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Algebraic Operations on Sequences by Diagonal Transforms

Operacje algebraiczne na ciągach określone transformacjami diagonalnymi

Let $\mathbf{K}$ be the field of all real or complex numbers, and $\mathbf{Z}_+$ be the set of all positive integers. λ is called a sequence space over $\mathbf{K}$ if λ is a linear space of sequences in $\mathbf{K}$ under coordinatewise operations. Some special sequence spaces are given in the following.

Notations: (i) $\omega = \prod_{n=1}^{+\infty} \mathbf{K} = \{(x_n)_{n\in\mathbf{Z}_+} : x_n \in \mathbf{K} \text{ for } n \in \mathbf{Z}_+\}$ and $\varphi = \coprod_{n=1}^{+\infty} \mathbf{K} = \{(x_n)_{n\in\mathbf{Z}_+} : x_n \in \mathbf{K}$ for any $n \in \mathbf{Z}_+$ and $x_n = 0$ for all but finitely many n's$\}$.

(ii) $\ell^1 = \{(x_n)_{n\in\mathbf{Z}_+} \in \omega : \sum_{n=1}^{+\infty} |x_n| < +\infty\}$ and $\ell^\infty = \{(x_n)_{n\in\mathbf{Z}_+} : \sup_{n\in\mathbf{Z}_+} |x_n| < +\infty\}$. Also, $\ell^p = \{(x_n)_{n\in\mathbf{Z}_+} : \sum_{n=1}^{+\infty} |x_n|^p < +\infty\}$ for $1 < p < +\infty$.

We also have the following definitions.

Definition 1. Let $x, y \in \omega$ and $\emptyset \neq \mu \subseteq \omega$.

(i) x is called positive iff $x_n \geq 0$ for any $n \in \mathbf{Z}_+$.

(ii) Let x, y be positive, then we define $x \leq y$ iff $x_n \leq y_n$ for any $n \in \mathbf{Z}_+$.

(iii) Let $|x| = (|x_n|)_{n\in\mathbf{Z}_+}$ and $x^{-1} = (\alpha_n)_{n\in\mathbf{Z}_+}$, where $\alpha_n = 1$ for $x_n = 0$, and $\alpha_n = x_n^{-1}$, otherwise.

(iv) Let $xy = (x_n y_n)_{n\in\mathbf{Z}_+}$ and $x/y = xy^{-1}$.

(v) μ is called solid (or normal on p.405,[2]) if $x \in \mu$ and $y \in \omega$ with $|y| \leq |x|$ implies $y \in \mu$.

(vi) Let $\mu^s = \{y \in \omega : |y| \leq |x| \text{ for any } x \in \mu\}$, then μ^s is convex, balanced and solid in ω. Furthermore, $\mu \subseteq \mu^s$ and μ^s is the smallest, solid subset of ω containing μ. μ^s is called the solid hull of μ in ω.

(vii) Let $\mu^\times = \{y \in \omega : \langle x, y \rangle = \sum_{n=1}^{+\infty} x_n y_n$ converges absolutely for any $x \in \mu\}$, then $\mu^{\bullet}$ is called the (1st order) summability polar of μ in ω.

(viii) If λ is a sequence space, then $\lambda^{\bullet}$ is called the α–dual of λ in ω (p.405,[2]).

(ix) μ is called perfect if $\mu^{\bullet\bullet} = \mu$ (p.406,[2]).

If L is a linear space over $\mathbb{K}$ and L^* is the algebraic dual of L, then, for any $\emptyset \neq A \subseteq L$ and $\emptyset \neq A^* \subseteq L^*$, we define $A^{o*} = \{f \in L^* : |f(x)| \leq 1 \quad \text{for } x \in A\}$ and ${}^o(A^*) = \{x \in L : |f(x)| \leq 1 \quad \text{for } f \in A^*\}$. A^{o*} (or ${}^o(A^*)$) is called the polar of A (or A^*) in L^* (or L). The summability polar $\mu^{\bullet}$ has some properties similar to those of A^{o*} as we will see in the following (cf. p.245,[2]).

Lemma 1. *Let $\emptyset \neq \mu, \quad \mu_1, \mu_2 \subseteq \omega$ and $\alpha \neq 0 \in \mathbb{K}$.*

(i) *$\omega^{\bullet} = \varphi$ and $\varphi^{\bullet} = \omega$.*

(ii) *$\mu^{\bullet}$ is a sequence space, $(\alpha\mu)^{\bullet} = |\alpha|\mu^{\bullet}$ and $\mu \subseteq \mu^{\bullet\bullet}$.*

(iii) *If $\mu_1 \subseteq \mu_2$, then $\mu_2^{\bullet} \subseteq \mu_1^{\bullet}$.*

(iv) *$(\mu^{\bullet})^{\bullet\bullet} = \mu^{\bullet}$ and $\mu^{\bullet}$ is a perfect sequence space.*

(v) *$\mu^{\bullet\bullet}$ is the smallest, perfect sequence space containing μ.*

(vi) *If μ is perfect, then μ is a sequence space, $\varphi \subseteq \mu$ and μ is solid.*

In particular, ω, φ, and ℓ^p for $1 \leq p \leq +\infty$ are perfect (p.406,[2]).

Proof. (i) If $x \in \varphi$, then $< x, y >$ converges absolutely for any $y \in \omega$. Thus $x \in \omega^{\bullet}$ and $\varphi \subseteq \omega^{\bullet}$. Conversely, if $x \in \omega$ with infinitely many non–zero x_n's, then for each of these x_n's, we can find an $y_n \in \mathbb{K}$ with $|x_n y_n| > 1$. For the other n's, we let $y_n = 0$. Thus $y \in \omega$ and $\sum_{n=1}^{+\infty} |x_n y_n| = +\infty$. Hence $x \notin \omega^{\bullet}$. This implies $\omega^{\bullet} \subseteq \varphi$. Thus $\varphi = \omega^{\bullet}$. $\varphi^{\bullet} \subseteq \omega$ is clear. But $\omega \subseteq \varphi^{\bullet}$ as we have shown in the beginning. Hence $\varphi^{\bullet} = \omega$.

(v) We note $(\mu^{\bullet\bullet})^{\bullet\bullet} = (\mu^{\bullet})^{\bullet\bullet\bullet} = \mu^{\bullet\bullet}$ and $\mu^{\bullet\bullet}$ is perfect. If λ is a perfect sequence space with $\mu \subseteq \lambda$, then $\mu^{\bullet\bullet} \subseteq \lambda^{\bullet\bullet} = \lambda$.

(vi) If μ is perfect, then $\mu = \mu^{\bullet\bullet}$ and $\mu^{\bullet\bullet}$ is a sequence space. Since $\mu^{\bullet} \subseteq \omega$, we have $\varphi = \omega^{\bullet} \subseteq \mu^{\bullet\bullet} = \mu$ by (i). If $x \in \mu$ and $y \in \omega$ with $|y| \leq |x|$, then $\sum_{n=1}^{+\infty} |y_n z_n| \leq \sum_{n=1}^{+\infty} |x_n z_n| < +\infty$ for any $z \in \mu^{\bullet}$. Thus $y \in \mu^{\bullet\bullet} = \mu$ and μ is solid.

If $\varphi \subseteq \mu_\gamma \subseteq \omega$ for any $\gamma \in \Gamma$, then we define $\sum_{\gamma\in\Gamma} \mu_\gamma = \{ \sum_{\gamma\in\Gamma} x^{(\gamma)} : x^{(\gamma)} \in \mu_\gamma$ for any $\gamma \in \Gamma$, and $x^{(\gamma)} = 0 \in \omega$ for all but finitely many γ's$\}$. The condition $\varphi \subseteq \mu_\gamma$ implies $x^{(\gamma)} \in \mu_\gamma$ for any $\gamma \in \Gamma$.

Proposition 1. *Let $\emptyset \neq \mu_\gamma \subseteq \omega$ for any $\gamma \in \Gamma$.*

(i) $\bigcap_{\gamma\in\Gamma} (\mu_\gamma)^{\bullet} = (\bigcup_{\gamma\in\Gamma} \mu_\gamma)^{\bullet}$.

(ii) *If $\varphi \subseteq \mu_\gamma$ for any $\gamma \in \Gamma$, then $(\sum_{\gamma\in\Gamma} \mu_\gamma)^{\bullet} = \bigcap_{\gamma\in\Gamma} (\mu_\gamma)^{\bullet}$. If Γ is finite, then the condition of $\varphi \subseteq \mu_\gamma$ for any $\gamma \in \Gamma$ is not necessary.*

(iii) $\bigcup_{\gamma\in\Gamma} (\mu_\gamma)^{\bullet} \subseteq (\bigcap_{\gamma\in\Gamma} \mu_\gamma)^{\bullet}$.

(iv) *If μ_γ is perfect for any $\gamma \in \Gamma$, then $(\bigcap_{\gamma\in\Gamma} \mu_\gamma)^{\bullet} = (\sum_{\gamma\in\Gamma} (\mu_\gamma)^{\bullet})^{\bullet\bullet}$, and $\bigcap_{\gamma\in\Gamma} \mu_\gamma$ is perfect.*

Proof. (ii) If $y \in (\sum_{\gamma\in\Gamma} \mu_\gamma)^*$, then $< x, y >$ converges absolutely for any $x \in \sum_{\gamma\in\Gamma} \mu_\gamma$. In particular, $< x^{(\gamma)}, y >$ converges absolutely for any $x^{(\gamma)} \in \mu_\gamma$ and $\gamma \in \Gamma$. Hence $y \in (\mu_\gamma)^*$ for any $\gamma \in \Gamma$ implies $y \in \bigcap_{\gamma\in\Gamma} (\mu_\gamma)^*$. Thus $(\sum_{\gamma\in\Gamma} \mu_\gamma)^* \subseteq \subseteq \bigcap_{\gamma\in\Gamma} (\mu_\gamma)^*$. Conversely, if $y \in \bigcap_{\gamma\in\Gamma} (\mu_\gamma)^*$, then $< x^{(\gamma)}, y >$ converges absolutely for any $x^{(\gamma)} \in \mu_\gamma$ and $\gamma \in \Gamma$. If $x \in \sum_{\gamma\in\Gamma} \mu_\gamma$, then $x = \sum_{j=1}^{k} x^{(\gamma_j)}$, where $x^{(\gamma_j)} = = (x_{\gamma_j,n})_{n\in\mathbb{Z}_+} \in \mu_{\gamma_j}$ for $j = 1, 2, \ldots, k$. Thus $\sum_{n=1}^{+\infty} |x_n y_n| = \sum_{n=1}^{+\infty} |(\sum_{j=1}^{k} x_{\gamma_j,n}) y_n| = = \sum_{n=1}^{+\infty} | \sum_{j=1}^{k} x_{\gamma_j,n} y_n| \le \sum_{n=1}^{+\infty} \sum_{j=1}^{k} |x_{\gamma_j,n} y_n| = \sum_{j=1}^{k} \sum_{n=1}^{+\infty} |x_{\gamma_j,n} y_n|$. Since $< x^{(\gamma_j)}, y >$ converges absolutely, we have $\sum_{n=1}^{+\infty} |x_{\gamma_j,n} y_n| < +\infty$ for $j = 1, 2, \ldots, k$. Thus $\sum_{n=1}^{+\infty} |x_n y_n| < < +\infty$ for any $x \in \sum_{\gamma\in\Gamma} \mu_\gamma$ and $y \in (\sum_{\gamma\in\Gamma} \mu_\gamma)^*$. This implies $\bigcap_{\gamma\in\Gamma} (\mu_\gamma)^* \subseteq (\sum_{\gamma\in\Gamma} \mu_\gamma)^*$. The given identity is proved.

(iii) This follows directly from Lemma 1(iii).

(iv) By (ii), $(\sum_{\gamma\in\Gamma} (\mu_\gamma)^*)^* = \bigcap_{\gamma\in\Gamma} (\mu_\gamma)^{**} = \bigcap_{\gamma\in\Gamma} \mu_\gamma$. Hence $(\bigcap_{\gamma\in\Gamma} \mu_\gamma)^{**} = = ((\sum_{\gamma\in\Gamma} (\mu_\gamma)^*)^*)^{**} = (\sum_{\gamma\in\Gamma} (\mu_\gamma)^*)^*$ by Lemma 1(iv). Thus $(\bigcap_{\gamma\in\Gamma} \mu_\gamma)^{**} = \bigcap_{\gamma\in\Gamma} \mu_\gamma$ and $\bigcap_{\gamma\in\Gamma} \mu_\gamma$ is perfect.

The simplest, algebraic operation on sequences is the following.

Definition 2. Let $\{n_k : k = 1, 2, \ldots\}$ be a strictly increasing sequence of positive integers. If $x \in \omega$, then we define $x^d = (x_{n_k})_{k\in\mathbb{Z}_+}$, which is called the sectional sequence of x associated with $\{n_k : k = 1, 2, \ldots\}$ (p. 410,[2]), where "d" means deletion for the obvious reason. If $\emptyset \neq \mu \subseteq \omega$, then we define $\mu^d = \{x^d : x \in \mu\}$.

Lemma 2. *For any $x \in \omega$ and any strictly increasing sequence $\{n_k : k = 1, 2, \ldots\}$ in $\mathbb{Z}_+$, there is an $y \in \omega$ with $y_{n_k} = x_k$ for $k \in \mathbb{Z}_+$ and $y_n = 0$, otherwise.*

Proof. Let $n_0 = 0$. For any $k \in \mathbb{Z}_+$, we put $n_k - n_{k-1} - 1$ 0's right before the coordinate x_k. Thus at the n_k–th coordinate, we add $\sum_{j=1}^{k} (n_j - n_{j-1} - 1) = n_k - k$ 0's. Let $y = (\underbrace{0, 0, \ldots, 0}_{n_1 - 1}, x_1, \underbrace{0, 0, \ldots, 0}_{n_2 - 1}, x_2, \ldots)$, then the n_k–th coordinate of y is x_k for any $k \in \mathbb{Z}_+$, and $y^d = x$.

Proposition 2. *Let $\emptyset \neq \mu$, $\mu_1, \mu_2 \subseteq \omega$ and $\{n_k : k = 1, 2, \ldots\}$ be a strictly increasing sequence in $\mathbb{Z}_+$.*

(i) *If $\mu_1 \subseteq \mu_2$, then $\mu_1^d \subseteq \mu_2^d$.*

(ii) *$\varphi^d = \varphi$. Thus if $\varphi \subseteq \mu$, then $\varphi \subseteq \mu^d$.*

(iii) *$(\mu^d)^* = (\mu^*)^d$.*

(iv) *If μ is solid (or perfect), so is μ^d.*

Proof. (ii) If $x \in \varphi^d$, then $x = y^d$ for some $y \in \varphi$. Since all but finitely many coordinates of y are 0, so are the coordinates of x. Thus $x \in \varphi$. This proves $\varphi^d \subseteq \varphi$. Conversely, if $x \in \varphi$, then $x = y^d$ for some $y \in \omega$ in Lemma 2. Since $y \in \varphi$, we have $x \in \varphi^d$. Thus $\varphi \subseteq \varphi^d$. The given identity is proved. Hence if $\varphi \subseteq \mu$, then $\varphi = \varphi^d \subseteq \mu^d$ by (i).

(iii) If λ is a sequence space, then so is λ^d. If $y \in (\mu^d)^*$, then $< x^d, y >$ converges absolutely for any $x \in \mu$. But $y = z^d$ for some $z \in \omega$ in Lemma 2. We can easily check $< x, z >=< x^d, y >$ for any $x \in \mu$. Hence $z \in \mu^*$ and $y \in (\mu^*)^d$. This proves $(\mu^d)^* \subseteq (\mu^*)^d$. Conversely, if $y \in (\mu^*)^d$, then $y = z^d$ for some $z \in \mu^*$. Since $z \in \mu^*$, $< x, z >$ converges absolutely for any $x \in \mu$. Thus $< x^d, y >$ converges absolutely for any $x \in \mu$, and $y \in (\mu^d)^*$. This proves $(\mu^*)^d \subseteq (\mu^d)^*$.

(iv) If μ is solid, $x \in \mu^d$ and $y \in \omega$ with $|y| \leq |x|$, then $x = z^d$ for some $z \in \mu$. But $y = w^d$ for some $w \in \omega$ in Lemma 2. We can easily check $|w| \leq |z|$. Since μ is solid, we have $w \in \mu$ and $y \in \mu^d$. Thus μ^d is solid. If μ is perfect, then $(\mu^d)^{**} = ((\mu^*)^d)^* = (\mu^{**})^d = \mu^d$ by the repeated applications of (iii).

We now discuss the main notion of this paper.

Definition 3. If $a = (a_n)_{n\in\mathbb{Z}_+} \in \omega$ and $x = (x_n)_{n\in\mathbb{Z}_+} \in \omega$, then $ax = (a_n x_n)_{n\in\mathbb{Z}_+}$ is called the diagonal transform of x by a for the following reason:

$$\text{If}\quad a = \begin{pmatrix} a_1 & & 0 \\ & a_2 & \\ 0 & & \ddots \end{pmatrix}, \qquad \text{then} \qquad ax = \begin{pmatrix} a_1 & & 0 \\ & a_2 & \\ 0 & & \ddots \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix}.$$

If $\emptyset \neq \mu \subseteq \omega$, then we define $a\mu = \{ax : x \in \mu\}$ which is called the diagonal transform of μ by a.

Theorem 1. *Let $a = (a_n)_{n\in\mathbb{Z}_+} \in \omega$ and $\emptyset \neq \mu \subseteq \omega$.*
(i) $a^{-1}\mu^* \subseteq (a\mu)^*$.
(ii) $(a\varphi)^* = a^{-1}\omega = \omega$.
(iii) *If $a_n \neq 0$ for all but finitely many n's, then $(a\mu)^* = a^{-1}\mu^*$.*

Proof. (i) If $y \in \mu^*$ and $x \in \mu$, then $a^{-1}y = (\alpha_n y_n)_{n\in\mathbb{Z}_+}$ and $\sum_{n=1}^{+\infty} |a_n x_n \alpha_n y_n| = \sum_{a_n \neq 0} |x_n y_n| < +\infty$. This is true for any $x \in \mu$. Thus $a^{-1}y \in (a\mu)^*$. This proves $a^{-1}\mu^* \subseteq (a\mu)^*$.

(ii) $(a\varphi)^* \subseteq \omega$ and $a^{-1}\omega \subseteq \omega$ are clear. If $x \in \omega$, then we claim $x \in a^{-1}\omega$ by considering the following cases, where $a^{-1} = (\alpha_n)_{n\in\mathbb{Z}_+}$.

(1) $a_n = 0$: Thus $\alpha_n = 1$. Let $y_n = x_n$.

(2) $a_n \neq 0$: Thus $\alpha_n = 1/a_n$. If $x_n = 0$, let $y_n = 0$ otherwise, let $y_n = a_n x_n$. For both cases, we have $\alpha_n y_n = x_n$. Thus $x \in a^{-1}\omega$, and we proved $\omega = a^{-1}\omega$. This proves $(a\varphi)^* \supseteq a^{-1}\varphi^* = a^{-1}\omega = \omega$.

(iii) It suffices to prove $(a\mu)^* \subseteq a^{-1}\mu^*$. Let $a_{n_j} = 0$ for $j = 1, 2, \ldots, k$. Let $y \in (a\mu)^*$, then we let $z_{n_j} = y_{n_j}$ for $j = 1, 2, \ldots, k$ and $z_n = a_n y_n$, otherwise. Hence $a^{-1}z = y$. If $x \in \mu$, then (for $y \in (a\mu)^*$ and $x \in \mu$)

$$\sum_{n=1}^{+\infty}|x_n z_n| = \sum_{j=1}^{k}|x_{n_j} z_{n_j}| + \sum_{n\neq n_j}|x_n z_n| = \sum_{j=1}^{k}|x_{n_j} y_{n_j}| + \sum_{n\neq n_j}|a_n x_n y_n| < +\infty.$$

This is true for any $x \in \mu$. Thus $z \in \mu^*$ and $(a\mu)^* \subseteq a^{-1}\mu^*$. The identity is proved.

Remark . We consider why $(a\mu)^* = a^{-1}\mu^*$ is not true for any $a \in \omega$: If $a_{n_k} = 0$ for any $k \in \mathbf{Z}_+$, then the infinite sum $\sum_{k=1}^{+\infty}|x_{n_k} y_{n_k}| < +\infty$ may not be true in the proof of the set–containment $(a\mu)^* \subseteq a^{-1}\mu^*$.

Diagonal transforms between two subsets of ω are more applicable, and we discuss them.

Definition 4. Let $\emptyset \neq \mu_1, \mu_2 \subseteq \omega$, then we define $D(\mu_1,\mu_2) = \{a \in \omega : a\mu_1 \subseteq \mu_2\}$. In other words, $D(\mu_1,\mu_2)$ is the set of all diagonal transforms from μ_1 to μ_2 (p.68, [1]).

Lemma 4. *Let* $\emptyset \neq \mu_1, \mu_2 \subseteq \omega$.

(i) $D(\mu_1,\mu_2) \subseteq (\mu_1(\mu_2^*))^*$.

(ii) *If* μ_2 *is solid, so is* $D(\mu_1,\mu_2)$.

(iii) *If* μ_2 *is perfect, then so is* $D(\mu_1,\mu_2)$, *and* $D(\mu_1,\mu_2) = (\mu_1(\mu_2^*))^*$.

Proof. (iii) is Proposition 1.2, [1], and (i) can be proved similarly. For (ii), we let $a \in D(\mu_1,\mu_2)$ and $b \in \omega$ with $|b| \leq |a|$, then $ax \in \mu_2$ for any $x \in \mu_1$. But $|b_n x_n| \leq |a_n x_n|$ for any $n \in \mathbf{Z}_+$. Thus $bx \in \mu_2$ and $b \in D(\mu_1,\mu_2)$. Hence $D(\mu_1,\mu_2)$ is solid.

Some special subsets μ_2 of ω will make $D(\mu_1,\mu_2)$ more applicable. For example, if $\emptyset \neq \mu \subseteq \omega$, then $D(\mu,\ell^1) = \{a \in \omega : ax \in \ell^1$ for any $x \in \mu\} = \{a \in \omega : \sum_{n=1}^{+\infty} a_n x_n$ converges absolutely for any $x \in \mu\} = \mu^*$ and $D(\mu^*,\ell^1) = D(D(\mu,\ell^1),\ell^1)$. Another example is in the following.

Definition 5. If $a \in \omega$, then $D(a,\ell^1) = \{x \in \omega : < a, x >$ converges absolutely$\} = \{a\}^* = \lambda_a$ which is called the dilation operation of a. Thus λ_a is a perfect sequence space. We note $(\lambda_a)^* = \{a\}^{**}$.

Lemma 5. *Let* $a, b \in \omega$ *and* $\emptyset \neq \mu \subseteq \omega$.

(i) $b \in \lambda_a$ *iff* $a \in \lambda_b$.

(ii) *If* $|a| \leq |b|$, *then* $\lambda_b \subseteq \lambda_a$ *and* $(\lambda_a)^* \subseteq (\lambda_b)^*$.

(iii) $\alpha a \in (\lambda_a)^*$ *for any* $\alpha \in \mathbf{K}$ *and* $\lambda_a, (\lambda_a)^*$ *are perfect sequence spaces.*

(iv) $\mu^* = D(\mu,\ell^1) = \bigcap_{a\in\mu} \lambda_a$.

Proof. (i) We note the following equivalences: $b \in \lambda_a$ iff $\sum_{n=1}^{+\infty}|a_n b_n| < +\infty$ iff $a \in \lambda_b$.

(iii) If $x \in \lambda_a$, then $\sum_{n=1}^{+\infty}|(\alpha a_n)x_n| = |\alpha|\sum_{n=1}^{+\infty}|a_n x_n| < +\infty$. Thus $\alpha a \in (\lambda_a)^*$.

That λ_a is perfect is a consequence of $\lambda_a = \{a\}^\circ$ or can be derived from this result: If $z \in (\lambda_a)^{\circ\circ}$, then $\sum_{n=1}^{+\infty} |y_n z_n| < +\infty$ for any $y \in (\lambda_a)^\circ$. In particular, $\sum_{n=1}^{+\infty} |a_n z_n| < +\infty$ implies $z \in \lambda_a$. This proves $(\lambda_a)^{\circ\circ} = \lambda_a$.

For any $\emptyset \neq \mu \subseteq \omega$, μ° and $\mu^{\circ\circ}$ can be represented by λ_c for c in some suitable subsets of ω.

Theorem 2. *Let $\emptyset \neq \mu \subseteq \omega$.*

(i) $\mu^\circ \subseteq \bigcap_{b\in\mu^\circ} \lambda_b = D(\mu^\circ, \ell^1) = \mu^{\circ\circ}$ *and* $\bigcap_{a\in\mu} \lambda_a = D(\mu, \ell^1) = \mu^\circ$.

(ii) $\mu^\circ \subseteq \bigcup_{a\in\mu} (\lambda_a)^\circ = \bigcup_{a\in\mu} D(a, \ell^1)^\circ \subseteq \mu^{\circ\circ}$ *and* $\bigcup_{b\in\mu^\circ} (\lambda_b)^\circ = \bigcup_{b\in\mu^\circ} D(b, \ell^1)^\circ = \mu^\circ$.

(iii) *If $a \in \omega$, then $\lambda_{a^d} = (\lambda_a)^d$, i.e. $D(a^d, \ell^1) = (D(a, \ell^1))^d$, where a^d is the sectional sequence of a associated with the strictly increasing sequence $\{n_k : k = 1, 2, \ldots\}$ in $\mathbb{Z}_+$.*

Proof. (i) We note $\mu^{\circ\circ} = D(\mu^\circ, \ell^1) = \bigcap_{b\in\mu^\circ} D(b, \ell^1) = \bigcap_{b\in\mu^\circ} \lambda_b$. Since $\mu^{\circ\circ}$ is perfect, $\mu^{\circ\circ}$ is solid and $\mu \subseteq \mu^{\circ\circ}$. Hence $\mu^\circ \subseteq \mu^{\circ\circ}$. Also, $\mu^\circ = D(\mu, \ell^1) = \bigcap_{a\in\mu} D(a, \ell^1) = \bigcap_{a\in\mu} \lambda_a$.

(ii) If $x \in \mu^\circ$, then $|x| \leq |a|$ for some $a \in \mu$. Since $a \in (\lambda_a)^\circ$ and $(\lambda_a)^\circ$ is perfect, we have $x \in (\lambda_a)^\circ$. This proves $\mu^\circ \subseteq \bigcup_{a\in\mu} (\lambda_a)^\circ$. We note $\bigcup_{a\in\mu} (\lambda_a)^\circ \subseteq (\bigcap_{a\in\mu} \lambda_a)^\circ = (\mu^\circ)^\circ = \mu^{\circ\circ}$ by Proposition 1(iii) and (i). Also, $\bigcup_{b\in\mu^\circ} (\lambda_b)^\circ \subseteq (\bigcap_{b\in\mu^\circ} \lambda_b)^\circ = (\bigcap_{b\in\mu^\circ} D(b, \ell^1))^\circ = (D(\mu^\circ, \ell^1))^\circ = ((\mu^\circ)^\circ)^\circ = \mu^\circ$ by Lemma 1(iii) and the identity $\mu^\circ = D(\mu, \ell^1) = \bigcap_{a\in\mu} \lambda_a$. Conversely, if $b \in \mu^\circ$, then $b \in (\lambda_b)^\circ$. Hence $\mu^\circ \subseteq \bigcup_{b\in\mu^\circ} (\lambda_b)^\circ$. This proves $\bigcup_{b\in\mu^\circ} (\lambda_b)^\circ = \mu^\circ$.

(iii) If $y \in \lambda_{a^d}$, then $\sum_{k=1}^{+\infty} |a_{n_k} y_k| < +\infty$. But $y = x^d$ for some $x \in \omega$ in Lemma 2. We can easily check $\sum_{n=1}^{+\infty} |a_n x_n| = \sum_{k=1}^{+\infty} |a_{n_k} y_k| < +\infty$. Hence $x \in \lambda_a$ and $y \in (\lambda_a)^d$. This implies $\lambda_{a^d} \subseteq (\lambda_a)^d$. Conversely, if $y \in (\lambda_a)^d$, then $y = x^d$ for some $x \in \lambda_a$. Thus $\sum_{k=1}^{+\infty} |a_{n_k} y_k| \leq \sum_{n=1}^{+\infty} |a_n x_n| < +\infty$ implies $y \in \lambda_{a^d}$. This implies $(\lambda_a)^d \subseteq \lambda_{a^d}$. Hence $\lambda_{a^d} = (\lambda_a)^d$.

We have the following characterizations of perfect sequence spaces.

Corollary 1. A sequence space λ is perfect iff $\lambda = \bigcap_{b\in\lambda^\circ} \lambda_b = D(\lambda^\circ, \ell^1)$.

We now consider the possibilities of $a \in \omega$ and its associated sequence space λ_a.

Theorem 3. *Let $a \in \omega$ be given.*

(i) *If $a_n \neq 0$ for all but finitely many n's, then $\lambda_a = \{a\}^\circ = D(a, \ell^1) = a^{-1}\ell^1$ and $(\lambda_a)^\circ = \{a\}^{\circ\circ} = b\ell^\infty$, where $a^{-1} = (\alpha_n)_{n\in\mathbb{Z}_+}$ and $b = (b_n)_{n\in\mathbb{Z}_+}$*

$\alpha_n = b_n = 1$ for $a_n = 0$, and $\alpha_n = a_n^{-1}$, $b_n = a_n$, otherwise.

(ii) If $a_n = 0$ for all but finitely many n's, then $\lambda_a = \{a\}^* = D(a, \ell^1) = \omega$ and $(\lambda_a)^* = \varphi$.

(iii) If $a_{m_k} \neq 0$ and $a_{n_k} = 0$, where $m_k < m_{k+1}$ and $n_k < n_{k+1}$ for any $k \in \mathbb{Z}_+$, then the sectional sequences of λ_a associated with $\{m_k : k = 1,2,\ldots\}$ and $\{n_k : k = 1,2,\ldots\}$ are $((a_{m_k}^{-1})_{k\in\mathbb{Z}_+})\ell^1$ and ω, respectively. Thus the sectional sequence spaces of $(\lambda_a)^*$ associated with $\{m_k : k = 1,2,\ldots\}$ and $\{n_k : k = 1,2,\ldots\}$ are $(a_{m_k})_{k\in\mathbb{Z}_+}\ell^\infty$ and φ respectively.

Proof. (i) Let $a_{n_j} = 0$ for $j = 1,2,\ldots,k$. If $x \in \lambda_a$, then we let $y_{n_j} = x_{n_j}$ for $j = 1,2,\ldots,k$ and $y_n = a_n x_n$, otherwise. Hence $x = a^{-1}y$ and $\sum_{n=1}^{+\infty} |y_n| = = \sum_{j=1}^{k} |y_{n_j}| + \sum_{n\neq n_j} |y_n| = \sum_{j=1}^{k} |x_{n_j}| + \sum_{n\neq n_j} |a_n x_n| < +\infty$. This implies $y \in \ell^1$. Hence $\lambda_a \subseteq a^{-1}\ell^1$. Conversely, if $y \in \ell^1$, then $a^{-1}y = (\alpha_n y_n)_{n\in\mathbb{Z}_+}$ and $\sum_{n=1}^{+\infty} |a_n\alpha_n y_n| = = \sum_{j=1}^{k} |a_{n_j}\alpha_{n_j} y_{n_j}| + \sum_{n\neq n_j} |a_n\alpha_n y_n| = \sum_{n\neq n_j} |y_n| < +\infty$. This implies $a^{-1}y \in \lambda_a$. Hence $a^{-1}\ell^1 \subseteq \lambda_a$. The identity $\lambda_a = a^{-1}\ell^1$ is proved. Since $\alpha_n \neq 0$ for any $n \in \mathbb{Z}_+$, we have $(\lambda_a)^* = (a^{-1}\ell^1)^* = b(\ell^1)^* = b\ell^\infty$ by Theorem 1(iii).

(ii) For any $x \in \omega$, $< a, x >$ converges absolutely, i.e. $x \in \lambda_a$. Thus $\lambda_a = \omega$ and $(\lambda_a)^* = \varphi$.

(iii) By Theorem 2(iii), $\lambda_{(a_{m_k})_{k\in\mathbb{Z}_+}} = (\lambda_a)^d$ which is thhe sectional sequence space of λ_a associated with $\{m_k : k = 1,2,\ldots\}$. Thus $(\lambda_a)^d = (a_{m_k}^{-1})_{k\in\mathbb{Z}_+}\ell^1$ by (i). Also, $\omega = \lambda_{(a_{n_k})_{k\in\mathbb{Z}_+}} = (\lambda_a)^{d'}$ which is the sectional sequence space of λ_a associated with $\{n_k : k = 1,2,\ldots\}$. Thus $((\lambda_a)^*)^d = ((\lambda_a)^d)^* = (\lambda_{(a_{m_k})_{k\in\mathbb{Z}_+}})^* = (a_{m_k})_{k\in\mathbb{Z}_+}\ell^\infty$ and $((\lambda_a)^*)^{d'} = ((\lambda_a)^{d'})^* = \omega^* = \varphi$.

Corollary 1. *For any $a \in \omega$, $a^{-1}\ell^1 \subseteq \lambda_a = D(a, \ell^1)$.*

Proof. We consider the following cases.

(i) $a_n \neq 0$ for all but finitely many n's: Thus $\lambda_a = a^{-1}\ell^1$ by Theorem 3(i).

(ii) $a_n = 0$ for all but finitely many n's: Thus $\lambda_a = \omega$ by Theorem 3(ii).

(iii) $a_{m_k} \neq 0$ and $a_{n_k} = 0$, where $m_k < m_{k+1}$ and $n_k < n_{k+1}$ for any $k \in \mathbb{Z}_+$: If $y \in \ell^1$, then $a^{-1}y = (\alpha_n y_n)_{n\in\mathbb{Z}_+}$ and $\sum_{n=1}^{+\infty} |a_n\alpha_n y_n| = \sum_{k=1}^{+\infty} |a_{m_k}\alpha_{m_k} y_{m_k}| + + \sum_{k=1}^{+\infty} |a_{n_k}\alpha_{n_k} y_{n_k}| = \sum_{k=1}^{+\infty} |y_{m_k}| < +\infty$ implies $a^{-1}y \in \lambda_a$. Thus $a^{-1}\ell^1 \subseteq \lambda_a$.

Corollary 2. *Let $a \in \omega$ satisfy the conditions of* Theorem 3(iii), *then $x \in \lambda_a$ iff $(x_{m_k})_{k\in\mathbb{Z}_+} \in (a_{m_k}^{-1})_{k\in\mathbb{Z}_+}\ell^1$ and $(x_{n_k})_{k\in\mathbb{Z}_+} \in \omega$. Also, $y \in (\lambda_a)^*$ iff $(y_{m_k})_{k\in\mathbb{Z}_+} \in \in (a_{m_k})_{k\in\mathbb{Z}_+}\ell^\infty$ and $(y_{n_k})_{k\in\mathbb{Z}_+} \in \varphi$.*

By the applications of Theorem 3 and the polar properties of μ^*, we can determine μ^* for some $\mu \subseteq \omega$. The following are two simple examples.

Corollary 3. *Let $\emptyset \neq \mu \subseteq \omega$.*

(i) *If $a_n \neq 0$ for all but finitely many n's for any $a \in \mu$, then $\mu^* = D(a, \ell^1) = = \bigcap_{a\in\mu} a^{-1}\ell^1$.*

(ii) *If $\mu \subseteq \varphi$, then $\mu^* = \omega$ and $\mu^{**} = \varphi$.*

Proof. (i) We note $\mu^* = (\bigcup_{a\in\mu} \{a\})^* = \bigcap_{a\in\mu} \{a\}^* = \bigcap_{a\in\mu} a^{-1}\ell^1$ by Proposition 1(i) and Theorem 3(i).

(ii) We note $\omega = \varphi^* \subseteq \mu^*$ and $\mu^{**} = \omega^* = \varphi$.

The polar properties of μ^* can simplify many computations on sequences. Another example is the following: If $a^{(j)} \in \omega$ for $j = 1, 2, \ldots, k$ and $a = \sum_{j=1}^{k} a^{(j)}$, then $\lambda_a = \{a\}^* = (\sum_{j=1}^{k} a^{(j)})^* = \bigcap_{j=1}^{n} \{a^{(j)}\}^* = \bigcap_{j=1}^{n} \lambda_{a^{(j)}}$ by Proposition 1(ii).

At the end of this paper, we will find $D(\ell^p, \ell^q)$ and $D(\ell^q, \ell^p)$ for $1 \leq p, q \leq +\infty$ (cf. Example 1.6,[1]).

Lemma 6. *For $1 \leq p, q \leq +\infty$ with $r = \frac{pq}{p+q} \geq 1$, we have $\ell^p\ell^q = \ell^r$.*

Proof. For any $t \geq 1$ and $x \in \omega$, we have $|x|^t = (|x_n|^t)_{n\in\mathbb{Z}_+}$. If $x \in \ell^p$ and $y \in \ell^q$, then $|x|^p \in \ell^1$ and $|y|^q \in \ell^1$. We note $(|xy|)^r = |x|^r|y|^r$ and $|x|^r \in \ell^{q/(p+q)}$ and $|y|^r \in \ell^{p/(p+q)}$. But $\frac{p}{p+q} + \frac{q}{p+q} = 1$ implies $|xy|^r = |x|^r|y|^r \in \ell^1$. Hence $xy \in \ell^r$ and we proved $\ell^p\ell^q \subseteq \ell^r$. Conversely, if $z \in \ell^r$, then $|z|^r \in \ell^1$ implies $|z|^{q/(p+q)} \in \ell^p$ and $|z|^{p/(p+q)} \in \ell^q$. Hence $|z| = |z|^{q/(p+q)}|z|^{p/(p+q)} \in \ell^p\ell^q$ and $z \in \ell^p\ell^q$. This implies $\ell^r \subseteq \ell^p\ell^q$.

The identity $\ell^p\ell^q = \ell^r$ is certainly not correct without the condition $r \geq 1$. This can be seen from the proof of Lemma 6 (i.e. $|x|^r$ is a $\frac{1}{r}$–th root of $|x|$) or can be disproved by the following example.

Proposition 3. (i) *$D(\ell^p, \ell^\infty) = \ell^\infty$ for any $1 \leq p \leq +\infty$.*

(ii) *$D(\ell^p, \ell^p) = \ell^\infty$ for any $1 \leq p \leq +\infty$.*

(iii) *$\ell^1\ell^1 \subseteq \ell^1$ and $\ell^1\ell^1$ is not perfect* (Remark (c), p.68,[1]).

Proof. (i) This can be proved by the similar arguments in Lemma 1.4,[1].

(ii) By Proposition 1.2,[1] we have $D(\ell^1, \ell^1) = (\ell^1(\ell^1)^*)^* = (\ell^1\ell^\infty)^* = (\ell^1)^* = = \ell^\infty$ and $D(\ell^\infty, \ell^\infty) = (\ell^\infty(\ell^\infty)^*)^* = (\ell^1\ell^\infty)^* = (\ell^1)^* = \ell^\infty$, where we use the fact $\lambda\ell^\infty = \lambda$ for any solid sequence space λ. For $1 < p < +\infty$, we have $D(\ell^p, \ell^p) = \{a \in \omega : \sum_{n=1}^{+\infty} |a_n x_n|^p < +\infty \quad \text{for any } x \in \ell^p\} = \{a \in \omega : \sum_{n=1}^{+\infty} |a_n|^p|x_n|^p < < +\infty \quad \text{for any } |x|^p \in \ell^1\} = \{a \in \omega : \sum_{n=1}^{+\infty} |a_n|^p|x_n| < +\infty \quad \text{for any } x \in \ell^1\}$. Thus $a \in D(\ell^p, \ell^p)$ iff $|a|^p \in (\ell^1)^* = \ell^\infty$ iff $a \in \ell^\infty$, i.e. $D(\ell^p, \ell^p) = \ell^\infty$.

(iii) We note $(\ell^1\ell^1)^* = (\ell^1(\ell^\infty)^*)^* = D(\ell^1, \ell^\infty) = \ell^\infty$ by Proposition 1.2 and Lemma 1.4, [1]. Thus $(\ell^1\ell^1)^{**} = \ell^1$ and $\ell^1\ell^1 \subseteq \ell^1$. This can also be seen by $\ell^1\ell^1 \subseteq \subseteq \ell^\infty\ell^1 = \ell^1$.

If Lemma 6 is true for any $1 \leq p, q < +\infty$, then $\ell^1 \ell^1 = \ell^{1/2}$. This is a contradiction.

Theorem 4. *Let* $1 \leq p \leq q \leq +\infty$.

(i) *If* $p = q = 1$, *then* $D(\ell^p, \ell^q) = D(\ell^q, \ell^p) = \ell^\infty$.

(ii) *If* $p = q = +\infty$, *then* $D(\ell^p, \ell^q) = D(\ell^q, \ell^p) = \ell^\infty$.

(iii) *If* $1 < p = q < +\infty$, *then* $D(\ell^p, \ell^q) = D(\ell^q, \ell^p) = \ell^\infty$.

(iv) *If* $p = 1$ *and* $q = +\infty$, *then* $D(\ell^p, \ell^q) = \ell^\infty$ *and* $D(\ell^q, \ell^p) = \ell^1$.

(v) *If* $p = 1$ *and* $q < +\infty$, *then* $D(\ell^p, \ell^q) = \ell^\infty$ *and* $D(\ell^q, \ell^p) = \ell^{q/(q-1)}$.

(vi) *If* $p > 1$ *and* $q = +\infty$, *then* $D(\ell^p, \ell^q) = \ell^\infty$ *and* $D(\ell^q, \ell^p) = \ell^p$.

(vii) *If* $p > 1$ *and* $q < +\infty$, *then* $D(\ell^q, \ell^p) = \ell^{pq/(q-p)}$ *and* $D(\ell^p, \ell^q)$ *is undecided.*

Proof. If $1 \leq r \leq +\infty$, then we let r' be the conjugate component of r, i.e. $\frac{1}{r} + \frac{1}{r'} = 1$. (i), (ii) and (iii) have been proved in Proposition 3.

(iv) We note $D(\ell^p, \ell^q) = D(\ell^1, \ell^\infty) = \ell^\infty$ by Lemma 1.4, [1], and $D(\ell^q, \ell^p) = = D(\ell^\infty, \ell^1) = (\ell^\infty)^* = \ell^1$.

(v) We note $D(\ell^p, \ell^q) = D(\ell^1, \ell^q) = (\ell^1(\ell^q)^*)^* = (\ell^1\ell^{q'})^* = ((\ell^\infty)^*\ell^{q'})^* = D(\ell^{q'}, \ell^\infty) = \ell^\infty$ by Proposition 3(i) and $D(\ell^q, \ell^p) = D(\ell^q, \ell^1) = (\ell^q)^* = \ell^{q/(q-1)}$.

(vi) We note $D(\ell^p, \ell^q) = D(\ell^p, \ell^\infty) = \ell^\infty$ and $D(\ell^q, \ell^p) = D(\ell^\infty, \ell^p) = = (\ell^\infty(\ell^p)^*)^* = ((\ell^{p'})\ell^\infty)^* = (\ell^{p'})^* = \ell^p$.

(vii) We note $D(\ell^q, \ell^p) = \ell^{pq/(q-p)}$ by Example 1.6, [1].

We also have the following simple consequences.

Corollary 1. *If* $1 \leq p, q \leq +\infty$, *then* $D(\ell^p, \ell^q) = D(\ell^{q'}, \ell^{p'})$.

Proof. This follows that $D(\ell^p, \ell^q) = (\ell^p(\ell^q)^*)^* = (\ell^{q'}(\ell^{p'})^*)^* = D(\ell^{q'}, \ell^{p'})$.

Corollary 2. $D(\ell^1, \ell^p) = \ell^\infty$ *and* $D(\ell^p, \ell^1) = \ell^{p'}$ *for* $1 \leq p \leq +\infty$.

Corollary 3. $D(\ell^\infty, \ell^p) = \ell^p$ *and* $D(\ell^p, \ell^\infty) = \ell^\infty$ *for* $1 \leq p \leq +\infty$.

If $1 < p < q < +\infty$, then $1 < q' < p' < +\infty$. Hence $D(\ell^{p'}, \ell^{q'})$ can be obtained by Example 1.6, [1]. In other words, the applications of Corollary 1 will simplify some results in Theorem 4. In §8 of the author's paper "Several basic theorems on locally convex spaces and their duality", various topologies on sequence spaces are briefly discussed.

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STRESZCZENIE

W pracy tej wprowadzono pojęcie zbioru polarnego ze względu na sumowalność, który jest analogonem zbioru polarnego w przestrzeniach lokalnie wypukłych. Wprowadzenie takiego zbioru pozwala uprościć niektóre działania na ciągach.

SUMMARY

In this paper polar sets w.r.t. summability are introduced which are counterparts of polar sets in locally convex spaces. This idea enables us to simplify some operations on sequences.