ANNALES UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA

LUBLIN-POLONIA

VOL. XLI, 2 SECTIO A 1987

Instytut Matematyki
Uniwersytet Marii Curie-Skłodowskiej

W. CIEŚLAK, J. ZAJĄC

The Visotropy Geometry of Curves

Geometria visotropowa krzywych

1. Introduction. In the paper [1] a complemented group of the isotropy group of a non-zero vector $v \in R^n$ has been considered. Here this group will be called the visotropy group and denoted by $B_n(v)$.

We recall that matrices which belong to $B_n(v)$ are of the form

$$[\delta^i_j + v^j c^i] \,, \tag{1}$$

where $c \in R^n$, $\det[\delta^i_j + v^j c^i] = 1 + < v, c > \neq 0$ and $<,>$ denotes the euclidean scalar product in R^n.

Affine mappings in R^n $x \to Ax + a$, where $A \in B_n(v)$ and $a \in R^n$, will be called visotropy mappings.

It is easy to verify that

$$< v, Ax > = \det A < v, x > \tag{2}$$

for the arbitrary $A \in B_n(v)$ and $x \in R^n$.

For $a_1, \dots, a_n \in R^n$ we put

$$(a_1, \dots, a_n) = \det[a^i_j] \,. \tag{3}$$

Let us consider a curve $t \to x(t) \in R^n$ of the class C^{n+1}. We note that the quantity

$$\begin{cases} \int_{t_1}^{t_2} \left(\frac{(x, \dot{x}, \dots, \overset{(n-1)}{x})}{< v, x >} \right)^{2/(n^2-n)} dt & \text{for } < v, x > \neq 0 \,, \\ 0 & \text{for } < v, x > = 0 \end{cases} \tag{4}$$

does not depend on parametrization and centrovisotropy mappings.

Similarly, the quantity

$$(5)\qquad \begin{cases} \displaystyle\int_{t_1}^{t_2}\left(\frac{(\dot{x},\ddot{x},\ldots,\overset{(n)}{x})}{<v,\dot{x}>}\right)^{2/(n^2+n-2)} dt & \text{for } <v,\dot{x}>\neq 0\,,\\ 0 & \text{for } <v,\dot{x}>=0 \end{cases}$$

does not depend on parametrization and visotropy mappings.

Using the invariants (4) and (5) we construct a theory of curves; the invariants will be found by the prolongation [3] of the visotropy group.

2. Theory of plane curves.

a. The visotropy arc length. The visotropy mappings in R^2 are of the form

$$(6)\qquad \begin{cases} X_1=(1+v^1a^1)X+\quad v^2a^1Y\quad +p^1\\ Y_1=\quad v^1a^2X\quad +(1+v^2a^2)Y+p^2\,, \end{cases}$$

where $a,p\in R^2$ and $1+<v,a>\neq 0$.

Now we find the arc length of a curve $X\to Y(X)$. To do this we introduce the notations $C=v^1+v^2Y'$, $\lambda=a^1$. By prolongation of (6) we obtain

$$(7)\qquad \begin{cases} Y_1'=\dfrac{Y'+Ca^2}{1+C\lambda}\\ Y_1''=\dfrac{1+<v,a>}{(1+C\lambda)^3}Y''\,. \end{cases}$$

Since

$$(8)\qquad dX_1=(1+C\lambda)\,dX\,,$$

so from the system of the equations (7) we must find λ. Then we have

$$C^4Y_1''\lambda^3+3C^3Y_1''\lambda^2+(3C^2Y_1''-Cv^1Y''-v^2CY_1'Y'')\lambda+$$
$$+CY_1''-CY''+v^2Y'Y''-v^2Y_1'Y''=0\,.$$

Substituting $\lambda=\frac{1}{C}\mu$ into the above equality we can write down

$$(9)\qquad CY_1''\mu^3+3CY_1''\mu^2+(3CY_1''-v^1Y''-v^2Y_1'Y_1'')\mu+$$
$$+CY_1''-CY''+v^2Y'Y''-v^2Y_1'Y''=0\,.$$

It is easy to see that $\mu_0=-1$ is a root of the equation (9) and we can rewrite (9) in the following form

$$(\mu+1)(CY_1''\mu^2+2CY_1''\mu+CY_1''-v^1Y''-v^2Y_1'Y'')=0\,.$$

Simple calculations show that

$$\mu_1=-1-\sqrt{\frac{C_1}{Y_1''}}\sqrt{\frac{Y''}{C}}\,,$$

$$\mu_2=-1+\sqrt{\frac{C_1}{Y_1''}}\sqrt{\frac{Y''}{C}}\,,$$

where $C_1 = v^1 + v^2 Y_1'$.

Substituting μ_2 to the formula (8) we see that

$$\sqrt{\frac{Y_1''}{C_1}}\, dX_1 = \sqrt{\frac{Y''}{C}}\, dX \, .$$

Hence we obtain the visotropy arc length of a curve $X \to Y(X)$ as

$$ds = \sqrt{\frac{Y''}{v^1 + v^2 Y'}}\, dX \, . \tag{10}$$

If a curve is given in the parametric form $t \to x(t) = \begin{bmatrix} x^1(t) \\ x^2(t) \end{bmatrix}$, then the formula (10) is

$$ds = \left(\frac{(\dot{x}, \ddot{x})}{< v, \dot{x} >} \right)^{1/2} dt \, . \tag{11}$$

The formula (11) coincides with (5) for $n = 2$.

b. The curvature of a plane curve and its geometric interpretation.

1^0 **The centrovisotropy curvature.** Consider a curve $t \to x(t) \in R^2$ such that $< v, x > \neq 0$ and $(x, \dot{x}) \neq 0$. Let

$$w = \begin{bmatrix} -v^2 \\ v^1 \end{bmatrix} . \tag{12}$$

It is easy to see that

$$< x, v > = (x, w) \tag{13}$$

for every $x \in R^2$.

For the natural centrovisotropy parameter s we have the identity

$$\frac{(x, x')}{< v, x >} = 1 \, , \tag{14}$$

where $'$ denotes differentiation with respect to the natural parameter.

From (13) and (14) it follows immediately

$$(x, x' - w) = 0$$

and

$$x' = \kappa_c x + w \, . \tag{15}$$

Hence

$$\kappa_c = \frac{(w, x')}{(w, x)} \tag{16}$$

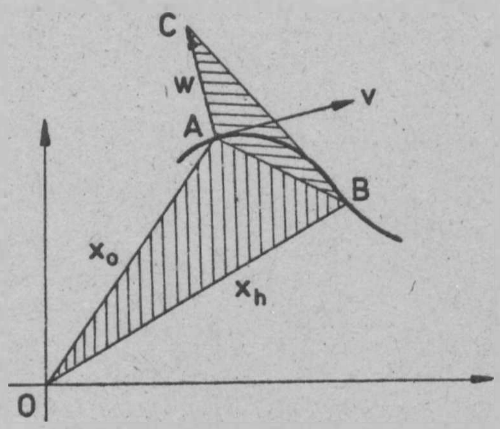
or in an initial parametrization

$$(17) \qquad \kappa_c = \frac{(w, \dot{x})}{(x, \dot{x})} \; ;$$

the function κ_c will be called a centrovisotropy curvature.

Now we will give a geometric interpretation of the centrovisotropy curvature. Let

$$(18) \qquad x_0 = x(t_0) \quad , \quad x_h = x(t_0 + h) \; .$$



We will show that

$$(19) \qquad \kappa_c(t_0) = \lim_{B \to A} \frac{*\text{area}\, \Delta CAB}{*\text{area}\, \Delta AOB} \; ,$$

where $*\text{area}\, \Delta PQR = \frac{1}{2}(\overrightarrow{QP}, \overrightarrow{QR})$.
Using the Taylor expansion $x_h = x_0 + \dot{x}_0 h + \cdots$ we obtain

$$\lim_{B \to A} \frac{*\text{area}\, \Delta CAB}{*\text{area}\, \Delta AOB} = \lim_{h \to 0} \frac{(w, x_h - x_0)}{(x_0, x_h)} = \lim_{h \to 0} \frac{(w, \dot{x}_0)h + \cdots}{(x_0, \dot{x}_0)h + \cdots} = \frac{(w, \dot{x}_0)}{(x_0, \dot{x}_0)} = \kappa_c(t_0) \; .$$

2° **The visotropy curvature.** For the natural visotropy parameter s we have

$$\frac{(x', x'')}{< v, x' >} = 1 \; .$$

Hence

$$(x', x'' - w) = 0$$

and

$$(20) \qquad x'' = \kappa x' + w \; ;$$

the function κ will be called a visotropy curvature.

Consider the indicatrix of tangents of the curve x (if the initial points of all the tangent vectors are shifted to the origin, their new end points trace out a curve called the indicatrix of tangents [2], [3]). Let's denote by $\hat{s}$ and $\hat{\kappa}_c$ the centrovisotropy arc length and curvature of the indicatrix. Using (20) we obtain

$$\frac{d\hat{s}}{ds} = \frac{(x', \frac{d}{ds} x')}{(x', w)} = \frac{(x', \kappa x' + w)}{(x', w)} = 1 .$$

Thus the visotropy arc length of a curve coincide (up to a constant) with the centrovisotropy arc length of its indicatrix.

Moreover we have

$$\hat{\kappa}_c = \frac{(w, \frac{d}{ds} x')}{(w, x')} = \frac{(w, \kappa x' + w)}{(w, x')} = \kappa .$$

It means that the visotropy curvature of a curve coincides with the centrovisotropy of its indicatrix.

c. Counterpart of Frenet formulas of plane curves. Let

$$\begin{cases} \mathbf{t} = x' \\ \mathbf{n} = w . \end{cases}$$

Then with respect to (20) we obtain

$$\begin{cases} \mathbf{t}' = \kappa \mathbf{t} + \mathbf{n} \\ \mathbf{n}' = 0 . \end{cases}$$

They are "Frenet formulas" of the plane visotropy geometry. Now we will prove the fundamental theorem of the visotropy theory of plane curves.

Theorem 1. *Let ξ be the function defined in an open interval I that contains 0. Further, let $\mathbf{n}_0 = \begin{bmatrix} w^1 \\ w^2 \end{bmatrix}$ be a non-zero vector and $x_0 \in R^2$. Then for $v = \begin{bmatrix} -w^2 \\ w^1 \end{bmatrix}$ there exists a curve x defined in I such that:*

1^0 $\quad x(0) = x_0$,

2^0 $\quad \dfrac{(\mathbf{t}, \mathbf{n})}{< v, \mathbf{t} >} = 1$ *in I, where* $\mathbf{t}, \mathbf{n}$ *are the moving frame of x,*

3^0 *the visotropy curvature κ of x is equal ξ.*

Proof. Consider a system of the differential equations

$$\begin{cases} \mathbf{t}' = \xi \mathbf{t} + \mathbf{n} \\ \mathbf{n}' = 0 \end{cases}$$

with an initial condition $\mathbf{n}(0) = \mathbf{n}_0$, $\mathbf{t}_0 = v$. It is easy to see that $\dfrac{(\mathbf{t}_0, \mathbf{n}_0)}{< v, \mathbf{t}_0 >} = 1$ and $\left(\dfrac{(\mathbf{t}, \mathbf{n})}{< v, \mathbf{t} >}\right)' = 0$. It implies 2^o.

By simple verification we can show that

$$x(s) = \int_0^s \mathbf{t}(u)\,du + x_0$$

is a required curve.

d. Curves with a constant visotropy curvature. Assume that $\kappa = 0$. Integrating (20) we obtain

$$x(s) = \frac{1}{2}s^2 w + sa + b \tag{21}$$

or

$$\begin{cases} X = -\frac{1}{2}s^2 v^2 + sa^1 + b \\ Y = \quad \frac{1}{2}s^2 v^1 + sa^2 + b^2 , \end{cases} \tag{22}$$

where $< v, a > \neq 0$.

If $v^2 \neq 0$, then from (22) we have

$$\begin{aligned} v^2(v^1)^2X^2 + (v^2)^3Y^2 + 2v^1(v^2)^2XY + \\ + (-2v^1v^2 < v,b > +2v^2a^2 < v,a >)X + \\ + (-2(v^2)^2 < v,b > -2a^1v^2 < v,a >)Y + \\ + v^2 < v,b >^2 + 2a^1 < v,b >< v,a > -2b^1 < v,a >^2 = 0 . \end{aligned} \tag{23}$$

The equation (23) represents a parabola.

Now we assume that $\kappa = \text{const} \neq 0$. By itegration of (20) we obtain

$$x(s) = -\frac{1}{\kappa}sw + \frac{1}{\kappa}e^{\kappa s}a + b \tag{24}$$

or

$$\begin{cases} X = \quad \frac{1}{\kappa}sv^2 + \frac{1}{\kappa}e^{\kappa s}a^1 + b^1 \\ Y = -\frac{1}{\kappa}sv^1 + \frac{1}{\kappa}e^{\kappa s}a^2 + b^2 , \end{cases} \tag{25}$$

where $< v, a > \neq 0$.

Example. Let's consider the logarithmic curve :

$$Y = \ln X , \qquad \begin{cases} x^1(t) = t \\ x^2(t) = \ln t \quad \text{for } t > 0 . \end{cases}$$

Because

$$(\dot{x}, \ddot{x}) = -\frac{1}{t^2}$$
$$< v, \dot{x} > = v^1 + v^2\frac{1}{t} ,$$

then we can take a vector v, which satisfies the inequality $< v, \dot{x} > < 0$.

We will consider two cases:

I. $v^1 = 0$.
Since $v^1 = 0$, we must take $v^2 < 0$. Let $p = -v^2$. Then we have

$$\left(\frac{(\dot{x}, \ddot{x})}{< v, \dot{x} >}\right)^{1/2} = \frac{1}{\sqrt{p}} \frac{1}{\sqrt{t}} .$$

Hence we obtain

$$s = \frac{2}{\sqrt{p}} \sqrt{t} \quad \text{and} \quad t = \frac{p}{4} s^2 .$$

Thus the logarithmic curve, in the natural visotropy parametrization, has the following form

$$\begin{cases} x^1(s) = \frac{p}{4} s^2 \\ x^2(s) = \ln\left(\frac{p}{4} s^2\right) \quad \text{for } s > 0 . \end{cases}$$

From these equations we can calculate the visotropy curvature: $\kappa = -\frac{1}{s} < 0$.

II. $v^2 = 0$.
$v^2 = 0$ implies that $v^1 < 0$. For $q = \sqrt{-v^1}$ we have

$$\left(\frac{(\dot{x}, \ddot{x})}{< v, \dot{x} >}\right)^{1/2} = \frac{1}{qt}$$

and

$$s = \frac{1}{q} \ln t \qquad \text{or} \qquad t = e^{qs} .$$

By that we obtain the parametrized form of the logarithmic curve

$$\begin{cases} x^1(s) = e^{qs} \\ x^2(s) = qs . \end{cases}$$

We can verify that $\kappa = q$. Thus the logarithmic curve has a constant visotropy curvature $\kappa = \sqrt{-v^1}$, for every vector v such that $v^1 < 0$ and $v^2 = 0$.

We note that by substituting $b^1 = b^2 = 0$, $a^1 = \kappa = \sqrt{-v^1}$, $a^2 = 0$ into the formula (25), we obtain our logarithmic curve, as well.

In the same way we can show that the exponential curve $Y = e^X$ has a constant visotropy curvature for every vector v such that $v^1 = 0$ and $v^2 > 0$.

3. Theory of curves in the 3–dimensional space.

a. The visotropy curvature and torsion. Let's consider a curve $s \to x(s) \in R^3$ such that $< v, x' > \neq 0$. Differentiating the identity

$$\frac{(x', x'', x''')}{< v, x' >} = 1 \qquad (26)$$

we can find

$$x^{IV} = \alpha x' + \beta x'' + \gamma x''' \tag{27}$$

where

$$\begin{cases} \beta = -\dfrac{(x', x''', x^{IV})}{< v, x' >} \\ \gamma = \dfrac{< v, x'' >}{< v, x' >} \, . \end{cases} \tag{28}$$

We can verify that

$$\alpha = \gamma'' + (\gamma^2)' - \beta\gamma \, . \tag{29}$$

From the above formulas it follows that α, β, γ are invariants of visotropy mappings and parametrizations.

We will denote by $x \wedge y$ the vector product of vectors $x, y \in R^3$. Now we can rewrite the formula (26) as follows $< x', x'' \wedge x''' - v >= 0$. Hence

$$x'' \wedge x''' - v = \kappa x' \wedge x''' + \lambda x' \wedge x'' \, . \tag{30}$$

It is easy to see that

$$\begin{cases} \kappa = \gamma \\ \lambda = -\dfrac{< v, x''' >}{< v, x' >} \, . \end{cases} \tag{31}$$

The function κ is said to be a visotropy curvature. Moreover, we can verify that

$$\lambda + \kappa' + \kappa^2 = 0 \tag{32}$$

The formulae (31) and (32) follow from (30).

The function

$$\tau = \beta + \lambda \tag{33}$$

will be called a visotropy torsion.

b. Counterpart of Frenet formulas in the visotropy geometry. Let

$$\begin{cases} \mathbf{t} = v \\ \mathbf{n} = \dfrac{1}{< v, x' >} v \wedge (x' \wedge x'') \\ \mathbf{b} = \dfrac{1}{< v, x' >} v \wedge (x' \wedge x''') \end{cases} \tag{34}$$

The vectors $\mathbf{t}, \mathbf{n}, \mathbf{b}$ are linearly independent, because

$$(\mathbf{t}, \mathbf{n}, \mathbf{b}) =< v, v > \neq 0 \, .$$

The formulas (27)–(34) imply

(35)
$$\begin{cases} \mathbf{t}' = 0 \\ \mathbf{n}' = -\kappa\mathbf{n} + \mathbf{b} \\ \mathbf{b}' = \tau\,\mathbf{n} + \kappa\mathbf{b}\,. \end{cases}$$

They are "Frenet formulas" of the visotropy geometry.

Now we prove the main theorem of our theory. Let I denote an open interval that contains 0.

Theorem 2. *Let's assume that*

– the functions ξ, η are defined in I, ξ has a continuous first derivative but η is continuous ;

– the vectors $\mathbf{n}_0, \mathbf{b}_0$ are linearly independent and $\mathbf{t}_0 = \mathbf{n}_0 \wedge \mathbf{b}_0$;

– the vector e satisfies a condition $< \mathbf{t}_0, e > \neq 0$.

Then for $v = \mathbf{t}_0$ there exists one and only one curve x defined in I, which passes through the arbitrary fixed point in R^3, with following properties:

1^0 *x has the natural visotropy parameter, $x'(0) = e$;*

2^0 $\mathbf{t}(0) = \mathbf{t}_0$ *and* $\dfrac{(\mathbf{t}, \mathbf{n}, \mathbf{b})}{< v, \mathbf{t} >} = 1$ *in I, where* $\mathbf{t}, \mathbf{n}, \mathbf{b}$ *are the moving frame of x ;*

3^0 *the visotropy curvature κ and the visotropy torsion τ of the curve x satisfy*
$$\kappa = \xi, \quad \tau = \eta\,.$$

Proof. Let's consider a system of differential equations

(36)
$$\begin{cases} \mathbf{t}' = 0 \\ \mathbf{n}' = -\xi\mathbf{n} + \mathbf{b} \\ \mathbf{b}' = \eta\mathbf{n} + \xi\mathbf{b}\,. \end{cases}$$

Since $\mathbf{t}_0 = \mathbf{n}_0 \wedge \mathbf{b}_0$ and $v = \mathbf{t}_0$, so $\dfrac{(\mathbf{t}_0, \mathbf{n}_0, \mathbf{b}_0)}{< v, \mathbf{t}_0 >} = 1.$

We note that $(\mathbf{n} \wedge \mathbf{b})' = 0$, so $\mathbf{n} \wedge \mathbf{b} = \text{const}$. We put

(37)
$$v = \mathbf{t} = \mathbf{n} \wedge \mathbf{b} = \mathbf{t}_0.$$

Now we have $(\mathbf{t}, \mathbf{n}, \mathbf{b}) = < v, \mathbf{t} > = < v, v > \neq 0$. Thus the solution of (36) is a system of linearly independent vectors.

We define a curve x by the differential equation

(38)
$$x'(s) = -\frac{1}{\mu(s)} \int_0^s \mu(u)\mathbf{n}(u)\,du + e\,,$$

where
$$\mu(s) = \exp\left(-\int_0^s \xi(u)\,du\right)\,.$$

Since $< v, \mathbf{n} > = 0$, so
$$< v, x' > = -\frac{1}{\mu}\int \mu < v, \mathbf{n} > + < v, e > = < v, e > \neq 0\,.$$

From (36) and (38) we can find

$$x'' = \xi x' - \mathbf{n} \,. \tag{39}$$

It implies that $\kappa = \xi$. Then in the same way we obtain

$$x''' = (\kappa' + \kappa^2)x' - \mathbf{b} \tag{40}$$

and hence

$$\lambda = -\kappa' - \kappa^2 \,.$$

Now we are able to show that

$$\frac{(x', x'', x''')}{< v, x' >} = 1 \,.$$

Using the formulas (39) and (40) we see that

$$(x', x'', x''') = (x', \xi x' - \mathbf{n}, -\lambda x' - \mathbf{b}) = (x', \mathbf{n}, \mathbf{b}) =< x', \mathbf{n} \wedge \mathbf{b} >=< x', v > \,.$$

Differentiating (40) and then making use of (36), (39), (40) we get

$$x^{IV} = (-\lambda' - \lambda\xi)x' + (\lambda - \eta)\mathbf{n} - \xi\mathbf{b} = -\frac{(x', x''', x^{IV})}{< v, x' >} = \frac{(x', \mathbf{b}, (\lambda - \eta)\mathbf{n})}{< v, x' >} =$$

$$= -(\lambda - \eta)\frac{(x', \mathbf{n}, \mathbf{b})}{< v, x' >} = \eta - \lambda.$$

It means that $\tau = \eta$. It ends our proof.

REFERENCES

[1] Cieślak, W., Kieres, A., *On a Complemented Group of the Isotropy Group*, Ann. Univ. Mariae Curie-Skłodowska, vol.XXXIV, 1 (1980), 1–8.

[2] Laugwitz, D., *Differential and Riemannian Geometry*, Academic Press, New York and London 1965.

[3] Širokow, P.A., Širokow, A.P., *Affine Differential Geometry*, Moscow 1959 (Russian).

STRESZCZENIE

W pracy tej podajemy teorię krzywych płaskich i trójwymiarowych w pewnej podgeometrii geometrii afinicznej. W badanej geometrii określono w sposób niezmienniczy parametr naturalny, krzywiznę i skręcenie krzywych oraz dowiedziono,że określają one krzywą z odpowiednią dokładnością.

SUMMARY

In this paper a theory of plane and space curves in a subgeometry of affine geometry is developed. Natural parameter, as well as curvature and torsion are defined which are invariant and define the curve to some extent.