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Ramanujan Institute
University of Madras

K.S. PADMANABHAN, R. MANJINI

Convolutions of Pre-starlike Functions with Negative Coefficients

Sploty funkcji pregwiaździstych o ujemnych współczynnikach

Let S denote the class of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ that are analytic and univalent in the unit disc $E = \{z : |z| < 1\}$. Let $S^*(A,B)$ denote the class of functions $f \in S$ such that

$$(1) \qquad \frac{z f'(z)}{f(z)} = \frac{1 + A\,\omega(z)}{1 + B\,\omega(z)}\,, \qquad -1 \le A < B \le 1\,,$$

for all z in E, where $\omega(z)$ is analytic in E with $\omega(0) = 0$ and $|\omega(z)| < 1$, $z \in E$. By the convolution $(f * g)$ of two functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, we mean the Hadamard product $(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n$. We say that $f \in R_\alpha(A,B)$, $0 \le \alpha < 1$, if $f * s_\alpha \in S^*(A,B)$ where $s_\alpha = z/(1-z)^{2(1-\alpha)}$, the extremal function for the class of functions starlike of order α. Let $R_\alpha[A,B]$ denote the class of functions $f \in R_\alpha(A,B)$ such that

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n\,, \qquad a_n \ge 0\,.$$

We investigate, in this paper, the family $R_\alpha[A,B]$ in terms of its coefficients, and then determine extreme points, radii of univalence, starlikeness, and convexity, and order of starlikeness. We also prove distortion theorems. Sharp results are obtained in each case. From our results many of the results of earlier papers can be deduced.

We observe that for $B = \beta$ and $A = \beta(2\gamma - 1)$, $(0 \le \gamma < 1\,,\ 0 < \beta \le 1)$, the class $R_\alpha[A,B]$ reduces to the class of functions f which are α-prestarlike of order γ and type β investigated in [1] and for $B = 1$, $A = 2\gamma - 1$, the class $R_\alpha(A,B)$ reduces to the class of prestarlike functions of order γ, introduced by St.Ruscheweyh [3]. If $\alpha = \frac{1}{2}$, $B = \beta$ and $A = \beta(2\gamma - 1)$, $(0 \le \gamma < 1\,,\ 0 < \beta \le 1)$, then the class $R_\alpha[A,B]$ is the class of functions f which are starlike of order γ and type β studied in [2]. The class $R_\gamma[2\gamma-1,1] \equiv R[\gamma]$ was studied in [5]. Also for $\alpha = \frac{1}{2}$, $B = 1$, and $A = 2\gamma - 1$, $(0 \le \gamma < 1)$, the class $R_\alpha[A,B]$ becomes the family $S^*[\gamma]$ studied in [4].

$s_\alpha(z)$ may be rewritten in the form $s_\alpha(z) = z + \sum_{n=2}^{\infty} c(\alpha,n) z^n$, where

$$c(\alpha,n) = \prod_{k=2}^{n} (k-2\alpha)/(n-1)! \qquad (n = 2,3,\ldots) . \tag{2}$$

Note that $c(\alpha,n)$ is a decreasing function of α , $0 \le \alpha \le 1$, with

$$\lim_{n\to\infty} c(\alpha,n) = \begin{cases} \infty , & \alpha < \frac{1}{2} \\ 1 , & \alpha = \frac{1}{2} \\ 0 , & \alpha > \frac{1}{2} \end{cases} .$$

Coefficient Inequalities. We begin by proving a characterization theorem for the class $R_\alpha[A,B]$.

Theorem 1. *$f \in R_\alpha[A,B]$ if and only if*

$$\sum_{n=2}^{\infty} \frac{[n(B+1)-(A+1)]\, c(\alpha,n)\, a_n}{B-A} \le 1 . \tag{3}$$

Proof. Suppose $f \in R_\alpha[A,B]$. Then, setting $g(z) = (f * s_\alpha)(z)$, we have

$$\frac{z\,g'(z)}{g(z)} = \frac{1 + A\,\omega(z)}{1 + B\,\omega(z)} , \qquad -1 \le A < B \le 1 ,$$

where ω is analytic in E with $\omega(0) = 0$ and $|\omega(z)| < 1$, $z \in E$. Thus we get

$$\omega(z) = \frac{g(z) - z\,g'(z)}{B\,z\,g'(z) - A\,g(z)}$$

and $|\omega(z)| < 1$ implies

$$\left| \frac{\sum_{n=2}^{\infty} (n-1)\, c(\alpha,n)\, a_n z^{n-1}}{B - A - \sum_{n=2}^{\infty} (Bn - A)\, c(\alpha,n)\, a_n z^{n-1}} \right| < 1 . \tag{4}$$

Thus

$$\operatorname{Re} \left[\frac{\sum_{n=2}^{\infty} (n-1)\, c(\alpha,n)\, a_n z^{n-1}}{(B-A) - \sum_{n=2}^{\infty} (Bn - A)\, c(\alpha,n)\, a_n z^{n-1}} \right] < 1 . \tag{5}$$

We consider real values of z and take $z = r$ with $0 \le r < 1$. Then, for $r = 0$, denominator of (5) is positive and so it is positive for all r with $0 \le r < 1$, since it cannot vanish for $z \in E$ because $\omega(z)$ is analytic for $|z| < 1$. Then (5) gives

$$\sum_{n=2}^{\infty}(n-1)\,c(\alpha,n)\,a_n r^{n-1} < (B-A) - \sum_{n=2}^{\infty}(Bn-A)\,c(\alpha,n)\,a_n r^{n-1}\,,$$
$$z = r\,,\ 0 \le r < 1\,.$$

Letting $r \to 1$ we get (3).

Conversely, suppose $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \ge 0$, satisfies (3). For $|z| = r$, $0 \le r < 1$, we have, since $r^{n-1} < 1$,

$$\sum_{n=2}^{\infty}[n(B+1)-(A+1)]\,c(\alpha,n)\,a_n r^{n-1} < B - A$$

by (3). So we have

$$\left|\sum_{n=2}^{\infty}(n-1)\,c(\alpha,n)\,a_n z^{n-1}\right| \le \sum_{n=2}^{\infty}(n-1)\,c(\alpha,n)\,a_n r^{n-1} <$$
$$< (B-A) - \sum_{n=2}^{\infty}(Bn-A)\,c(\alpha,n)\,a_n r^{n-1} \le$$
$$\le \left|(B-A) - \sum_{n=2}^{\infty}(Bn-A)\,c(\alpha,n)\,a_n z^{n-1}\right|.$$

Hence (4) holds and therefore follows that

$$\frac{z(f * s_\alpha)'(z)}{(f * s_\alpha)(z)} = \frac{1 + A\,\omega(z)}{1 + B\,\omega(z)}$$

where ω is regular in E, $\omega(0) = 0$ and $|\omega(z)| < 1$. That is, $f \in R_\alpha[A,B]$.

Corollary 1. *If* $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ *is in* $R_\alpha[A,B]$, *then* $a_n \le (B-A)/[n(B+1)-(A+1)]\,c(\alpha,n)$, $n \ge 2$, *with equality for functions of the form*

$$f_n(z) = z - (B-A)z^n/[n(B+1)-(A+1)]\,c(\alpha,n)\,.$$

Corollary 2. $f \in S^*[\eta]$, $0 \le \eta < 1$, *if and only if*

$$\sum_{n=2}^{\infty}(n-\eta)a_n \le 1 - \eta\,.$$

Proof. The Corollary follows on choosing $A = (2\eta - 1)$, $(0 \le \eta < 1)$, $B = 1$ and $\alpha = \frac{1}{2}$ in Theorem 1. Corollary 2 is nothing but Theorem 2 in [4].

Theorem 2. *Let* $f_1(z) = z - \sum_{n=2}^{\infty} a_{n,1} z^n$, $f_2(z) = z - \sum_{n=2}^{\infty} a_{n,2} z^n$ *be in the same class* $R_\alpha[A, B]$, $0 \le \alpha \le \frac{1}{2}$. *Then* $(f_1 *_1 f_2)(z) = z - \sum_{n=2}^{\infty} a_{n,1} a_{n,2} z^n$ *is also in the class* $R_\alpha[A, B]$.

Proof. By Theorem 1,

$$\sum_{n=2}^{\infty} \frac{[n(B+1) - (A+1)]\, c(\alpha, n)\, a_{n,i}}{B - A} \le 1 , \qquad i = 1, 2 .$$

Therefore, noting that $c(\alpha, n) \ge c(\alpha, 2)$ for $n \ge 2$, $\alpha \le \frac{1}{2}$, we get

$$a_{n,i} \le \frac{B - A}{(2B - A + 1)\, c(\alpha, 2)} , \qquad i = 1, 2 .$$

Using Theorem 1 and the above inequality we obtain

$$\sum_{n=2}^{\infty} [n(B+1) - (A+1)]\, c(\alpha, n)\, a_{n,1} a_{n,2} \le \frac{(B - A)^2}{(2B - A + 1)\, c(\alpha, 2)} =$$

$$= \frac{(B - A)^2}{2(2B - A + 1)(1 - \alpha)} \le B - A \qquad \text{for } \alpha \le \frac{1}{2} .$$

Hence $f_1 *_1 f_2 \in R_\alpha[A, B]$, $0 \le \alpha \le \frac{1}{2}$.

Theorem 3. *The class* $R_\alpha[A, B]$ *is closed under convex linear combinations.*

Proof. Let $f, g \in R_\alpha[A, B]$ and let $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $g(z) = z - \sum_{n=2}^{\infty} b_n z^n$. For η such that $0 \le \eta \le 1$, it suffices to show that $h(z) = (1 - \eta) f(z) + \eta g(z)$, $z \in E$ is also a function of $R_\alpha[A, B]$. Now $h(z) = z - \sum_{n=2}^{\infty} [(1 - \eta) a_n + \eta b_n] z^n$. Applying Theorem 1 to $f, g \in R_\alpha[A, B]$, we have

$$\sum_{n=2}^{\infty} \frac{[n(B+1) - (A+1)]\, c(\alpha, n) \{(1 - \eta) a_n + \eta b_n\}}{B - A} =$$

$$= (1 - \eta) \sum_{n=2}^{\infty} \frac{[n(B+1) - (A+1)]\, c(\alpha, n)\, a_n}{B - A} + \eta \sum_{n=2}^{\infty} \frac{[n(B+1) - (A+1)]\, c(\alpha, n)\, b_n}{B - A} \le$$

$$\le (1 - \eta) + \eta = 1.$$

This implies $h \in R_\alpha[A, B]$.

It is shown in the following theorem that the extreme points of the closed convex hull of $R_\alpha[A, B]$ are those that maximize the coefficients.

Theorem 4. *Define*

(6) $f_1(z) = z$ and $f_n(z) = z - (B - A) z^n / [n(B+1) - (A+1)]\, c(\alpha, n)$, $n = 2, 3, 4, \ldots$.

Then $f \in R_\alpha[A,B]$ *if and only if* f *can be expressed as* $f(z)=\sum_{n=1}^{\infty}\mu_n f_n(z)$, *where* $\mu_n \geq 0$ *and* $\sum_{n=1}^{\infty}\mu_n = 1$.

Proof. If $f(z)=\sum_{n=1}^{\infty}\mu_n f_n(z)$, then

$$\sum_{n=2}^{\infty}\frac{[n(B+1)-(A+1)]c(\alpha,n)}{B-A}\mu_n\frac{B-A}{[n(B+1)-(A+1)]c(\alpha,n)}=\sum_{n=2}^{\infty}\mu_n = 1-\mu_1 \leq 1$$

and hence $f \in R_\alpha[A,B]$.

Conversely, let $f(z)=z-\sum_{n=2}^{\infty}a_n z^n \in R_\alpha[A,B]$. Define $\mu_n=\dfrac{[n(B+1)-(A+1)]c(\alpha,n)a_n}{B-A}$, $n=2,3,\ldots$, and define $\mu_1 = 1-\sum_{n=2}^{\infty}\mu_n$. We see from Theorem 1 that $\sum_{n=2}^{\infty}\mu_n \leq 1$ and so $\mu_1 \geq 0$. Since $\mu_n f_n(z)=\mu_n z - a_n z^n$, $\sum_{n=1}^{\infty}\mu_n f_n(z) = z-\sum_{n=2}^{\infty}a_n z^n = f(z)$.

Distortion Theorems. Now we determine bounds on the modulus of f and f' for $f \in R_\alpha[A,B]$.

Theorem 5. *If* $f \in R_\alpha[A,B]$, $-1 \leq A < B \leq 1$, *and either* $r \leq \dfrac{3B-A+2}{2(2B-A+1)}$ *or* $0 \leq \alpha \leq \dfrac{5B-A+4}{2(3B-A+2)}$, *then for* $|z|=r$,

$$\mathrm{Max}\left\{0, r-\frac{(B-A)r^2}{2(1-\alpha)(2B-A+1)}\right\} \leq |f(z)| \leq r+\frac{(B-A)r^2}{2(1-\alpha)(2B-A+1)}.$$

The bounds are sharp for the extremal function $f_2(z)=z-\dfrac{(B-A)z^2}{2(1-\alpha)(2B-A+1)}$.

Proof. Since $|f(z)| \leq \sum_{n=1}^{\infty}\mu_n|f_n(z)| \leq \max_n |f_n(z)|$, we have

$$|f(z)| \leq r+\max_n\frac{(B-A)r^n}{[n(B+1)-(A+1)]c(\alpha,n)}$$

and

$$|f(z)|=\left|\sum_{n=1}^{\infty}\mu_n f_n(z)\right| = \left|z-\sum_{n=2}^{\infty}\frac{\mu_n(B-A)z^n}{[n(B+1)-(A+1)]c(\alpha,n)}\right| \geq$$

$$\geq r-\max_n\frac{(B-A)r^n}{[n(B+1)-(A+1)]c(\alpha,n)}.$$

Therefore to prove the theorem it is enough to show that

(7) $$\Phi(A,B,\alpha,r,n)=(B-A)r^n/[n(B+1)-(A+1)]c(\alpha,n)$$

is a decreasing function of n for $n \geq 2$ under the given conditions for α and r. From the definition of $c(\alpha, n)$ in (2) we have $c(\alpha, n+1) = [(n+1-2\alpha)/n]c(\alpha, n)$. Now $\Phi(A, B, \alpha, r, n) \geq \Phi(A, B, \alpha, r, n+1)$ if and only if

(8) $h(A, B, \alpha, r, n) = (n+1-2\alpha)[(n+1)(B+1)-(A+1)] - rn[n(B+1)-(A+1)] \geq 0$.

For A and B fixed, the function h is a decreasing function of α and r is an increasing function of n. Hence $h(A, B, \alpha, r, n) \geq h\left(A, B, \frac{5B-A+4}{2(3B-A+3)}, 1, 2\right) = 0$ for $0 \leq \alpha \leq \frac{5B-A+4}{2(3B-A+3)}$, $r < 1$ and $n \geq 2$. Similarly, $h(A, B, \alpha, r, n) \geq$ $\geq h\left(A, B, 1, \frac{3B-A+2}{2(2B-A+1)}, 2\right) = 0$ for $0 \leq \alpha < 1$, $r \leq \frac{3B-A+2}{2(2B-A+1)}$, and $n \geq 2$. Therefore $\max_{n \geq 2} \Phi(A, B, \alpha, r, n)$ is attained for $n = 2$. This completes the proof of the theorem.

Remark. The function $f_2(z) = 0$ in Theorem 5, when $z = 2(1-\alpha)(2B-A+1)/(B-A)$. Let $z \to 1^-$, we obtain $|f(z)| \geq r - (B-A)r^2/2(1-\alpha)(2B-A+1)$ for all z in E if and only if $0 \leq \alpha \leq (3B-A+2)/2(2B-A+1)$.

The upper bound for $|f|$ when $\alpha > \frac{5B-A+4}{2(3B-A+3)}$ and $r > (3B-A+2)/2(2B-A+1)$ is not known by the above theorem. We deal with this case in the following theorem.

Theorem 6. *Define* $r(n_0, A, B, \alpha) = \dfrac{(n_0+1-2\alpha)[(n_0+1)(B+1)-(A+1)]}{n_0[n_0(B+1)-(A+1)]}$. *If* $f \in R_\alpha[A, B]$, $-1 \leq A < B \leq 1$,

$$\alpha_0 = \frac{(2n_0+1)(B+1)-(A+1)}{2[(n_0+1)(B+1)-(A+1)]} < \alpha \leq \frac{(2n_0+3)(B+1)-(A+1)}{2[(n_0+2)(B+1)-(A+1)]} = \alpha_1, \quad (n_0 = 2, 3, \ldots)$$

and $r(n_0, A, B, \alpha) < r < 1$, *then*

$$|f(z)| \leq r + (B-A)r^{n_0+1}/[(n_0+1)(B+1)-(A+1)]\,c(\alpha, n_0+1) \qquad (|z| = r).$$

Equality holds for functions f_{n_0+1} *given in* (6).

Proof. To prove the theorem we have to determine when $\Phi(A, B, \alpha, r, n)$ defined by (7), is maximized for $n = n_0 + 1 > 2$. The function Φ attains its maximum value at $n = n_0 + 1$ if the function, defined by (8), is negative for $n = n_0$ and positive for $n = n_0 + 1$, which occurs for $r(n_0, A, B, \alpha) < r < r(n_0+1, A, B, \alpha)$. However, $r(n_0, A, B, \alpha) < 1$ if and only if $\alpha > \alpha_0$ and $r(n_0+1, A, B, \alpha) \geq 1$ for $\alpha \leq \alpha_1$. Therefore, $\max_n \Phi(A, B, \alpha, r, n)$ is attained at $n = n_0 + 1$ for $r(n_0, A, B, \alpha) < r < 1$ and $\alpha_0 < \alpha \leq \alpha_1$. Hence the theorem is proved.

Theorem 7. *If* $f \in R_\alpha[A, B]$, $-1 \leq A < B \leq 1$, *and either* $0 \leq \alpha \leq \frac{1}{2}$ *or* $r \leq \dfrac{3B-A+2}{3(2B-A+1)}$, *then*

$$1 - \frac{(B-A)r}{(1-\alpha)(2B-A+1)} \leq |f'(z)| \leq 1 + \frac{(B-A)r}{(1-\alpha)(2B-A+1)}.$$

Equality holds for $f_2(z) = z - \dfrac{(B-A)z^2}{2(1-\alpha)(2B-A+1)}$.

Proof. By Theorem 4, we have

$$1-\max_{n\geq 2}\frac{n(B-A)r^{n-1}}{[n(B+1)-(A+1)]c(\alpha,n)} \leq |f'(z)| \leq 1+\max_{n\geq 2}\frac{n(B-A)r^{n-1}}{[n(B+1)-(A+1)]c(\alpha,n)}$$

It is sufficient to show that $\Psi(A,B,\alpha,r,n) = \dfrac{n(B-A)r^{n-1}}{[n(B+1)-(A+1)]c(\alpha,n)}$ is a decreasing function of n for $n \geq 2$. Ψ is a decreasing function if and only if

$$h_1(A,B,\alpha,r,n) = (n+1-2\alpha)[(n+1)(B+1)-(A+1)]-(n+1)r[n(B+1)-(A+1)] \geq 0.$$

The function h_1 is a decreasing function of r and α for $\alpha \leq \frac{1}{2}$ and is an increasing function of n. Therefore we have

$$h_1(A,B,\alpha,r,n) \geq h_1(A,B,\tfrac{1}{2},1,2) = A+1 \geq 0$$

for $0 \leq \alpha \leq \frac{1}{2}$, $r < 1$ and $n \geq 2$. Also

$$h_1(A,B,\alpha,r,n) \geq h_1(A,B,1,\frac{3B-A+2}{3(2B-A+1)},2) = 0$$

for $0 \leq \alpha < 1$, $r \leq \frac{3B-A+2}{3(2B-A+1)}$ and $n \geq 2$. Hence $\max_{n\geq 2}\Psi(A,B,\alpha,r,n)$ is attained for $n = 2$. This completes the proof.

Remark . Since $h_1(A,B,1,r,2) < 0$ for $r > \frac{3B-A+2}{3(2B-A+1)}$ and $\Psi(A,B,\alpha,1,n) >$ $> \Psi(A,B,\alpha,1,2)$ for each fixed $\alpha > \frac{1}{2}$ and $n = n(\alpha)$ sufficiently large, Theorem 7 is the best possible.

Radii of Univalence, Starlikeness, and Convexity. The function $f_2(z) = 0$, in Theorem 5, when $z = 2(1-\alpha)(2B-A+1)/(B-A)$. It is, therefore, possible to have $f(z_0) = 0$, $0 < |z_0| < 1$ for f in $R_\alpha[A,B]$. Hence if $f \in R_\alpha[A,B]$, then f need not be univalent. The next theorem discusses the question of univalence of members of $R_\alpha[A,B]$.

Theorem 8. $R_\alpha[A,B] \subset S$ *if and only if* $0 \leq \alpha \leq \frac{1}{2}$.

Proof. Let $\alpha \leq \frac{1}{2}$ and let $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in R_\alpha[A,B]$. Since $z + \sum_{n=2}^{\infty} a_n z^n \in S$ if $\sum_{n=2}^{\infty} n|a_n| \leq 1$ (Theorem 1 in [4]), by Theorem 1, it is sufficient to show for $\alpha \leq \frac{1}{2}$ that

$$[n(B+1)-(A+1]c(\alpha,n)/(B-A) \geq n \qquad \text{for } n = 2,3,4,\ldots . \tag{9}$$

But if $\alpha \leq \frac{1}{2}$, $c(\alpha,n) \geq c(\frac{1}{2}.n) = 1$. So it is enough to prove (9) for $\alpha = \frac{1}{2}$. When $\alpha = \frac{1}{2}$, (9) becomes

$$n(A+1) \geq A+1$$

which is true for all $n \geq 2$.

To prove the converse, we take $f_n(z)$ defined by (6). Then we have

$$f_n'(z) = 1 - n(B-A)z^{n-1}/[n(B+1)-(A+1)]\,c(\alpha,n) = 0$$

for

$$z^{n-1} = [n(B+1)-(A+1)]\,c(\alpha,n)/n(B-A)\ ,$$

which is less than 1 for n sufficiently large because as $n \to \infty$, $c(\alpha, n) \to 0$ for $\alpha > \frac{1}{2}$. Hence, $f_n(z)$ is not univalent for $\alpha > \frac{1}{2}$ and $n = n(\alpha)$ sufficiently large. The proof is complete.

Corollary 3. *$f \in R_\alpha[A,B]$ is starlike if and only if $0 \leq \alpha \leq \frac{1}{2}$.*

Proof. Since functions of the form $z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$, are starlike if and only if they are univalent [4], the corollary follows by Theorem 8.

We now proceed to determine the largest disc centred at the origin for functions in $R_\alpha[A,B]$, $0 \leq \alpha \leq \frac{1}{2}$, to be starlike of specified order η , $0 \leq \eta < 1$.

Theorem 9. *If $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in R_\alpha[A,B]$, $-1 \leq A < B \leq 1$, $0 \leq \alpha \leq \frac{1}{2}$, then f is starlike of order η , $0 \leq \eta < 1$, in the disc $|z| < r_0$, where*

$$r_0 = \inf_n \left[\frac{(1-\eta)[n(B+1)-(A+1)]c(\alpha,n)}{(B-A)(n-\eta)}\right]^{1/(n-1)} .$$

Equality holds for the functions $f_n(z)$ defined in (6).

Proof. It suffices to show that $|(zf'/f) - 1| < 1 - \eta$ for $|z| < r_0$. But

$$\left|\frac{zf'}{f} - 1\right| \leq \frac{\sum_{n=2}^{\infty}(n-1)a_n|z|^{n-1}}{1 - \sum_{n=2}^{\infty} a_n|z|^{n-1}} \leq 1 - \eta \qquad (|z| = r)$$

if and only if

$$\sum_{n=2}^{\infty} \frac{n-\eta}{1-\eta} a_n r^{n-1} \leq 1\ .$$

By Theorem 1 and Corollary 2, we need only find values of r for which

$$\left(\frac{n-\eta}{1-\eta}\right) r^{n-1} \leq \frac{[n(B+1)-(A+1)]\,c(\alpha,n)}{B-A} \qquad (n = 2,3,\ldots)\ .$$

The above inequality will be true when $r \leq r_0$. This completes the proof.

Corollary 4. *If $f \in R_\alpha[A,B]$, $-1 \leq A < B \leq 1$, $0 \leq \alpha \leq \frac{1}{2}$, then f is convex of order η , $0 \leq \eta < 1$, in the disc $|z| < r_1$, where*

$$r_1 = \inf_n \left[\frac{(1-\eta)[n(B+1)-(A+1)]c(\alpha,n)}{(B-A)n(n-\eta)}\right]^{1/(n-1)} .$$

Proof. Since $f = z + \sum_{n=2}^{\infty} a_n z^n$ is convex of order η if and only if $z + \sum_{n=2}^{\infty} n a_n z^n = zf'$ is starlike of order η, the proof follows that of Theorem 9, with a_n replaced by na_n.

By taking $\eta = 0$ in Theorem 9, we may determine the radius of univalence (and starlikeness) of $R_\alpha[A, B]$ when $\alpha > \frac{1}{2}$.

Corollary 5. *If $f \in R_\alpha[A, B]$, $-1 \le A < B \le 1$, $\frac{1}{2} < \alpha < 1$, then f is univalent and starlike for $|z| < r_2$, where*

$$r_2 = \inf_n \left[\frac{[n(B+1) - (A+1)]c(\alpha, n)}{n(B-A)} \right]^{1/(n-1)} .$$

Order of Starlikeness. Since functions in $R_\alpha[A, B]$, $0 \le \alpha \le \frac{1}{2}$, are starlike, it is of interest to determine the order of starlikeness of this class of functions.

Theorem 10. *If $f \in R_\alpha[A, B]$, $-1 \le A < B \le 1$, $0 \le \alpha \le \frac{1}{2}$, then f is starlike of order*

$$\delta = \frac{(2B - A + 1)(1 - \alpha) - (B - A)}{(2B - A + 1)(1 - \alpha) - (B - A)/2} .$$

Equality holds for the functions $f_2(z) = z - \dfrac{(B - A)z^2}{2(1 - \alpha)(2B - A + 1)}$.

Proof. From Theorem 1 and Corollary 2 , it is sufficient to show , for $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in R_\alpha[A, B]$, that $\sum_{n=2}^{\infty} \frac{[n(B+1) - (A+1)]\, c(\alpha, n)\, a_n}{B - A} \le 1$ implies $\sum_{n=2}^{\infty} (\frac{n - \delta}{1 - \delta}) a_n \le 1$. This will be true if

$$g(A, B, \alpha, n) = \frac{[n(B+1) - (A+1)]\, c(\alpha, n)(1 - \delta)}{(B - A)(n - \delta)} \ge 1 \qquad (n = 2, 3, \ldots) .$$

For fixed A and B, g is a decreasing function of α , $0 \le \alpha \le \frac{1}{2}$, and an increasing function of n , $n \ge 2$. So that

$$g(A, B, \alpha, n) \ge g(A, B, \tfrac{1}{2}, 2) = 1$$

for $0 \le \alpha \le \frac{1}{2}$ and $n \ge 2$. The Theorem is proved.

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STRESZCZENIE

Niech $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$, będzie funkcją analityczną w kole jednostkowym i niech $s_\alpha(z) = z/(1-z)^{2(1-\alpha)}$, $0 \leq \alpha < 1$. Autorzy badają klasę $R_\alpha[A, B]$ funkcji f takich, że pochodna logarytmiczna splotu $f * s_\alpha$ jest podporządkowana homografii $z \mapsto (1+Az)/(1+Bz)$, gdzie $-1 \leq A < B \leq 1$.

SUMMARY

Let $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$, be analytic in the unit disk and let $s_\alpha(z) = z/(1-z)^{2(1-\alpha)}$, $0 \leq \alpha < 1$. The authors are concerned with the class $R_\alpha[A, B]$ of functions f such that the logarithmic derivative of the convolution $f * s_\alpha$ is subordinate to the homography $z \mapsto (1 + Az)/(1 + Bz)$, where $-1 \leq A < B \leq 1$.