ANNALES UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN-POLONIA
VOL. XLI, 6 SECTIO A 1987

Zakład Zastosowań Matematyki
Instytut Teorii Rozwoju Społeczno Ekonomicznego
Uniwersytet Marii Curie-Skłodowskiej

Z. LEWANDOWSKI

New Remarks on Some Univalence Criteria

Nowe uwagi o pewnych kryteriach jednolistności

1. Introduction. This paper contains an improvement and extension of some univalence criteria contained in my earlier papers [1] and [2]. Section 2 of this article contains general results while Section 3 includes some corollaries. We conclude with remarks and information about some misprints contained in [1] and [2], although they were of no consequence for all results of the above mentioned articles.

We begin with some notations : $\mathbb{C}$ is the complex plane ; $\bar{A}$, ∂A denote the closure or the boundary of the set $A \subset \bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, respectively ; $\mathbb{R} = (-\infty, \infty)$; $K(S; R)$ is an open disc of centre S and radius R ; $E_r = \{z : |z| < r\}$, $r \in (0; 1]$, $E_1 = E$; $E_r^0 = \{w \in \bar{\mathbb{C}} : |w| > r \geq 1\}$, $E_1^0 = E^0$.

2. Main results. Before the formulation of general results we shall give a trivial but useful

Remark 1. Let $D \subset \mathbb{C}$ be a convex domain such that ∂D does not contain any rectilinear segment. Suppose that $A \in \bar{D}$ and $w(\lambda_0) = \lambda_0 A + (1 - \lambda_0)B \in \bar{D}$, where $A \neq B$ are fixed points. Then it is easy to see that

$$\text{a)}\quad [\lambda_0 \in (0; 1)] \Longrightarrow w(\lambda) \in D \qquad \text{for each } \lambda \in (\lambda_0; 1) ,$$
$$\text{b)}\quad [\lambda_0 > 1] \Longrightarrow w(\lambda) \in D \qquad \text{for each } \lambda \in (1; \lambda_0) .$$

We come now to the formulation and proofs of general results.

Theorem 1. *Let $a \geq 1/2$, $s = \alpha + \beta i$, $\alpha > 0$, $\beta \in \mathbb{R}$ be fixed numbers and let $f(z) = z + a_2 z^2 + \cdots$ and $g(z)$ be regular in E with $f'(z) \neq 0$ for $z \in E$. Suppose that the following inequalities*

$$\left| \frac{z f'(z)}{f(z) g(z)} - \frac{as}{\alpha} \right| \leq \frac{a|s|}{\alpha} \quad , \tag{1}$$

and

$$\left| |z|^{2a/\alpha} \frac{z f'(z)}{f(z) g(z)} + \left(1 - |z|^{2a/\alpha}\right) \left[\frac{z f'(z)}{f(z)} + s \frac{z g'(z)}{g(z)} \right] - \frac{as}{\alpha} \right| \leq \frac{a|s|}{\alpha} \tag{2}$$

hold for $z \in E$. Then f is univalent in E.

Proof. Theorem 1 was proved in [1] for $a > 1/2$ by using Pommerenke's subordinations chains. It remains to prove Theorem 1 in the limit case $a = 1/2$ for which the mentioned method cannot be applied directly. In this case from (1) and (2) we obtain

$$\left|\frac{z f'(z)}{f(z)g(z)} - \frac{s}{2\alpha}\right| \leq \frac{|s|}{2\alpha} \quad , \tag{3}$$

and

$$\left||z|^{1/\alpha}\frac{z f'(z)}{f(z)g(z)} + \left(1 - |z|^{1/\alpha}\right)\left[\frac{z f'(z)}{f(z)} + s\frac{z g'(z)}{g(z)}\right] - \frac{s}{2\alpha}\right| \leq \frac{|s|}{2\alpha} \quad . \tag{4}$$

Let us put $f_r(z) = r^{-1}f(rz)$, $g_r(z) = g(rz)$ where $r \in (0;1)$ is a fixed number. Then (4) implies the following inequality

$$\left||rz|^{1/\alpha}\frac{z f_r'(z)}{f_r(z)g_r(z)} + \left(1 - |rz|^{1/\alpha}\right)\left[\frac{z f_r'(z)}{f_r(z)} + s\frac{z g_r'(z)}{g_r(z)}\right] - \frac{s}{2\alpha}\right| \leq \frac{|s|}{2\alpha} \quad . \tag{5}$$

Let us set $A_r(z) = z f_r'(z)/[f_r(z)g_r(z)]$, $B_r(z) = z f_r'(z)/f_r(z) + sz g_r'(z)/g_r(z)$. From the definition and by (3) $A_r(z) \in K(s/2\alpha; |s|/2\alpha)$ for $z \in E$. Applying Remark 1,a) with $D = K(s/2\alpha; |s|/2\alpha)$, $A = A_r(z)$, $B = B_r(z)$, $\lambda_0 = |rz|^{1/\alpha}$ to conditions (3) and (4) we obtain the following inequality

$$\left||z|^{1/\alpha}A_r(z) + \left(1 - |z|^{1/\alpha}\right)B_r(z) - s/2\alpha\right| \leq |s|/2\alpha \tag{6}$$

which is equivalent to the following one

$$\left||z|^{1/\alpha}[A_r(z) - B_r(z)] + N_r(z) + 1 - s/2\alpha\right| \leq |s|/2\alpha \tag{7}$$

where $N_r(z) = B_r(z) - 1$. In what follows we will show that there exists $e \in (0;1)$ such that the inequalities

$$\left|A_r(z) - \frac{(1+e)s}{2\alpha}\right| \leq \frac{(1+e)|s|}{2\alpha} \quad , \tag{8}$$

$$\left||z|^{(1+e)/\alpha}A_r(z) + \left(1 - |z|^{(1+e)/\alpha}\right)B_r(z) - s/2\alpha\right| \leq |s|/2\alpha \tag{9}$$

hold for $z \in E$. In such a case by Theorem 1 for $a = (1+e)/2 > 1/2$ $f_r(z)$ would be univalent in E. Inequality (8) is an easy consequence of (3). From (5) by Remark 1,a) we obtain (9) for $|z| \geq r^{1/e}$ because $|rz|^{1/\alpha} \leq |z|^{(1+e)/\alpha}$ and $K(s/2\alpha; |s|/2\alpha) \subset K((1+e)s/2\alpha; (1+e)|s|/2\alpha)$ for each $e \in (0;1)$. Now in order to complete the proof we ought to show that there exists $e \in (0;1)$ such that (9) holds for $|z| \leq r^{1/e}$. From (3) we obtain $z^{-1}f(z)g(z) \neq 0$ for $z \in E$ and hence $z^{-1}f_r(z)g_r(z) \neq 0$ in E. Thus there exists $M(r) > 0$ such that $|A(z) - B(z)| \leq M(r)$,

$|N(z)| \le M(r)$. Moreover in view of $N(0) = 0$ and the Schwarz lemma $|N(z)| \le M(r)|z|$. Similarly as (6) and (7) inequality (9) is equivalent to the following one

$$\tag{10} \left| |z|^{(1+\varepsilon)/\alpha}[A_r(z) - B_r(z)] + N_r(z) + 1 - (1+\varepsilon)s/2\alpha \right| \le (1+\varepsilon)|s|/2\alpha \quad .$$

It follows from the above considerations that

$$\left| |z|^{(1+\varepsilon)/\alpha}[A_r(z) - B_r(z)] + N_r(z) \right| \le M(r)\left(|z|^{(1+\varepsilon)/\alpha} + |z|\right) < M(r)\left(|z|^{1/\alpha} + |z|\right)$$

and (10) will be fulfilled for $|z| \le r^{1/\varepsilon}$ if $M(r)\left(|z|^{1/\alpha}+|z|\right)$ is smaller than the distance $d(\varepsilon)$ of the point $w = 1$ from the boundary of $K((1+\varepsilon)s/2\alpha; (1+\varepsilon)|s|/2\alpha)$ and if the point $w = 1$ is in that disc. Further we have $d(\varepsilon) = (1+\varepsilon)|s|/2\alpha - |(1+\varepsilon)s/2\alpha - 1| =$ $=2\varepsilon/\left[(1+\varepsilon)\left(\sqrt{1+(\beta/\alpha)^2}+\sqrt{(1-\varepsilon)^2/(1+\varepsilon)^2 + (\beta/\alpha)^2}\right)\right] > \varepsilon/\left[(1+\varepsilon)\sqrt{1+(\beta/\alpha)^2}\right] =$ $= \varepsilon\cos\gamma/(1+\varepsilon)$ where $\gamma = \arg s \in (-\pi/2; \pi/2)$, $s = \alpha+i\beta$. Hence we deduce that the point $w = 1$ lies in the mentioned disc and $d(\varepsilon) > \varepsilon\cos\gamma/2$. Since $\lim\limits_{0<x\to\infty} (b^x/x) = 0$ for $0 < b < 1$ we obtain $M(r)\left(|z|^{1/\alpha} + |z|\right) \le M(r)\left(r^{1/\varepsilon\alpha} + r^{1/\varepsilon}\right) < \varepsilon\cos\gamma/2 < d(\varepsilon)$ for $|z| < r^{1/\varepsilon}$ and for sufficiently small $\varepsilon \in (0;1)$. Thus (10) and so (9) is fulfilled in E for this ε and then f_r is univalent there. Obviously $f(z) = \lim\limits_{r\to 1} f_r(z)$ is univalent in E as well. The proof of Theorem 1 has been completed.

Theorem 2. *Suppose that* $g(w) = w + b_0 + b_1 w^{-1} + \cdots$, $g'(w) \ne 0$, $h(w) = 1 + c_n w^{-n} + \cdots$ *are regular in* $E^0 \setminus \{\infty\}$ *or* E^0 *respectively. For some fixed numbers* $a > 1/2$, $s = \alpha + i\beta$, $\alpha > 0$, $\beta \in \mathbb{R}$, *let the following inequalities*

$$\tag{11} \left| \frac{w\, g'(w)}{g(w)h(w)} - \frac{a\,s}{\alpha} \right| \le \frac{a|s|}{\alpha} \quad ,$$

$$\tag{11'} \left| |w|^{2a/\alpha}\frac{w\, g'(w)}{g(w)h(w)} + \left(1 - |w|^{2a/\alpha}\right)\left[\frac{w\, g'(w)}{g(w)} + s\frac{w\, h'(w)}{h(w)}\right] - \frac{as}{\alpha} \right| \le \frac{a|s|}{\alpha}$$

hold for $w \in E^0$. *Then* g *is univalent in* E^0.

The main tool in our proof is the following

Pommerenke's lemma [3]. *Let* $r_0 \in (0;1]$ *and let* $f(z,t) = a_1(t)z + \cdots$, $a_1(t) \ne 0$, *be regular in* E_{r_0} *for each* $t \in [0;\infty)$ *and locally absolutely continuous in* $[0;\infty)$, *local uniformly in* E_{r_0}. *Suppose that for almost all* $t \in [0;\infty)$ f *satisfies the equation* $f'_t(z,t) = z f'_z(z,t)p(z,t)$ *for* $z \in E_{r_0}$, *where* $p(z,t)$ *is regular in* E *and* $\operatorname{Re} p(z) > 0$ *for* $z \in E$. *If* $|a_1(t)| \to \infty$ *for* $t \to \infty$ *and if* $\{f(z,t)/a_1(t)\}$ *forms a normal family in* E_{r_0}, *then for each* $t \in [0;\infty)$ $f(z,t)$ *has a regular and univalent extension to the whole disc* E.

Proof of Theorem 2. From the normalizations of g and h we infer that (11') has the form

$$\left| |w|^{2a/\alpha}\left[(ns - 1)c_n w^{-n} + o(w^{-n})\right] + 1 + O(w^{-1}) - \frac{as}{\alpha} \right| \le \frac{a|s|}{\alpha}, \qquad w \to \infty$$

and this in turn implies the following inequality

$$(11'') \qquad a \le n\alpha/2$$

From $g'(w) \neq 0$ for $w \in E^0$ and (11) we obtain $g(w)h(w) \neq 0$ in E^0. For $t \in [0;\infty)$ let us put formally

$$(12) \qquad f(z,t) = \frac{1}{g(e^{\alpha t}z^{-1})}\left[1-(1-e^{-2at})h(e^{\alpha t}z^{-1})\right]^{-s}, \qquad z \in E.$$

Then we have

$$(13) \qquad \begin{cases} g(e^{\alpha t}z^{-1}) = \dfrac{e^{\alpha t}}{z} + b_0 + b_1 z e^{-\alpha t} + \cdots, \\ h(e^{\alpha t}z^{-1}) = 1 + c_n z^n e^{-n\alpha t} + \cdots \end{cases}$$

Putting $A(z;a,s,t) = 1-(1-e^{-2at})h(e^{\alpha t}z^{-1}) = e^{-2at} - (1-e^{-2at})(c_n z^n e^{-n\alpha t} + \cdots)$ we obtain that $A(z;a,s,t) \neq 0$ for $z \in E_{r_1}$ and for each $t \in [0;\infty)$, where $r_1 \in (0;1]$ is a fixed number. For example r_1 may be chosen so that $|c_n z^n + c_{n+1}z^{n+1} + \cdots| \le 1$ for $z \in E_{r_1}$. Then $|A(z;a,s,t)| \ge e^{-2at} - (1-e^{-2at})e^{-\alpha nt} =$ $= e^{-2at}\left[1-(1-e^{-2at})e^{(2a-n\alpha)t}\right] > 0$ for $t \in [0;\infty)$ because $2a - n\alpha \le 0$ by (11''). Hence, for each fixed $t \in [0;\infty)$, each fixed single-valued branch of $f(z,t)$ is regular in E_{r_1}. Further from (13) we obtain $a_1(t) = [e^{-t}e^{2at}]^{s}$. In what follows we choose that fixed branch of power in $a_1(t)$ for which $|a_1(t)| = e^{-\alpha t}e^{2a\alpha t}$. Thus $|a_1(t)| = e^{(2a-1)\alpha t} \to \infty$ as $t \to \infty$ because $a > 1/2$ and $\alpha > 0$. By the definition of $A(z;a,s,t)$ and (12), (13) we obtain

$$(14) \qquad \frac{f(z,t)}{a_1(t)} =$$

$$= \frac{z}{\left(1+b_0e^{-\alpha t}z + b_1z^2e^{-2\alpha t}+\cdots\right)\left[1-(e^{2at}-1)(c_nz^ne^{-n\alpha t} + c_{n+1}ze^{-(n+1)\alpha t} + \cdots)\right]^{s}}$$

It follows from (14) and from (11'') that there exists r_0, $0 < r_0 < r_1$ such that $\{f(z,t)/a_1(t)\}$ forms a normal family in E_{r_0}. Furthermore, from the definition of $f(z,t)$, its regularity in E_{r_0} it follows that $f'_t(z,t)$ is uniformly bounded in E_{r_0} for $t \in [0;T]$, where $T > 0$ is an arbitrarily chosen fixed number. Thus $f(z,t)$ is absolutely continuous in $[0;T]$, unformly in E_{r_0}. Now from (12) after some computations we obtain

$$\frac{f'_t(z,t)}{z f'_z(z,t)} = p(z,t) =$$

$$= -s + \frac{2ase^{-2at}g(we^{t\alpha})h(we^{t\alpha})}{we^{t\alpha}g'(we^{t\alpha})\left[1-(1-e^{-2at})h(we^{t\alpha})\right] - s\left[(1-e^{-2at})we^{t\alpha}g(we^{t\alpha})h'(we^{t\alpha})\right]},$$

where $w = z^{-1}$. Thus

$$(14') \qquad p(z,t) = -s + \frac{2as}{e^{2at}A(we^{t\alpha}) + (1-e^{2at})B(we^{t\alpha})}$$

where $A(w) = wg'(w)/[g(w)h(w)]$, $B(w) = wg'(w)/g(w) + swh'(w)/h(w)$. (11) implies that $A(we^{ts}) \in \bar{K}(as/\alpha; a|s|/\alpha)$ for each $w \in E^0$ and $t \in [0;\infty)$. Moreover $A(w) \neq 0$, because $f'(w) \neq 0$ for $w \in E^0$. It follows from (11') that the quantity $|we^{ts}|^{2a/\alpha}A(we^{ts}) + (1-|w|^{2a/\alpha})B(we^{ts})$ lies in $\bar{K}(as/\alpha; a|s|/\alpha)$, and in addition $|we^{ts}|^{2a/\alpha} = |w|^{2a/\alpha}e^{2at} > e^{2at}$. Hence, by Remark 1,b) with $\lambda_0 = |we^{ts}|^{2a/\alpha}$ and $\lambda = e^{2at}$ we see that the denominator d of the r.h.s. of (14') lies in $K(as/\alpha; a|s|/\alpha)$ for each $w \in E^0$ and $t \in (0;\infty)$. Thus $p(z,t)$ is regular in E^0 for each $t \in [0;\infty)$. The inequality $\operatorname{Re} p(z,t) > 0$ and the relation $d \in K(as/\alpha; a|s|/\alpha)$ are equivalent by (14'). Then $\operatorname{Re} p(z,t) > 0$ for $z \in E$ and $t \in (0;\infty)$. Thus we see from the above considerations that all assumptions of Pommerenke's lemma are fulfilled. Hence $f(z,t)$ is univalent in E for each $t \in [0;\infty)$ and so is g because $f(z,0) = 1/g(z^{-1})$. The proof of Theorem 2 has been completed.

In the special case $a = 2$ Theorem 2 was proved in [2].

3. Corollaries. We infer from (1) that there exists a function ω which is regular in E and $|\omega(z)| \leq 1$, $\omega(z) \neq 1$ there and such that $[1-\omega(z)]as/\alpha = zf'(z)/[f(z)g(z)]$ for $z \in E$. Taking logarithm of both sides of the last equality and differentiating we obtain by (2) after simple calculation the following equivalent form of Theorem 1

Theorem 3. *Let $f(z) = z + a_2z^2 + \cdots$, $f'(z) \neq 0$, be regular in E. If there exists a function ω regular in E with $|\omega(z)| \leq 1$, $\omega(z) \neq 1$ for $z \in E$ and such that the inequality*

$$\left||z|^{2a/\alpha}\omega(z) - (1-|z|^{2a/\alpha})\left\{\frac{\alpha - a}{a} + \frac{\alpha}{as}\left[(1-s)\frac{zf'(z)}{f(z)} + s\left(\frac{zf''(z)}{f'(z)} + \frac{z\omega'(z)}{1-\omega(z)}\right)\right]\right\}\right| \leq 1$$

holds for some fixed numbers $a \geq 1/2$, $s = \alpha + i\beta$, $\alpha > 0$, $\beta \in \mathbb{R}$ then f is univalent in E.

If we assume $h(w) = wg'(w)/g(w)$ in Theorem 2 then by simple calculation we obtain

Corollary 1. *Suppose that $g(w) = w + b_0 + b_1w^{-1} + \cdots$ is regular in $E^0 \setminus \{\infty\}$ and $g'(w) \neq 0$ there. For some fixed numbers $a > 1/2$, $s = \alpha + i\beta$, $\alpha > 0$, $\beta \in \mathbb{R}$ let the following inequality*

$$(15) \qquad \left||w|^{2a/\alpha} + (1-|w|^{2a/\alpha})\left[(1-s)\frac{wg'(w)}{g(w)} + s\left(1 + \frac{wg''(w)}{g'(w)}\right)\right] - \frac{as}{\alpha}\right| \leq \frac{a|s|}{\alpha}$$

holds for $w \in E^0$. Then g is unvalent in E^0.

Note that inequality (11) is satisfied automatically in this case because $\partial K(as/\alpha; a|s|/\alpha)$ passes through the points $w = 0$, $w = 2a$ and this in turn implies that $wg'(w)/[g(w)h(w)] \equiv 1 \in K(as/\alpha; a|s|/\alpha)$.

Now we will give Theorem 4 which is equivalent to Theorem 2. (11) implies that there exists a function ω, $|\omega(w)| \leq 1$, $\omega(w) \neq 1$, regular in E^0 and such that

$$(16) \qquad \frac{as}{\alpha}(1-\omega(w)) = \frac{wg'(w)}{g(w)h(w)}$$

Thus by simple calculation we obtain from (11′) and (16), similarly as previously, the following

Theorem 4. *Let* $g(w) = w + b_0 + b_1 w^{-1} + \cdots$, $g'(w) \neq 0$, *be regular in* $E^0 \setminus \{\infty\}$ *and let* $\omega(w)$, $|\omega(w)| \le 1$, $\omega(w) \neq 1$, *be regular in* E^0. *If for some fixed numbers* $a > 1/2$, $s = \alpha + i\beta$, $\alpha > 0$, $\beta \in \mathbb{R}$ *the following inequality*

$$\Bigg| |w|^{2a/\alpha}\omega(w) - \\ - \left(1 - |w|^{2a/\alpha}\right)\left\{\frac{\alpha - a}{a} + \frac{\alpha}{as}\left[(1-s)\frac{w\, g'(w)}{g(w)} + s\left(\frac{w\, g''(w)}{g'(w)} + \frac{w\, \omega'(w)}{1-\omega(w)}\right)\right]\right\}\Bigg| \le 1$$

holds for $w \in E^0$ *then* g *is univalent in* E^0.

It is easily seen from (16) that $\omega(\infty) = 1 - \alpha/as$. If we assume in Theorem 4 $\omega(w) = \text{const} = 1 - \alpha/as$ then we obtain

Corollary 2. *For the previous assumptions let the inequality*
(17)

$$\left| |w|^{2a/\alpha}(1 - \alpha/as) - \left(1 - |w|^{2a/\alpha}\right)\left\{\frac{\alpha - a}{a} + \frac{\alpha}{as}\left[(1-s)\frac{w\, g'(w)}{g(w)} + s\frac{w\, g''(w)}{g'(w)}\right]\right\}\right| \le 1$$

holds in E^0. *Then* g *is univalent in* E^0.

In the case $s = \alpha = a = 1$ we obtain from (17) the well known Becker's univalence criterion, cf. p.ex. [3], p.173.

Similarly as in Theorem 1 we come now to present the limit case $a = 1/2$ in Theorem 2. It must be emphasized that this limit case is somewhat different than the mentioned one of Theorem 1. By definition of g and h we obtain $w\, g'(w)/[g(w)h(w)] = 1$ at the point $w = \infty$. A simple geometrical observation tells us that the point $u = 1$ lies on the $\partial K(s/2\alpha; |s|/2\alpha)$. Thus (11) and the regularity of the quantity $w\, g'(w)/[g(w)h(w)]$ in E^0 implies that $h(w) \equiv w\, g'(w)/g(w)$ in E^0. This leads to the limit case $a = 1/2$ of the Corollary 1. Hence (15) implies the following inequality

$$(18)\quad \left| |w|^{1/\alpha} + \left(1 - |w|^{1/\alpha}\right)\left[(1-s)\frac{w\, g'(w)}{g(w)} + s\left(1 + \frac{w\, g''(w)}{g'(w)}\right)\right] - s/2\alpha \right| \le |s|/2\alpha\,.$$

Let $A(w)$ denote the expression in square bracket of (18). The function $A(w)$ is regular in E^0 and $A(\infty) = 1$. If $A(w) \neq 1$ then there exists a $w_0 \in E^0 \setminus \{\infty\}$ such that $A(w_0) = 1 - \varepsilon$ for some $\varepsilon \in (0;1)$. Further we obtain from (18)
$|w_0|^{1/\alpha} + \left(1 - |w_0|^{1/\alpha}\right)A(w_0) = |w_0|^{1/\alpha} + \left(1 - |w_0|^{1/\alpha}\right)(1-\varepsilon) = 1 + \varepsilon\left(|w_0|^{1/\alpha} - 1\right) > 1$. Thus $|w_0|^{1/\alpha} + \left(1 - |w_0|^{1/\alpha}\right)A(w_0)$ lies outside the disc $\bar{K}(s/2\alpha; |s|/2\alpha)$ in spite of (18). Therefore $A(w) \equiv 1$ in E^0. Solving the suitable differential equation we obtain $g(w) = \left(c + w^{1/s}\right)^s$ with $|c| \le 1$. These functions are regular in $E^0 \setminus \{\infty\}$ and univalent in E^0 if and only if $c = 0$ or $s = 1$. Thus we obtain

Corollary 3. *For* $a = 1/2$, $s = \alpha + i\beta$, $\alpha > 0$, $\beta \in \mathbb{R}$ *only the function* $g(w) = w$ *satisfies* Theorem 2 *and in addition for* $s = 1$ $g(w) = w + c_1$ *does so.*

4. Concluding remarks.

Remark 2. We infer from (2) and (11) for $z = 0$ or $w = \infty$ respectively that $1 \in K(as/\alpha; a|s|/\alpha)$ if $a \geq 1/2$ but this cannot be true if $0 < a < 1/2$. Then the assumption $a \geq 1/2$ is essential in our previous considerations.

Remark 3. We shall list here misprints in paper [1]. They are
$88_{12}, f'_z(0,t) = 1^s = 1$; $88_6, \varsigma f'(\varsigma)/[f(\varsigma)g(\varsigma)]$; $89^{12}, zf''(z)/f'(z) - z\omega'(z)/[e^{i\gamma} - \omega(z)]$; $92^4, zf''(z)/f'(z)$; $93^1, |s|^2$; $93^6, \alpha/(2a-1)$; $93^9, zf''(z)/f'(z) - z\omega'(z)/[e^{i\gamma} - \omega(z)]$. They ought to be replaced by $f'_z(0,0) = 1^s = 1$; $\varsigma f'(\varsigma)/f(\varsigma)$; $zf''(z)/f'(z) + z\omega'(z)/[e^{i\gamma} - \omega(z)]$; $1 + zf''(z)/f'(z)$; $|z|^2$; $\alpha/(2a-\alpha)$; $zf''(z)/f'(z) + z\omega'(z)/[e^{i\gamma} - \omega(z)]$, respectively.

Remark 4. Similarly, there is $b_0 z + b_1 z^2 e^{-st}$ on p.179[11] and $z \in E^0$ on p.180[5] in the paper [2]. It should be $b_0 z e^{-st} + b_1 z^2 e^{-2st}$ and $z \in E$, respectively.

REFERENCES

[1] Lewandowski, Z., *Some Remarks on Univalence Criteria*, Ann. Univ. Mariae Curie-Skłodowska, Sect.A, 36/37 (1982/1983), 87–95.

[2] Lewandowski, Z., *Some remarks on univalence criteria for functions meromorphic in the exterior of the unit disc*, Ann. Polon. Math. vol 46 (1985), 177–181.

[3] Pommerenke, Ch., *Über die Subordination analytischer Functionen*, J. Reine Angew. Math. 218 (1965), 159–173.

STRESZCZENIE

Wcześniej w pracach [1] i [2] otrzymano dwa następujące główne wyniki, które cytuje się tutaj zgodnie z oznaczeniami przyjętymi w tych pracach. Dla ustalonych liczb $a > 1/2$, $s = \alpha + i\beta$, $\alpha > 0$, $\beta \in (-\infty; \infty)$, $\kappa = 2a/\alpha$ prawdziwe są twierdzenia

Twierdzenie 2[1]. *Niech* $f(z) = z + a_2 z^2 + \cdots$, $f'(z) \neq 0$, *i* $g(z)$ *będą funkcjami regularnymi w* $E = \{z : |z| < 1\}$ *takimi, że* $|zf'(z)/[f(z)g(z)] - as/\alpha| \leq a|s|/\alpha$ *dla* $z \in E$. *Jeżeli prócz tego zachodzi nierówność*

$$\left| |z|^{2\kappa} \frac{zf'(z)}{f(z)g(z)} + \left(1 - |z|^{2\kappa}\right) \left[\frac{zf'(z)}{f(z)} + s\frac{zg'(z)}{g(z)} \right] - \frac{as}{\alpha} \right| \leq \frac{a|s|}{\alpha} \tag{A}$$

dla $z \in E$ *to* f *jest jednolistna w* E.

Twierdzenie 2[2]. *Niech* $g(\varsigma) = \varsigma + b_0 + b_1 \varsigma^{-1} + \cdots$, $g'(\varsigma) \neq 0$, *i* $h(\varsigma) = 1 + c_2 \varsigma^{-2} + \cdots$ *będą funkcjami regularnymi w* $E^0 \setminus \{\infty\} = \{\varsigma : |\varsigma| > 1\} \setminus \{\infty\}$ *takimi, że* $|\varsigma g'(\varsigma)/[g(\varsigma)h(\varsigma)] -$

$-as/\alpha| \le a|s|/\alpha$ *dla* $\varsigma \in E^0$. *Jeżeli prócz tego zachodzi nierówność*

$$\text{(B)} \qquad \left| |\varsigma|^{2\kappa} \frac{\varsigma g'(\varsigma)}{g(\varsigma)h(\varsigma)} + \left(1 - |\varsigma|^{2\kappa}\right) \left[\frac{\varsigma g'(\varsigma)}{g(\varsigma)} + s\frac{\varsigma h'(\varsigma)}{h(\varsigma)} \right] - \frac{as}{\alpha} \right| \le \frac{a|s|}{\alpha}$$

dla $\varsigma \in E^0$ *i* $a \le \alpha$ *to* g *jest jednolistna w* E^0.

W niniejszej pracy rozszerza się te wyniki dowodząc, że twierdzenie 2[1] zachodzi również w przypadku granicznym $a = 1/2$ (twierdzenie 1) oraz, że twierdzenie 2[2] zachodzi również w przypadku ogólnym, gdy $h(\varsigma) = 1 + c_n\varsigma^{-n} + \cdots$, $n = 1, 2, \ldots$. Również dla twierdzenia 2 rozważa się przypadek graniczny $a = 1/2$. W p.3 podaje się pewne wnioski oraz twierdzenia 3 i 4 równoważne, odpowiednio, twierdzeniu 1 i 2. W zakończeniu formułuje się pewne uwagi oraz podaje się usterki drukarskie jakie znajdują się w pracach [1] i [2].

SUMMARY

In the papers [1],[2] the following results have been obtained. For fixed $a > 1/2$, $s = \alpha + i\beta$, $\alpha > 0$, $\beta \in (-\infty; \infty)$, $\kappa = 2a/\alpha$ we have

Theorem 2[1]. *Let* $f(z) = z + a_2z^2 + \cdots$, $f'(z) \neq 0$ *and* $g(z)$ *be regular in* $E = \{z : |z| < 1\}$ *and such that* $|zf'(z)/[f(z)g(z)] - as/\alpha| \le a|s|/\alpha$ *for* $z \in E$. *If the inequality* (A) *holds for all* $z \in E$ *then* f *is univalent in* E.

Theorem 2[2]. *Let* $g(\varsigma) = \varsigma + b_0 + b_1\varsigma^{-1} + \cdots$, $g'(\varsigma) \neq 0$ *and* $h(\varsigma) = 1 + c_2\varsigma^{-2} + \cdots$ *be regular in* $E^0 \setminus \{\infty\} = \{\varsigma : |\varsigma| > 1\} \setminus \{\infty\}$ *and such that* $|\varsigma\, g'(\varsigma)/[g(\varsigma)h(\varsigma)] - as/\alpha| \le a|s|/\alpha$ *for all* $\varsigma \in E^0$. *Then, if the inequality* (B) *holds for* $\varsigma \in E^0$ *and* $a \le \alpha$, *the function* g *is univalent in* E^0.

In this paper the above mentioned results are extended as follows. Theorem 2[1] holds in the limiting case $a = 1/2$ (Thm. 1) and Theorem 2[2] holds for $h(\varsigma) = 1 + c_n\varsigma^{-n} + \cdots$, $n = 1, 2, \ldots$ Also the limiting case $a = 1/2$ is considered. In Sect.3 some conclusions and Thms 3,4 equivalent to Thms 1,2, resp. are given. Finally some misprints appearing in [1] and [2] are corrected.