ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN — POLONIA
VOL. XIV, 1 SECTIO A 1960

Z Katedry Zespołowej Matematyki Wydziału Mat.-Fiz.-Chem. UMCS
Kierownik: prof. dr A. Bielecki

JAN KRZYŻ

Distortion Theorems for Bounded Convex Functions II

Twierdzenia o zniekształceniu dla ograniczonych funkcji wypukłych II

Теоремы искажения для выпуклых ограниченных функций II

1. Introduction. Notations

This paper contains detailed proofs of some results announced in an earlier note under the same title [2] presented to the Polish Academy of Science. Let $C(M)$ denote the class of functions $f(z)$ regular and univalent in the unit circle $K = \{z: |z| < 1\}$ with $f(0) = 0$, $|f'(0)| = 1$, mapping the circle K onto a convex domain $f(K) = \Omega(f)$ contained in $K(M) = \{w: |w| < M\}$, $M > 1$.

It is easy to see that the boundary of $\Omega(f)$ is a simple closed convex Jordan curve $\Gamma(f)$ having the one-sided tangents everywhere. Besides, the set of points with different one-sided tangents is at most enumerable. In fact, the intersection $\Omega(f) \frown \{w: \Re w = u\}$ $(-M < u < M)$ if not empty, is a single open segment. Let $h(u)$ denote the ordinate of its lower end point, $h(u)$ being defined and bounded for $u \in (\alpha, \beta) \subset (-M, M)$. The convexity of $\Omega(f)$ implies that $h(u)$ is a function convex downwards, i. e.

$$h(u) \leqslant \frac{u_2 - u}{u_2 - u_1} h(u_1) + \frac{u - u_1}{u_2 - u_1} h(u_2), \qquad u_1 < u < u_2.$$

This inequality involves ([6], p. 172) the continuity of $h(u)$. An analogous statement for the upper end points of $\Omega(f) \frown \{w: \Re(w) = u\}$ holds, and hence we conclude that the boundary of $\Omega(f)$ is a closed Jordan curve $\Gamma(f)$ consisting of two convex arcs $v = g(u)$, $v = h(u)$ $(\alpha < u < \beta)$

and of two segments of the straight lines $\Re(w) = \alpha$, $\Re(w) = \beta$ which may possibly degenerate to points. Besides, we have $h(u) = \int_{\alpha}^{u} \gamma(t)dt$, where $\gamma(t)$ is a bounded and non-decreasing function ([6], p. 372). Since the set of discontinuity points of $\gamma(t)$ is at most enumerable, the derivative $h'(u)$ and therefore the tangent of $\Gamma(f)$ exist everywhere apart from discontinuity points of $\gamma(t)$. Finally, the one-sided limits of the monotonic function $\gamma(t)$ exist everywhere and this implies the existence of one-sided tangents. Besides, $\Gamma(f)$ being a convex Jordan curve is obviously rectifiable. In the previous paper [2] we have found by elementary methods the Koebe constant for the class $C(M)$, i. e. the radius $\delta(M)$ of the largest circular disc with the centre at the origin which is contained in $\Omega(f)$ for every $f \epsilon C(M)$. We have

$$\delta(M) = M \sin\theta \tag{1.1}$$

where θ is the unique solution of the equation

$$(\pi + 2\theta)\sin\frac{4\pi\theta}{\pi + 2\theta} = \frac{2\pi}{M}\cos\theta, \ (M > 1), \tag{1.2}$$

included in the open interval $(0, \pi/2)$. The extremal function $f^*(z, M)$ for which the intersection $\Gamma(f) \frown \{w: |w| = \delta(M)\}$ is not empty, maps K onto $\Omega^* = \Omega^*(M) = K(M) \frown \{w: \Re(w) > -\delta(M)\}$ and is unique apart from rotations of K and $\Omega^*(M)$ about the origin. Supposing that $f_z^{*\prime}(0, M) = 1$, we have

$$w = f^*(z, M) = \frac{M}{i}\,\frac{e^{-i\theta} H(z) - e^{i\theta}}{1 + H(z)} \tag{1.3}$$

where

$$H(z) = e^{2i\theta}\left(\frac{1 - \tau z}{1 - \tau z}\right)^{1/\mu}, \quad \mu = \frac{2\pi}{\pi + 2\theta}, \quad \tau = e^{2i\theta\mu},$$

θ being defined by (1.2).

In this paper we shall deduce by variational methods precise bounds for $|f(z)|$, $(1 - |z|^2)|f'(z)|$, $|a_2| = \frac{1}{2}|f''(0)|$ $(f \epsilon C(M))$. In each case considered the function $f^*(z, M)$ is extremal. The author is very much indebted to Prof. Z. Charzyński for suggesting these problems.

2. An extremal problem connected with bounds for $|f|$ in $C(M)$

Suppose that $\eta (0 < \eta < M)$ being fixed, we wish to determine such a function $f \epsilon C(M)$ which attains the value η for $z \epsilon K$ with the least possible modulus. If the function $\varphi(z)$ so obtained is the same for every

$\eta\in(0, M)$ then $\varphi(z)$ provides evidently the extremal value for the upper bound of $|f|$ in $C(M)$ if $|z|$ is fixed. Let A denote the class of closed convex domains Ω containing the points 0 and η (A depends on η) included in the circle $K(M)=\{w:|w|<M\}$ and such that the inner conformal radius $r(0, \Omega)=1$. It is easy to see that the problem of determining $\sup g(0, \eta, \Omega)$, $\Omega\in A$, is equivalent to that of determining such a $\varphi\in C(M)$ which attains the value η for $z\in K$ with the least modulus. The expression $g(w, w_0, \Omega)$ denotes here the classical Green's function of Ω. We may confine ourselves to the classical Green's function because the boundary of Ω is a simple closed Jordan curve as pointed out in sect. 1. In fact, if $\psi\in C(M)$, $\psi(re^{i0})=\eta$, $\psi(K)=\Omega$, then $\Omega\in A$. On the other hand, if $\Omega\in A$, there exists a function $\psi\in C(M)$ mapping K onto Ω, such that $\psi(re^{i\theta})=\eta$. Then $g(0, \eta, \Omega)=\log r^{-1}$. Hence the problem of minimizing r with $\psi(re^{i\theta})=\eta$, $\psi\in C(M)$ is equivalent to that of finding the domain $\Omega\in A$ with the greatest possible value of $g(0, \eta, \Omega)$ (η being fixed). Similarly the problem of determining the function $\psi\in C(M)$ which attains the given fixed value $\eta(-\delta(M)<\eta<0)$ for z with the greatest possible modulus, may be reduced to that of finding $\inf g(0, \eta, \Omega)$, $\Omega\in A$. The assumption $-\delta(M)<\eta<0$ is essential since for $|\eta|\geqslant\delta(M)$ the infimum to be determined is obviously equal zero.

In order to obtain the extremal domain, we shall use the Hadamard's formulae for the variations of the Green's function and of the Robin's constant $\gamma(\zeta, \Omega)=\log r(\zeta, \Omega)$ (see e. g. [5]). Next, we bring these formulae to a form more convenient for our purposes. Let $z=\varphi(w)$ map conformally the domain $\Omega\in A$, with the boundary being an analytical curve Γ, onto the unit circle K in such a way that $\varphi(0)=0$. Then

$$g(w, \eta, \Omega)=\log\left|\frac{1-\varphi(w)\overline{\varphi(\eta)}}{\varphi(w)-\varphi(\eta)}\right|=\log|\Phi(w)|$$

If the relative orientations of the outward pointing normal and of the tangent of Γ are like those of the x and y axes respectively, then by analyticity of Γ and by Cauchy-Riemann equations we have

$$\frac{\partial g}{\partial n_w}=\frac{\partial}{\partial n_w}\log|\Phi(w)|=\frac{\partial}{\partial s}\arg\Phi(w)=\frac{\partial\theta}{\partial s}=\left|\frac{dz}{dw}\right|=|\Phi'(w)|$$

and therefore

$$\frac{\partial g(w, \eta, \Omega)}{\partial n_w}=\frac{|\varphi'(w)|(1-|\varphi(\eta)|^2)}{|\varphi(w)-\varphi(\eta)|^2}, \quad w\in\Gamma. \tag{2.1}$$

In view of (2.1) we may bring the Hadamard's formulae to the following forms

$$(2.2) \qquad \delta\gamma(0) = \frac{1}{2\pi}\int_{\Gamma}\left[\frac{\partial g(w, 0, \Omega)}{\partial n_w}\right]^2 \delta n(s)ds = \frac{1}{2\pi}\int_{\Gamma}|\varphi'(w)|^2\delta n(s)ds$$

$$(2.3) \qquad \delta g(0, \eta, \Omega) = \frac{1}{2\pi}\int_{\Gamma}\frac{\partial g(w, 0, \Omega)}{\partial n_w}\frac{\partial g(w, \eta, \Omega)}{\partial n_w}\delta n(s)ds =$$

$$= \frac{1}{2\pi}\int_{\Gamma}|\varphi'(w)|^2\frac{1-|\varphi(\eta)|^2}{|\varphi(w)-\varphi(\eta)|^2}\delta n(s)ds.$$

Here $\delta n(s) = \varepsilon p(s)$ is the normal displacement which is to be taken positive, if the displacement vector coincides with the outward pointing normal, and negative, if it has the opposite direction. Besides, $p(s)$ is a piecewise continuous function of the arc length s on Γ. The above given formulae are obviously also valid, when $p(s) \neq 0$ on a finite system of analytic boundary arcs and $p(s) = 0$ on the remainder of boundary of a convex domain.

If the domain Ω_0 yields the extremal value for the Green's function $g(0, \eta, \Omega)$ within a class of domains fulfilling the condition $r(0, \Omega) = 1$ (resp. $\gamma(0, \Omega) = \log r(0, \Omega) = 0$) there exists a constant λ such that for any variation of Ω_0 leading to domains of the considered class, we have $\delta g + \lambda\delta\gamma = 0$. This implies that, if under an admissible variation $\delta\gamma = 0$, and in the same time $\delta g > 0$, the domain Ω_0 cannot yield the extremal value for the Green's function. For our further considerations it is very important that the expression

$$(2.4) \qquad \sigma(w) = \frac{1-|\varphi(\eta)|^2}{|\varphi(w)-\varphi(\eta)|^2} = \frac{1-|z_0|^2}{|z-z_0|^2},$$

occurring in the formula (2.3) varies in a certain monotonic manner for fixed η and for w moving on Γ. The boundary Γ is a Jordan curve and therefore there exists a homeomorphism between the boundaries of K and Ω.

The equality (2.4) involves the existence of two points w_1, w_2 on Γ dividing Γ into two arcs Γ_1, Γ_2 such that $\sigma(w)$ decreases strictly as w is moving on each of two arcs Γ_1, Γ_2 from w_1 to w_2. The function $\sigma(w)$ attains at the points w_1, w_2 its extremal values with respect to Γ. In order to find the extremal domain with respect to a class of domains, we shall show that some domains cannot be extremal. We distinguish two arcs l_1, l_2 on the boundary Γ such that

$$\text{(A)} \qquad \min_{w \in l_1} \sigma(w) \geqslant \max_{w \in l_2} \sigma(w).$$

The existence of such arcs is secured by the lemma 3.1 based on the above mentioned monotonic behaviour of $\sigma(w)$. We now choose $p(s) > 0$ and $p(s) < 0$ on the open arcs l_1 and l_2 respectively, so that

$$\text{(B)} \qquad \int_{l_1} |\varphi'(w)|^2 p(s)\, ds = \int_{l_2} |\varphi'(w)|^2 [-p(s)]\, ds.$$

Then the inequality (A), in view of strict monotonity of $\sigma(w)$ implies

$$\text{(C)} \qquad \int_{l_1} |\varphi'(w)|^2 \sigma(w)\, \delta n(s)\, ds > \int_{l_2} |\varphi'(w)|^2 \sigma(w) [-\delta n(s)]\, ds$$

(B) means that $\delta\gamma = 0$ whereas (C) yields $\delta g > 0$ for the variation of Ω defined by $p(s)$.

Such a process will be referred to as a construction of positive and negative variations, on l_1 and l_2 respectively, which do not change $\gamma(0, \Omega)$ while increasing the Green's function. If such a construction is possible, the domain subject to it, cannot evidently yield the maximal value to the Green's function. Similarly, putting $p_1(s) = -p(s)$, $p(s)$ being defined as above, we obtain a variation of the boundary which, not changing $\gamma(0, \Omega)$, decreases the Green's function, and such a domain cannot minimize the Green's function.

3. Boundary variations within A_n and A. Auxilary lemmata

Let A_n denote the class of closed convex polygonal domains Ω with at most n vertices and such that $0 \in \Omega$, $\eta \in \Omega$, $r(0, \Omega) = 1$; $\Omega \subset \{w: |w| \leqslant M\}$ ($\eta \in (0, M)$ being fixed). To every domain $\Omega \in A_n$ we may attach a function $\psi \in C(M)$ with $\psi'(0) = 1$ and so we may consider compact and everywhere dense sets of domains. Clearly A_n is a compact set of domains. Thus A_n contains an extremal domain Ω_n such that

$$g(0, \eta, \Omega_n) = \sup g(0, \eta, \Omega), \quad \Omega \in A_n.$$

Similarly A is a compact class and the domain Ω_0 for which $g(0, \eta, \Omega)$ has a maximum within A may be approximated by polygons, the convergence being understood in the sense of nucleus convergence (see e. g. [3], p. 373, or [4], p. 140). It is easily verified that a suitably chosen subsequence $\{\Omega_{n_k}\}$ converges into its nucleus being the extremal domain Ω_0 for the class A.

Let us suppose that DA, AB and BC are any three adjacent sides of $\Omega \in A_n$, γ being the remainder of the boundary and that the vertex B is inside the circle $K(M)$, whereas the angles at A and B are less than π. Let us draw through B the outward perpendicular BB'' to AB so that the angle $\sphericalangle B''AD < \pi$ and that B'' lies inside the circle $K(M)$. If $\sphericalangle ABC > \pi/2$, then the prolongation of CB meets AB'' at B' lying inside $K(M)$. We now define $p(s) = MM'' = AM \tan\alpha$ ($\alpha = \sphericalangle BAB''$) for $M \in AB$, $p(s) = 0$ outside AB, and compare the Green's function for Ω and for the varying domain Ω'' defined by the normal displacement $\varepsilon p(s)$ of Ω. Since the Hadamard's formula may be obviously applied in this case, we have

$$\delta g = g(w, w_0, \Omega'') - g(w, w_0, \Omega) + O(\varepsilon^2) = \frac{1}{2\pi} \int_{AB} |\varphi'(w)|^2 \sigma(w)\, \delta n(s)\, ds .$$

If Ω' is the varying domain with the boundary $\gamma DAB'BC$ (B' is the varying point where the prolongation of CB meets the boundary of Ω''); then the difference $g(w, w_0, \Omega') - g(w, w_0, \Omega'')$ ($w_0 \in \Omega'$ being fixed) is a harmonic function of $w \in \Omega'$ which is equal $O(\varepsilon)$ for $w \in BB'$ and vanishes on the remainder of the boundary of Ω'. By the Green's formula we have $g(w, w_0, \Omega'') - g(w, w_0, \Omega') = O(\varepsilon^2)$ since the boundary values on the boundary of Ω' are equal to $O(\varepsilon)$ on the segment BB' (the length of which is equal to $O(\varepsilon)$), and vanish on the remainder of the boundary. Comparing the Green's functions of Ω and Ω', we obtain therefore

$$\delta g(0, \eta, \Omega) = \frac{1}{2\pi} \int_{AB} |\varphi'(w)|^2 \sigma(w)\, \delta n(s)\, ds \tag{2.31}$$

where $\delta n(s) = \varepsilon \cdot MM'' = \varepsilon \cdot AM \tan\alpha$ for $M \in AB$. If $\sphericalangle ABC \leqslant \pi/2$, we put $\Omega' = \Omega''$ and the same formula holds. In both cases such a variation leads to domains Ω' within A_n, once B is an inner point of $K(M)$, and it will be referred to as an outward rotation of the side AB about A.

We can also draw BB'' — the inward perpendicular to AB — (B may not be now an inner point of $K(M)$), and define $p(s) = -MM'' = -AM \tan\alpha$ ($\alpha = \sphericalangle BAB''$) for $M \in AB$, $p(s) > 0$ on the remainder of the boundary of Ω. The variable domain Ω'' will be determined by the normal displacement $\varepsilon p(s)$ of the boundary, whereas Ω' is the varying domain with the boundary $\gamma DAB'C$ (where B' is the varying point at which the segment BC meets the rotating side AB'', resp. its prolongation. Comparing the Green's functions of Ω and Ω' we obtain similarly

(2.31). Such a variation also leads to domains within A_n, and it will be referred to as an inward rotation of the side AB about A.

We can define quite similarly the inward and outward rotations of a rectilinear side for a domain the boundary of which is composed of a system of arcs of the circumference $\{w: |w| = M\}$ and of straight line segments connecting their end points. An outward rotation of the side AB about A may be now defined also for B situated on the circumference $\{w: |w| = M\}$. The variable domain $\Omega \in A$ arises by adjoining to Ω the curvilinear triangle ABB' with variable B' outside Ω on the circumference $\{w: |w| = M\}$. The formula (2.31) holds also in this case. It C is an inner point of the side AB, we shall consider a variation of boundary referred to as an outward shifting of the point C. The function $p(s)$ is now defined as $AM\tan\alpha$ for $M \in AC$ and $BM\tan\beta$ for $M \in BC$. The condition $CB\tan\alpha = AC\tan\beta$ implies the continuity of $p(s)$. If the boundary of Ω contains "superfluous" vertices with the angles equal to π, the outward shifting of such a superfluous vertex provides a variation within A_n, respectively within A.

If the boundary of a domain $\Omega \in A$ contains the chord AB of the circle $K(M)$, we also consider a variation of boundary referred to as bending of the side AB at the point C. The function $p(s)$ is now defined as follows: $p(s) = CM\tan\alpha$ for $M \in AC$, $p(s) = -CM\tan\beta$ for $M \in CB$ and $p(s) = 0$ on the remainder of boundary, $0 < \alpha \leqslant \beta < \pi/2$. Let Ω' be the convex domain the boundary of which consists of a suitable part of the boundary of Ω, of two rays with the origin at C and the varying arc AA' of the circumference $\{w: |w| = M\}$. We obtain quite similarly that (2.31) also holds in this case and this gives a variation within A.

Lemma 3.1. *Let ABC be a triangle with the boundary L and let $\sigma(w)$ be a function defined and continuous for $w \in L$ which attains its greatest and least values at the points M and m respectively. Besides, let us suppose that $\sigma(M) > \sigma(m)$ and that $\sigma(w)$ decreases strictly as w is moving on L from M to m. Then we can distinguish two closed sides L_1, L_2 of the triangle such that*

$$\min_{w \in L_1} \sigma(w) \geqslant \max_{w \in L_2} \sigma(w). \tag{A1}$$

Proof. If both points m, M are on the same side AC, $\sigma(w)$ varies monotonically on ABC and we may take $L_1 = AB, L_2 = BC$, or conversely.

Let us now suppose that m, M are on different sides of the triangle, say $M \in AB, m \in AC$, and that $M \neq A$ (the case $M = A$ has been already considered). There exists the unique point $A_1 \neq A$ such that $\sigma(A) = \sigma(A_1)$. If $A_1 \in BC$, then $L_1 = AB$, $L_2 = AC$. If $A_1 \in AC$, then we can find $C_1 \in MA$ such that $\sigma(C_1) = \sigma(C)$. Then we have

$$\min_{w \in CB} \sigma(w) \geqslant \max_{w \in C_1 AmC} \sigma(w) \leqslant \max_{w \in AC} \sigma(w)$$

and we may take $L_1 = BC$, $L_2 = AC$. Finally, if $A_1 \in AB$, we can find $B_1 \in Am$, $B_1 \neq B$, such that $\sigma(B) = \sigma(B_1)$. Then we have

$$\min_{k \in AB} \sigma(w) \geqslant \min_{w \in B_1 AB} \sigma(w) \geqslant \max_{w \in BC} \sigma(w)$$

and we may take $L_1 = AB$, $L_2 = BC$.

Corollary. The lemma holds obviously, if we replace the triangle ABC by three adjacent arcs of a simple closed Jordan curve. Besides, the arcs L_1, L_2 may be replaced by their arbitrary non void closed subsets l_1, l_2.

Lemma 3.2. *All the n angles of the polygon Ω_n providing a maximum for the Green's function $g(0, \eta, \Omega)$ within A_n are less than π. At most one vertex of Ω_n is inside $K(M)$ and all the remaining vertices are situated on the circumference $\{w: |w| = M\}$.*

Proof. We first prove that the boundary of Ω_n cannot have two vertices with angles less than π inside $K(M)$. Suppose that, contrary to this, A and B are such vertices. Let C be an arbitrary vertex of Ω_n different from A, B. The points A, B, C split Γ_n, the boundary of Ω_n, into three parts and in view of lemma 3.1 there exist two polygonal lines L_1 and L_2 each having A or B as one of its end points, such that (A1) holds. The polygonal lines L_1 and L_2 may be replaced by two segments l_1 and l_2 respectively, each having A or B as one of its end points and such that (A) holds. We now turn l_1 outwards and l_2 inwards about their end points by moving A or B and the angles of rotations are chosen so that (B) holds. Such a variation leads to domains within the class A_n and does not change γ while increasing the Green's function. We see that the Green's function cannot attain a maximum within A_n for such a domain. Next we prove that Ω_n cannot have "superfluous" vertices with angles equal to π. We choose two segments l_1, l_2 on Γ_n such that (A) is fulfilled, then we remove the superfluous vertex C and situate it on l_1 without changing the domain. We now shift C outwards and turn l_2 inwards, $p(s)$ being chosen so that (B) holds. Then $\delta\gamma = 0$, $\delta g > 0$ and this means that the Green's function cannot have a maximum for such a domain. The lemma 3.2 is proved.

4. The structure of the domain Ω_0

In view of lemma 3.2 the extremal domain Ω_n has all n vertices with angles less than π, and at most one of them is situated inside $K(M)$. The sequence $\{\Omega_n\}$ is a compact set of domains and therefore a convergent subsequence $\{\Omega_{n_k}\}$ can be selected which converges into its nucleus Ω_0, Ω_0 being the extremal domain within A. Since n_k-1, resp. n_k vertices of Ω_{n_k} are situated on the circumference $\{w:|w| = M\}$, the set F of accumulation points of vertices of Ω_{n_k} is a closed set all points of which (with at most one exception) lie on $\{w:|w| = M\}$. If the set F is dense on an arc γ of the circumference $\{w:|w| = M\}$, the arc γ must be a boundary arc of Ω_0. Since $M > 1$, $r(0, \Omega_0) = 1$, we see that Ω_0 cannot be identical with the closed disc $K(M)$. Thus the set $G = \{w:|w| = M\}\setminus F$ is non-void and open with respect to $\{w:|w| = M\}$. Therefore G must be an at most enumerable sum of open arcs. Let γ be an arbitrary component of G. We see that the chord connecting both its end points must be a part of boundary of Ω_0, with perhaps one exception, where the corresponding part of boundary is composed of two straight line segments. Therefore the boundary of Ω_0 consists of an at most enumerable system of rectilinear segments and arcs of the circumference $\{w:|w| = M\}$.

We first prove that on the boundary Γ_0 of Ω_0 there are at most two straight line segments. For suppose that, contrary to this, there are three segments on Γ_0. Let us split Γ_0 into three parts each of them containing one segment. In view of lemma 3.1 and the corollary there exist two segments l_1, l_2 such that (A) is fulfilled. We now turn l_1 outwards and l_2 inwards and take $p(s)$ so that (B) holds. The variation of Γ_0 corresponding to such $p(s)$ provides $\delta\gamma = 0$, $\delta g > 0$ which is impossible. Therefore the set G also consists of at most two components. Thus Γ_0 consists of at most two circular arcs on $\{w:|w| = M\}$ and of at most two rectilinear segments.

Finally, we prove that Γ_0 cannot contain two straight line segments. Let $\sigma(w)$ attain its minimal value at m and let us suppose that l_1, L_2 are different boundary segments of Γ_0. If m is an inner point of a boundary segment, say $m \in L_2$, we shall move two points C_1, C_2 on L_2 so that $\sigma(C_1) = \sigma(C_2)$ having started at m. As one of them attains the end point of L_2, the other is located at $C, C \in L_2$. The point C divides L_2 into two parts, one of them l_2 containing m. Obviously (A) holds. We now turn l_2 inwards about C, whereas l_1 is turned about one of its end points outwards so that (B) holds. This implies $\delta\gamma = 0$, $\delta g > 0$ which is impossible.

We now suppose that m is situated on $\{w:|w| = M\}$, or, that m is the common end point of both boundary segments. It is easy to see that in both cases $\sigma(w)$ varies in a strictly monotonic manner, as w is moving on the one suitably chosen boundary segment, say on L_1. We choose arbitrary fixed numbers a, β $(0 < a < \beta < \pi/2)$ and a point $C \epsilon L_2$. The point C splits L_2 into two segments l_1, l_2 such that (A) holds. We now draw two rays emanating from C, one of them going inwards Ω_0 and inclined at an angle β to l_2, the other going outwards Ω_0 and inclined at an angle a to l_1. Both rays determine the function $p(s)$ positive on l_1, negative on l_2 and equal to zero on the remainder of the boundary. We now locate C so that the equality (B) holds. This is possible, because during a contraction of $l_2(l_1)$ to a point by suitable moving of $C(a, \beta$ being fixed) the right (left) hand side tends to zero, whereas the other side tends to a positive limit. In this way we obtain a bending of the side L_2 providing $\delta\gamma = 0$, $\delta g > 0$ which is impossible. We have thus proved that the boundary of the extremal domain Ω_0 within A is composed of one rectilinear segment and, consequently, of one circular arc on $\{w:|w| = M\}$. This implies, in view of $r(0, \Omega_0) = 1$ that $\Omega_0 = \Omega^*$, apart from rotations about $w = 0$.

Taking $p(s)$ positive on these parts of boundary where $\sigma(w)$ is small, and negative where $\sigma(w)$ is large, we can prove by an analogous argument that the same domain also minimizes the Green's function.

Since the extremal domains in both cases do not depend on η, we see, in view of sect. 2, that the function $f^*(z, M)$ defined in sect. 1 is extremal for upper and lower bounds of $|f|$ within $C(M)$, $|z|$ being fixed. $f^*(z, M)$ is a circularly symmetric function (see [1]) with respect to the positive real axis and therefore the modulus $|f^*(z, M)|$ attains, $|z| = r$ being fixed, its maximal and minimal values for $z = r$ and $z = -r$ respectively. In view of this we obtain

Theorem 4.1. *Suppose that $f \epsilon C(M)$. Then*

$$-f^*(-|z|, M) \leqslant |f(z)| \leqslant f^*(|z|, M), \tag{4.1}$$

where $f^*(z, M)$ is defined by the formulae (1.3) *and* (1.2)

The fact that the function f^* provides the upper bound for all $z \epsilon K$ may be used to obtain the precise upper bound for $|a_2|$. We have

Theorem 4.2. *Suppose that $f \epsilon C(M), f(z) = a_1 z + a_2 z^2 + \ldots$, $|a_1| = 1$. Then*

$$|a_2| \leqslant A_2 = A_2(M) = \frac{\sin\theta}{M} + \cos\frac{4\pi\theta}{\pi + 2\theta} \tag{4.2}$$

where θ is the unique solution of (1.2) contained in the open interval $(0, \pi/2)$.

Proof. Put $M(r,f) = \sup|f(re^{i\theta})|$, $\theta \in \langle 0, 2\pi\rangle$; $g(z) = e^{i\alpha}f(ze^{i\beta})$ (α, β arbitrary real numbers). Obviously $g \in C(M)$ and $M(r,f) = M(r,g)$. After a suitable choice of α, β we have $g'(0) = 1$, $\frac{1}{2}g''(0) = |a_2|$, and this implies

$$M(r,f) = M(r,g) = r + |a_2|r^2 + O(r^3) \leqslant f^*(r, M) =$$

$$= r + A_2(M)r^2 + O(r^3).$$

$A_2(M)$ is a positive number because f^* is a circularly symmetric function such that $f^*(z, M) \neq z$, [1], and therefore we obtain $|a_2| \leqslant A_2(M)$. Now $A_2(M)$ can be easily calculated explicitly and the inequality (4.2) follows. If $M \to +\infty$, then $\theta \to 0$ and $A_2(M) \to \cos 0 = 1$, if $M \to 1$, then $\theta \to \pi/2$ and $A_2(M) \to 1 + \cos\pi = 0$ in accordance with the well known facts.

Let us now suppose that $f \in C(M)$ and $\eta = f(z)$. If $\Omega = f(K)$ then $r(\eta, \Omega) = (1-|z|^2)|f'(z)|$. Putting $\gamma(\eta, \Omega) = \log r(\eta, \Omega)$ we obtain, in wiew of (2.2), the following expression for the variation of the Robin's constant of Ω with an analytic boundary Γ:

$$\delta\gamma(\eta, \Omega) = \frac{1}{2\pi}\int_{\Gamma} |\varphi'(w)|^2 \sigma^2(w)\,\delta n(s)\,ds. \tag{4.3}$$

The same formula is valid, if a part of boundary of a convex domain where $p(s) \neq 0$ is a finite system of analytic arcs. The function $\sigma^2(w)$ has similar property of monotonity like $\sigma(w)$, and an analogous argumentation yields

$$(1-|z|^2)|f'(z)| \leqslant r(|f(z)|, \Omega^*). \tag{4.4}$$

Besides, for $|f(z)| \leqslant \delta(M)$ we obtain

$$r(-|f(z)|, \Omega^*) \leqslant (1-|z|^2)|f'(z)|. \tag{4.5}$$

Putting $G(w) = \left(\dfrac{iw + Me^{i\theta}}{-iw + Me^{-i\theta}}\right)^{\mu}$, we have for real w: $|G(w)| = 1$ and

$$r(|w|, \Omega^*) = \frac{G^2(|w|)-1}{G'(|w|)} = \frac{G-\bar{G}}{G'/G}. \tag{4.6}$$

By substituting for r the value (4.6) we obtain, in view of (4.4) and (4.5) the precise bounds for $|f'(z)|$ which depend, however, on $|f(z)|$ and $|z|$.

REFERENCES

[1] Jenkins, J. A., *On circularly symmetric functions*, Proc. Amer. Math. Soc. **6** (1955), p. 620-624.

[2] Krzyż, J., *Distortion theorems for bounded convex functions* I, Bull. Acad. Polon. d. Sci. **8** (1960), p. 625-627.

[3] Leja, F., *Teoria funkcji analitycznych*, Warszawa 1957.

[4] Littlewood, J. E., *Lectures on the Theory of Functions*, Oxford 1944.

[5] Nehari, Z., *Conformal Mapping*, New York-Toronto-London 1952.

[6] Titchmarsh, E. C., *The Theory of Functions*, Oxford 1947.

Streszczenie

W pracy tej rozważam klasę $C(M)$ funkcji $f(z)$ holomorficznych i jednolistnych w kole jednostkowym K, o rozwinięciu $f(z) = a_1 z + a_2 z^2 + \ldots$ $|Q_1| = 1$, odwzorowujących koło K na obszar wypukły $\Omega(f)$ zawarty w kole $K(M) = \{w: |w| < M\}$, $M > 1$. Posługując się wzorami wariacyjnymi Hadamarda, znajduję dokładne oszacowania wielkości $|f(z)|$, $(1-|z|^2)|f'(z)|$, $|a_2|$ dla funkcji klasy $C(M)$. Funkcją ekstremalną jest przy tym funkcja $f^*(z, M)$ odwzorowująca koło K na obszar $\Omega^*(M) = K(M) \frown \{w: \Re(w) > -\delta(M)\}$, przy czym $\delta(M)$ jest stałą Koebego dla klasy $C(M)$, której wartość liczbowa została przeze mnie znaleziona poprzednio w pracy [2].

Резюме

Обозначим через $C(M)$ класс функций вида $f(z) = a_1 z + a_2 z^2 + \ldots$ ($|a_1| = 1$), регулярных и однолистных в единичном круге K, которые отображают этот круг на выпуклую область Ω, заключенную в круге $K(M) = \{w: |w| \leqslant M\}$.

Пользуясь формулами Адамара вариации функции Грина и постоянной Робена, я получаю по методу множителей Лагранжа следующие результаты:

a) точную оценку сверху и снизу для $|f(z)|$ при установленном $z \in K$, когда $f \in C(M)$;

п) $\sup |a_2|$ в классе $C(M)$;

c) строгую оценку сверху и снизу для $(1-|z|^2)|f'(z)|$, $f \in C(M)$.

Во всех этих случаях экстремальная функция та же самая. Она отображает круг K на область

$$\Omega^* = K(M) \frown \{w: \Re w > -\delta(M)\},$$

где $\delta(M)$ есть постоянная Кёбе для класса $C(M)$.