ANNALES UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN-POLONIA
VOL. XLI, 9 SECTIO A 1987

Instytut Matematyki
Uniwersytet Marii Curie-Skłodowskiej

J. MICHALSKI

A Sufficient Condition for Univalence of Functions Meromorphic in the Unit Disc

Pewien warunek dostateczny jednolistności funkcji meromorficznej w kole jednostkowym

1. Introduction. The class S_p , $0 < p < 1$, of functions f meromorphic and univalent in the unit disk D with a simple pole at p and the normalization : $f(0) = f'(0) - 1 = 0$ has been investigated by many authors, cf.[1]. S. M. Zemyan was first to investigate the subclasses $S_p(a)$ of S_p consisting of all $f \in S_p$ with a fixed residue $\operatorname{res}(f;p) = a$, cf. [2], [3], [4]. It is easily seen that $F(\varsigma) = \dfrac{1}{f(1/\varsigma)}$ is a function of the familiar class Σ which satisfies $F(1/p) = 0$. Hence $a = -F'(1/p)/p^2$ and this determines the range Ω_p of values of $a = \operatorname{res}(f;p)$, $f \in S_p(a)$:

$$(1.1) \qquad \Omega_p = \{a \in \mathbb{C} : a = -p^2(1-p^2)^s , \ s \in \bar{D}\} .$$

Consequently, any $f \in S_p(a)$ has the form:

$$(1.2) \qquad f(z) = \frac{a}{z-p} + \frac{a}{p} + \Big(1 + \frac{a}{p^2}\Big)z + \sum_{k=2}^{\infty} \alpha_k z^k ; \qquad a \in \Omega_p , \ z \in D .$$

According to Zemyan [3] the following area theorem holds for $f \in S_p(a)$:

$$(1.3) \qquad \frac{|a|^2}{(1-p^2)^2} \geq \Big|1 + \frac{a}{p^2}\Big|^2 + \sum_{k=2}^{\infty} k|\alpha_k|^2 .$$

This implies that

$$(1.4) \qquad |a| \geq (1-p^2)\Big|1 + \frac{a}{p^2}\Big|$$

holds for any $a \in \Omega_p$.

On the other hand, a more restrictive condition

$$|a| \geq (1+p)^2 \left|1 + \frac{a}{p^2}\right| \tag{1.5}$$

implies that the set of all a satisfying (1.5) is a proper subset of Ω_p. In fact, in Sect.2 we show that the function

$$F(z,p,a) = \frac{a}{z-p} + \frac{a}{p} + \left(1 + \frac{a}{p^2}\right)z \tag{1.6}$$

with a satisfying (1.5) is univalent which implies $a \in \Omega_p$. As shown in [3], the area of the set of values omitted by $f \in S_p(a)$ is a maximum for $F(z,p,a)$, as soon as (1.5) holds.

Since (1.5) implies the univalence of (1.6) , i.e. the univalence of f whose all coefficients α_k in the expansion (1.2) vanish, it seems natural to ask whether a suitably modified condition (1.5) involving the coefficients α_k does imply the univalence of f as given by (1.2). A positive answer is given in the next section. For the suggestions concerning the problem I am much indebted to Prof. J.Krzyż.

2. A sufficient condition for the univalence of f. We have the following

Theorem . *Suppose that* (1.5) *holds for some* $a \in \mathbb{C}$ *and* $0 < p < 1$. *If, moreover*

$$\sum_{k=2}^{\infty} k|\alpha_k| \leq |a|(1+p)^{-2} - \left|1 + \frac{a}{p^2}\right| , \tag{2.1}$$

then the function

$$f(z) = \frac{a}{z-p} + \frac{a}{p} + \left(1 + \frac{a}{p^2}\right)z + \sum_{k=2}^{\infty} \alpha_k z^k , \qquad z \in D - \{p\} , \tag{2.2}$$

is univalent. Consequently, $f \in S_p(a)$ *and* $a \in \Omega_p$.

Proof. If $z_1, z_2 \in D - \{p\}$, then

$$\begin{aligned} &\frac{f(z_2) - f(z_1)}{z_2 - z_1} = \\ &= 1 + \frac{a}{p^2} - \frac{a}{(z_1 - p)(z_2 - p)} + \sum_{k=2}^{\infty} \alpha_k \left(z_1^{k-1} + z_1^{k-2} z_2 + \cdots + z_2^{k-1}\right) . \end{aligned} \tag{2.3}$$

It follows from (1.5) that

$$\frac{|a|p^2}{|a+p^2|} \geq (1+p)^2 > |(z_1 - p)(z_2 - p)| ; \qquad z_1, z_2 \in D .$$

Hence

$$\left|1 + \frac{a}{p^2}\right| \leq |a|(1+p)^{-2} < \frac{|a|}{|(z_1 - p)(z_2 - p)|} ,$$

and cosequently,

$$(2.4)\qquad \left|1+\frac{a}{p^2}-\frac{a}{(z_1-p)(z_2-p)}\right| \geq \frac{|a|}{|(z_1-p)(z_2-p)|}-\left|1+\frac{a}{p^2}\right| > \\ > |a|(1+p)^{-2}-\left|1+\frac{a}{p^2}\right| .$$

In view of (2.1) and (2.4) we have

$$\left|1+\frac{a}{p^2}-\frac{a}{(z_1-p)(z_2-p)}\right| > \sum_{k=2}^{\infty} k|\alpha_k| > \left|\sum_{k=2}^{\infty} \alpha_k\left({z_1}^{k-1}+{z_1}^{k-2}z_2+\cdots+{z_2}^{k-1}\right)\right|$$

and from this and (2.3) we readily see that $f(z_2)-f(z_1)\neq 0$ for $z_2\neq z_1$; $z_1,z_2\in D-\{p\}$. This ends the proof.

REFERENCES

[1] Goodman, A. W. , *Univalent Functions* , Vol. II, Mariner Publ. Company, Tampa, Florida 1983.

[2] Zemyan, S. , *A minimal outer area problem in conformal mapping* , J. Analyse Math. 39 (1981), 11-23.

[3] Zemyan , S. , *On a maximal outer area problem for a class of meromorphic univalent functions* , Bull. Austral. Math. Soc. 34 (1986), 433–445.

[4] Zemyan, S. , *The range of the residue functional for the class S_p* , Michigan Math. J. 31 (1984), 73–77.

STRESZCZENIE

W pracy tej wykazano następujące twierdzenie : jeśli $0<p<1$ oraz dla $a\in \mathbb{C}$ spełniona jest nierówność (1.5), to funkcja $f(z)$ określona wzorem (2.2), w którym współczynniki α_k spełniają nierówność (2.1), jest jednolistna w obszarze $D\setminus\{p\}$.

SUMMARY

In this paper the following theorem is proved : If $0<p<1$ and for some $a\in \mathbb{C}$ the inequality (1.5) holds, then $f(z)$ as defined by (2.2) with coefficients α_k satisfying (2.1), is univalent in $D\setminus\{p\}$.