve{y}\ ,\ \breve{L}\rangle_{\mathcal{H}_m} \quad \text{and} \quad |\breve{L}(t)| \leq |\breve{L}|_{\mathcal{H}_m} \sqrt{|t|}\ .$$

c) *For arbitrary elements $\breve{f} \in C'(T_m)$, $\breve{y} \in H$ and $\breve{L} \in \mathcal{H}_m$, $\langle \breve{y}, \breve{L}(\cdot)\rangle \in C'(T_m)$,*

$$\langle\, \langle \breve{y}, \breve{L}(\cdot)\rangle\ ,\ \breve{f}\,\rangle_{C'(T_m)} = \langle \breve{L}\ ,\ \breve{f}\ \breve{y}\rangle_{\mathcal{H}_m} \quad \text{and} \quad |\langle \breve{y}\ ,\ \breve{L}(\cdot)\rangle|_{C'(T_m)} \leq |\breve{L}|_{\mathcal{H}_m}\ |\breve{y}|\ .$$

d) *Let $\{\breve{y}_j,\ j \geq 1\}$ be any CONS in $H$. Then for each $\breve{L} \in \mathcal{H}_m$ we have*

$$\breve{L} = \sum_j \langle \breve{y}_j\ ,\ \breve{L}(\cdot)\rangle\ \breve{y}_j$$

*where the series converges in the norm $|\cdot|_{\mathcal{H}_m}$ .*

e) *If $\{\breve{y}_j,\ j \geq 1\}$ is a CONS in $H$ such that $y_j^* \in B^*$, $j \geq 1$, then for $W_m$–a.e. $x \in C(T_m, B)$ we have*

$$\sum_j (y_j^*, x)\ \breve{y}_j = x\ ,$$

*and the last series converges in the norm $\|\cdot\|_m$.*

**Proof.** As an example of methods exploited for the proof of this theorem we present here only the proof of the last statement, because the demonstration of the analogous assertion was omitted by Kuelbs and LePage [14]. The other parts of our theorem follow easily from the construction of Brownian sheets in a Banach space.

At the beginning we quote some basic facts concerning convergence of double series in a Banach space that are applied in a further fragment of the proof.

Let $\mathcal{N}$ be a collection of all finite subsets of the product $N \times N$ ordered partially by inclusion. We say that a double series

$$\sum_{(i,j)\in N\times N} x_{ij}$$

of elements of a Banach space $X$ converges strongly with respect to the family $\mathcal{N}$ to an element $x \in X$ and write

$$\lim_{D\in\mathcal{N}} \sum_{(i,j)\in D} x_{ij} = x ,$$

iff given any $\varepsilon > 0$ there is a set $D \in \mathcal{N}$ such that for every $D' \supseteq D$ , $D' \in \mathcal{N}$,

$$\Big\| \sum_{(i,j)\in D'} x_{ij} - x \Big\|_X < \varepsilon .$$

It can be shown that if $\lim_{D\in\mathcal{N}} \sum_{(i,j)\in D} x_{ij} = x$, then

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} x_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} x_{ij} = x$$

strongly in $X$. The proof of this result may be obtained by a slight modification of arguments used by Singer [19] – Ch. II, Lemma 16.1 p. 458–461.

Let now $\{\check{f}_j,\ j \geq 1\}$ be a CONS in $C'(T_m)$ and let $\{\check{y}_n,\ n \geq 1\}$ be a CONS in $H$. Obviously $\{\check{f}_j\,\check{y}_n,\ j, n \geq 1\}$ forms then a CONS in $\mathcal{X}_m$. Suppose that $f^* \in C^*(T_m)$ and $y^* \in B^*$. If $x \in C(T_m, B)$, then the map $(y^*, x(\cdot)) : T_m \to R$ is an element of $C(T_m)$, for $|(y^*, x(t)) - (y^*, x(s))| \leq \|y^*\|_{B^*} \|x(t) - x(s)\|$ and $(y^*, x(t)) = (y^*, 0) = 0$ provided $t \in T_m \cap \partial T$. Thus we can define the functional $(fy)^\vee \in C^*(T_m, B)$ by the formula $((fy)^\vee, x) = (f^*, (y^*, x))$. Note that $(fy)^\vee = \check{f}\,\check{y}$. Indeed, to see this it is enough to show that for each $z^* \in B^*$ and $t \in T_m$, $(G_{z^*,t}, (fy)^\vee) = (G_{z^*,t}, \check{f}\,\check{y})$, where $G_{z^*,t} = G_{z^*,V}$ for $V$ of the form $(0, t)$. Evidently, we have

$$(6.1) \qquad (G_{z^*,t}, \check{f}\,\check{y}) = (z^*, \check{f}(t)\check{y}) = \check{f}(t)\ \langle \check{z}, \check{y}\rangle ,$$

and on the other side

$$(6.2) \quad (G_{z^*,t}, (fy)^\vee) = \int_{C(T_m,B)} (G_{z^*,t}, x)\,(fy, x)\, dW_m(x) = (fy, \check{G}_{z^*,t}) =$$

$$= (fy, |t \wedge \cdot|\, \check{z}) = (f^*, (y^*, |t\wedge\cdot|\, \check{z})) = \langle \check{f}, |t\wedge\cdot|\rangle_{C'(T_m)} \langle \check{y}, \check{z}\rangle = \check{f}(t)\ \langle \check{y}, \check{z}\rangle ,$$

because $\check{G}_{z^*,t} = \dot{G}_{z^*,t} = |t \wedge \cdot|\, \check{z}$ .

It can be proved moreover that $(y^*, \cdot) : C(T_m, B) \to C(T_m)$ is a Gaussian random element. Clearly

$$\sup_{t \in T_m} |(y^*, x(t))| \leq \|y^*\|_{B^*} \|x\|_m .$$

Hence we conclude that the mappping $(y^*, \cdot) : C(T_m, B) \to C(T_m)$ is continuous, and consequently it is a random element defined on $(C(T_m, B), \mathcal{B}(C(T_m, B)), W_m)$ with values in $(C(T_m), \mathcal{B}(C(T_m)))$. This is a Gaussian random element since for an arbitrary $f^* \in C^*(T_m)$,

$$W_m\Big[x \in C(T_m, B) : (f^*, (y^*, x)) < a\Big] = W_m\Big[x \in C(T_m, B) : (f\, y, x) < a\Big] = \\ = \Phi(a; 0, |\check{f}\, \check{y}|_{\mathcal{H}_m}) .$$

Let $W_m \circ (y^*)^{-1}$ denote the distribution of $(y^*, \cdot)$. Assume now that $y^* \in B^*$ and $f^*_{kj} \in C^*(T_m)$, where $\check{f}_{kj} \to \check{f}_j$ in $C'(T_m)$. The scalar product of two functions $(f^*, \cdot)$ and $(g^*, \cdot)$, $f^*, g^* \in C^*(T_m)$, in the space $L^2(C(T_m), \mathcal{B}(C(T_m)), W_m \circ (y^*)^{-1})$ is equal to

$$(6.3) \quad \int_{C(T_m)} (f^*, u)\,(g^*, u)\, d\, W_m \circ (y^*)^{-1}(u) = \int_{C(T_m, B)} (f\, y, x)\,(g\, y, x)\, d\, W_m(x) = \\ = (\check{f}\, \check{y}, \check{g}\, \check{y})_{\mathcal{H}_m} = (\check{f}, \check{g})_{C'(T_m)}\, |\check{y}|^2 .$$

Therefore the mapping $\check{f} \to (f^*, \cdot)$ defined on the dense subset $\{\check{f} : f^* \in C^*(T_m)\}$ of $C'(T_m)$ with values in $L^2(C(T_m), \mathcal{B}(C(T_m)), W_m \circ (y^*)^{-1})$ preserves the scalar product up to the positive factor $|\check{y}|^2$. Since $\check{f}_{kj} \to \check{f}_j$ in $C'(T_m)$, the sequence of random variables $(f^*_{kj}, \cdot)$ converges in $L^2(C(T_m), \mathcal{B}(C(T_m)), W_m \circ (y^*)^{-1})$ to a r.v. denoted by $(\check{f}_j, \cdot)_{C'(T_m)}$. This r.v. is determined $W_m \circ (y^*)^{-1}$ – a.e. on $C(T_m)$, hence $W_m$ – a.e. on $C(T_m, B)$ the r.v. $(\check{f}_j, (y^*, \cdot))_{C'(T_m)}$ is defined as well. On the other side, to each functional $f^* \in B^*$ there corresponds the r.v. $(f y, \cdot)$ being an element of the space $L^2(C(T_m, B), \mathcal{B}(C(T_m, B)), W_m)$ and (6.3) implies that the mapping $\check{f} \to (f y, \cdot)$ also preserves the scalar product up to the positive factor $|\check{y}|^2$. Since $\check{f}_{kj} \to \check{f}_j$ in $C'(T_m)$, and consequently $(f_{kj} y)^\vee \to \check{f}_j \check{y}$ in $\mathcal{H}_m$, it follows that $(f_{kj} y, \cdot) \to (\check{f}_j \check{y}, \cdot)_{\mathcal{H}_m}$ in $L^2(C(T_m, B), \mathcal{B}(C(T_m, B)), W_m)$. However, $(f^*, (y^*, x)) = (f y, x)$ for all $x \in C(T_m, B)$, thus $W_m$ – a.e. on $C(T_m, B)$ we have the equality

$$(6.4) \quad (\check{f}_j, (y^*, x))_{C'(T_m)} = (\check{f}_j\, \check{y}, x)_{\mathcal{H}_m} .$$

We observe next that on the basis of our construction the double series

$$\sum_{(j,n) \in N \times N} (\check{f}_j\, \check{y}_n, x)_{\mathcal{H}_m}\, \check{f}_j\, \check{y}_n$$

converges strongly with respect to the family $\mathcal{N}$ to $x \in C(T_m, B)$ $W_m$ – a.e. . Hence, taking into account the quoted already result concerning double series we conclude that for $W_m$ – a.e. $x \in C(T_m, B)$,

$$(6.5) \quad \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} (\check{f}_j\, \check{y}_n, x)_{\mathcal{H}_m}\, \check{f}_j\, \check{y}_n = x .$$

Applying a very well-known expansion into a series of real Brownian sheet (cf. Kuelbs [12] Theorem 3.1) we see that $W_m \circ (y^*)^{-1}$ – a.e. in the usual sup norm of $C(T_m)$

$$\sum_{j=1}^{\infty} \langle \tilde{f}_j, f \rangle_{C'(T_m)} \tilde{f}_j = f . \tag{6.6}$$

However $\|\tilde{f}\, \tilde{y}\|_m = \|\tilde{f}\|_{C(T_m)} \|\tilde{y}\|$, so in view of (6.4) and (6.6), $W_m$ – a.e. strongly on $C(T_m, B)$ ,

$$\sum_{j=1}^{\infty} \langle \tilde{f}_j\, \tilde{y}, x \rangle_{\chi_m} \tilde{f}_j\, \tilde{y} = \sum_{j=1}^{\infty} \langle \tilde{f}_j, (y^*, x) \rangle_{C'(T_m)} \tilde{f}_j\, \tilde{y} = (y^*, x)\, \tilde{y} . \tag{6.7}$$

Neglecting a set of $W_m$ – measure zero determined by $\{y^*_n,\ n \geq 1\}$, on account of (6.5) and (6.7) we obtain

$$\sum_{n=1}^{\infty} (y^*_n, x)\, \tilde{y}_n = x \qquad W_m - \text{a.e.}$$

and the proof is complete.

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W artykule przedstawiona została elementarna metoda konstrukcji wieloparametrowego procesu Wienera o wartościach w rzeczywistej nieskończenie wymiarowej przestrzeni Banacha. Opisano też podstawowe własności tego procesu, np. strukturę kowariancji, mocną własność Markowa i.t.p. . Ponadto scharakteryzowana została przestrzeń Hilberta generująca rozkład procesu w przestrzeni jego ciągłych trajektorii i wyprowadzono rozwinięcie procesu w szereg niezależnych jednowymiarowych powierzchni brownowskich.

## SUMMARY

This paper deals with an elementary construction of a multiparameter Wiener process with values in a real separable infinitely dimensional Banach space. Basic properties of this process such as covariance structure, strong Markov property, etc. are described. Moreover, a Hilbert space

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