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Radial and Optimal Selections of Metric Projections onto Balls

Abstract. We characterize differentiability of radial selections of metric projections onto balls, and derive (estimations of) their best Lipschitz constants for Banach spaces L^p (2-convex spaces, respectively). Moreover, the optimal selections are determined for several normed lattices, which enabled to prove Ky Fan's approximation principle for order intervals in the Banach lattice L^∞.

1. Introduction. Let X be a normed linear space, and let

$$B = \{x \in X : \|x\| \le 1\}$$

be the unit ball in X. Denote by $\mathcal{P} : X \to 2^B$ the *metric projection* onto B,

$$\mathcal{P}(x) = \{z \in B : \|x - z\| = \inf_{y \in B} \|x - y\|\}.$$

Since

$$\|x - x/\|x\|\| = \|x\| - 1 \le \|x\| - \|y\| \le \|x - y\|,$$

whenever $x \notin B$ and $y \in B$, it follows that $\mathcal{P}(x) \neq \emptyset$ for every $x \in X$, and that the mapping

$$R(x) = \begin{cases} x/\|x\|, & \text{if } x \notin B, \\ x, & \text{if } x \in B, \end{cases} \tag{1.1}$$

is a selection of the metric projection $\mathcal{P}$, which is said to be a *radial projection* [4,16]. Clearly, $\mathcal{P}$ is a multivalued mapping if and only if X is not strictly convex.

It is well-known, and elementary to prove that the radial selection R is Lipschitz continuous, and that the *best Lipschitz constant*

$$k(X) = k_R(X) := \sup\left\{ \frac{\|R(x) - R(y)\|}{\|x - y\|} : x \neq y \right\} \tag{1.2}$$

satisfies the inequality $1 \le k(X) \le 2$. Moreover, de Figueiredo and Karlovitz [4] and Thele [16] proved that identities $k(X) = 1$ and $k(X) = 2$ hold if and only if the Birkhoff's orthogonality is symmetric (this is equivalent to X being an inner-product space, whenever the dimension of X is greater than 2), and iff X is not uniformly non-square, respectively.

If X is not strictly convex, then we define the *optimal Lipschitz constant* by

$$k_o(X) = \inf k_P(X),$$

where the infimum is taken over all selections P of $\mathcal{P}$ and $k_P(X)$ is defined as in (1.2). Further, a metric selection T of $\mathcal{P}$ is said to be *optimal* if $k_o(X) = k_T(X)$. Clearly, we have $1 \leq k_o(X) \leq k(X) \leq 2$, and $k_o(X) = k(X)$ if X is strictly convex.

In this paper, we first characterize differentiability of radial selections, and derive the constants $k(L^p)$ for $1 < p < \infty$ and estimates of $k(X)$, whenever X is 2-convex. Next, we show that there exist optimal selections $T \neq R$ of metric projections $\mathcal{P} : X \to 2^B$ in several normed linear spaces with $k(X) = 2$ for which $k_T(X)$ is equal to 1. The result is applied to prove Ky Fan's approximation principle for nonexpansive mappings on order intervals in the Banach lattice L^∞.

2. The differentiability of radial projections. Denote by $\tau(x,h)$ and $R'(x)h$ directional derivatives of the norm and radial selection R which are defined by

$$\tau(x,h) = \lim_{t\to 0^+} \frac{\|x+th\| - \|x\|}{t} \tag{2.1}$$

and

$$R'(x)h = \lim_{t\to 0^+} \frac{R(x+th) - R(x)}{t}, \tag{2.2}$$

respectively. Clearly, if $\|x\| < 1$, then $R'(x)h = h$. In the following, we study the derivative $R'(x)h$ for $x \in X\backslash B$, where B is the unit ball.

Lemma 2.1. *Let $x \notin B$ be an element of a normed linear space X. Then the derivative $R'(x)h$ exists and*

$$R'(x)h = \frac{h - \tau(x,h)R(x)}{\|x\|}.$$

for all $h \in X$.

Proof. Let $x \notin B$ and $h \in X$. Since $\tau(x,h)$ exists [12], and $x + th \notin B$ for sufficiently small t, we have

$$\begin{aligned}
\lim_{t\to 0^+} \frac{R(x+th) - R(x)}{t} &= \lim_{t\to 0^+} \left[\frac{x+th}{\|x+th\|} - \frac{x}{\|x\|}\right] / t \\
&= \frac{1}{\|x\|^2} \lim_{t\to 0^+} \frac{t\|x\|h + x(\|x\| - \|x+th\|)}{t} \\
&= \frac{1}{\|x\|}(h - \tau(x,h)R(x)),
\end{aligned}$$

which completes the proof. ■

Theorem 2.1. *Let X be a normed linear space. Then the radial projection R is Gateaux differentiable on $X\backslash B$ if and only if X is a smooth space.*

Proof. The operator $R'(x) : X \to X$ from Lemma 2.1 is continuous, whenever $\|x\| > 1$. Indeed, by (2.1) we have

$$\|R'(x)h_1 - R'(x)h_2\| \le \frac{1}{\|x\|}\Big(\|h_1 - h_2\| + \|Rx\||\tau(x,h_1) - \tau(x,h_2)|\Big)$$

$$\le \frac{1 + \|Rx\|}{\|x\|}\|h_1 - h_2\|.$$

Next, the operator $R'(x)$ is linear if and only if $h \to \tau(x,h)$ is a linear functional on X. Since $\tau(\lambda x, h) = \tau(x,h)$ for every $\lambda > 0$, it follows that $h \to \tau(x,h)$ is linear for all $x \ne 0$. Finally, the last statement is equivalent to smoothness of X [7]. ■

Recall that a (smooth) normed linear space X is said to have the Fréchet differentiable norm if

$$\lim_{h\to 0} \frac{\|x+h\| - \|x\| - \tau(x,h)}{\|h\|} = 0 \tag{2.3}$$

for all $x \ne 0$. For such spaces X, the above characterization can be improved as follows.

Theorem 2.2. *Let X be a normed linear space. Then the radial projection R is Fréchet differentiable on $X \setminus B$ if and only if the norm of X is Fréchet differentiable.*

Proof. For the proof of sufficiency, we have to show that

$$f(h) := \frac{R(x+h) - Rx - R'(x)h}{\|h\|} \to 0 \text{ as } h \to 0, \tag{2.4}$$

whenever $\|x\| > 1$. By Lemma 2.1 and (1.1) we obtain

$$\|f(h)\| = \left\| \frac{h}{\|h\|}\left(\frac{1}{\|x+h\|} - \frac{1}{\|x\|}\right) - \frac{x(\|x+h\| - \|x\| - \tau(x,h))}{\|x\|^2\|h\|}\right.$$

$$\left. - \frac{x(\|x+h\| - \|x\|)}{\|x\|\|h\|}\left(\frac{1}{\|x+h\|} - \frac{1}{\|x\|}\right)\right\| \tag{2.5}$$

$$\le 2\left|\frac{1}{\|x+h\|} - \frac{1}{\|x\|}\right| + \left|\frac{\|x+h\| - \|x\| - \tau(x,h)}{\|h\|}\right|.$$

This in conjunction with (2.3) proves (2.4). Conversely, suppose that (2.4) holds for all $x \in X$ with $\|x\| > 1$. Then we get

$$\|f(h)\| \geq \left| \frac{\|x+h\| - \|x\| - \tau(x,h)}{\|h\|\|x\|} \right| - 2\left| \frac{1}{\|x+h\|} - \frac{1}{\|x\|} \right| \tag{2.6}$$

in a similar way as (2.5). Hence we obtain (2.3) in the case when $\|x\| > 1$. This directly implies that $\tau(\lambda x, h) = \tau(x,h)$ exists in the Fréchet sense for all $\lambda > 0$, which completes the proof of (2.3) for all $x \neq 0$. ■

An additional property of R can be established if X is uniformly smooth, which is equivalent [2] to the fact that the limit

$$\lim_{t \to 0} \frac{\|x+th\| - \|x\|}{t}$$

exists uniformly for all x and h in the unit sphere. Clearly, this is equivalent to the existence of this limit uniformly for all x and h in each sphere $S_r = \{z : \|z\| = r\}$ of radius $r > 0$. In this case, the norm of X is said to be uniformly Fréchet differentiable. By analogy, we say that R is uniformly Fréchet differentiable if the limit

$$\lim_{t \to 0} \frac{R(x+th) - Rx}{t}$$

exists uniformly for all x, h in each sphere S_r with $r > 1$.

Theorem 2.3. *The radial projection R is uniformly Fréchet differentiable if and only if X is uniformly smooth.*

Proof . If $\|x\| = \|y\| = r > 1$ and $|t| < 1$, then we have

$$\left| \frac{1}{\|x+th\|} - \frac{1}{\|x\|} \right| = \frac{\big|\|x\| - \|x+th\|\big|}{\|x+th\|\|x\|} \leq \frac{|t|}{(1-|t|)r}.$$

Hence one can insert th for h in (2.5) and (2.6) to finish the proof. ■

3. Best Lipschitz constants for 2-convex spaces. A normed linear space X is said to be 2-convex [13] if there exists a constant $c > 0$ such that the inequality

$$\left\| \frac{x+y}{2} \right\|^2 \leq \frac{1}{2}\left(\|x\|^2 + \|y\|^2\right) - c\left\| \frac{x-y}{2} \right\|^2 \tag{3.1}$$

holds for all x, $y \in X$. Clearly, we always have $c \leq 1$. The estimation $k(X) < 2$ can be improved, whenever X is 2-convex. In order to do this, we need the following lemma.

Lemma 3.1. *If X is 2-convex, then*

$$\|(1-t)x + ty\|^2 \leq (1-t)\|x\|^2 + t\|y\|^2 - ct(1-t)\|x-y\|^2 \tag{3.2}$$

for all $x,\ y \in X$ and $0 < t < 1$, where c is as in (3.1).

Proof. The inequality was proved in [15] for an abstract L^p-space X with $1 < p \leq 2$ and $c = p-1$. However, the proof applies without any change to our more general case. ∎

Theorem 3.1. *Let X be a 2-convex normed linear space. Then we have*

$$k(X) \leq \frac{2}{c+1},$$

where c is as in (3.1).

Proof. By the Thele formula [16], we have

$$k(X) = \sup\left\{\frac{1}{\|y-\lambda x\|} : x, y \in X,\ \|x\| = \|y\| = 1,\ x \perp y,\ \lambda \in \mathbb{R}\right\},$$

where $x \perp y$ means that the distance $dist(x, \hat{y})$ of x to the one-dimensional subspace $\hat{y} = span\{y\}$ spanned by y is equal to 1. Therefore, the Thele formula can be rewritten in the form

$$(3.3)\qquad k(X) = \sup\left\{\frac{1}{dist(y,\hat{x})} : x, y \in X,\ \|x\| = \|y\| = dist(x,\hat{y}) = 1\right\},$$

where $\hat{x} = span\{x\}$. Now, suppose that $x,\ y \in X$ and $\|x\| = \|y\| = dist(x, \hat{y}) = 1$. Next, insert $y = x - z$ into (3.2) and use $\|x\| = 1$ to get

$$t \leq t\|x-z\|^2 - ct(1-t)\|z\|^2 - \left(\|x-tz\|^2 - \|x\|^2\right).$$

Dividing this inequality by t and letting $t \to 0$, we obtain

$$1 \leq \|x-z\|^2 - c\|z\|^2 - 2\tau(x,-z).$$

Since $0 \in \hat{y}$ and $\|x\| = 1$, we conclude that $m = 0$ is a best approximation in $\hat{y}$ to x. Hence we get $\tau(x,-z) \geq 0$ and

$$(3.4)\qquad 1 \leq \|x-z\|^2 - c\|z\|^2$$

for all $z \in \hat{y}$. Now, suppose additionally that the best approximation to y in $\hat{x}$ is equal to βx with $\beta \neq 0$. Then it follows from (3.4) that

$$d^2 = \|y-\beta x\|^2 = |\beta|^2\left\|x - \frac{y}{\beta}\right\|^2 \geq |\beta|^2\left(1 + c\left\|\frac{y}{\beta}\right\|^2\right) = |\beta|^2 + c,$$

where $d = dist(y, \hat{x})$. On the other hand, we have

$$|\beta| = \|\beta x\| \geq \|y\| - \|y - \beta x\| = 1 - d.$$

Therefore, we get

$$d^2 \geq (1-d)^2 + c,$$

which yields

$$\frac{1}{d} = \frac{1}{dist(y,\hat{x})} \leq \frac{2}{c+1}. \tag{3.5}$$

Note that this inequality is also true when $\beta = 0$, which follows directly from the fact that

$$d = 1 \geq \frac{c+1}{2}$$

in this case. Hence one can take the supremum in (3.5) to finish the proof. ■

The theorem yields the following estimate for best Lipschitz constants of Banach spaces $L^p = L^p(\Omega, \Sigma, \mu)$, where (Ω, Σ, μ) is a positive measure space.

Corollary 3.1. *The estimate*

$$k(L^p) \leq \frac{2}{p} \max\{p-1, 1\}$$

holds, whenever $1 < p < \infty$.

Proof. The best constant $c = c(L^p)$ in (3.1) is equal to $p-1$, whenever $1 < p \leq 2$ [15]. Hence Theorem 3.1 gives

$$k(L^p) \leq 2/p.$$

If $p > 2$, then we can apply the Franchetti identity $k(X) = k(X^*)$ [5] and the last inequality to get

$$k(L^p) = k(L^{p/(p-1)}) \leq \frac{2(p-1)}{p},$$

which completes the proof. ■

Note that the estimate of $k(L^p)$ is exact, whenever $p = 2$, and that it is asymptotically sharp as $p \to 1$ and $p \to \infty$.

4. Best Lipschitz constants for L^p. In this section we derive $k(L^p)$ for the real Banach spaces $L^p = L^p(\Omega, \Sigma, \mu)$, whenever $1 < p < \infty$ and (Ω, Σ, μ) is a positive measure space. By usual isometric embeddings [11], it follows that the assumption - L^p is over the real field - does not restrict the generality. Since $k(X) = 1$ for each space X of dimension 1, it will be also assumed below that the dimension of L^p is greater than 1, which is equivalent to the existence of disjoint measurable sets A and B in Ω such that $A \cup B = \Omega$ and $\mu(A)\mu(B) > 0$. The main result of this section is included in the following theorem.

Theorem 4.1. *If* $1 < p < \infty$, *then*

$$k(L^p) = \max_{0 \leq t \leq 1} \left[t^{p-1} + (1-t)^{p-1}\right]^{1/p} \left[t^{1/(p-1)} + (1-t)^{1/(p-1)}\right]^{(p-1)/p}.$$

For the proof of Theorem 4.1, we need the following results about best Lipschitz constants $k(l_n^p)$ of the Banach spaces l_n^p which consists of all real n-tuples $x = (x_1, \ldots, x_n)$ equipped with the norm

$$\|x\| = \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p} .$$

Lemma 4.1. *The inequality*

$$k(L^p) \geq k(l_2^p)$$

holds for each space $L^p = L^p(\Omega, \Sigma, \mu)$.

Proof. Choose disjoint measurable sets A and B such that $A \cup B = \Omega$ and $\mu(A)\mu(B) > 0$, and define the subspace

$$M = \{\alpha\chi_A/[\mu(A)]^{1/p} + \beta\chi_B/[\mu(B)]^{1/p} : \alpha, \beta \in \mathbb{R}\}$$

of L^p. Since we have

$$\|x\| = (|\alpha|^p + |\beta|^p)^{1/p}$$

for every $x \in M$, it follows that M is isometrically isomorphic to l_2^p. Hence Thele's formula (3.3) yields

$$k(L^p) \geq \sup\left\{\frac{1}{dist(y, \hat{x})} : x, y \in M,\ \|x\| = \|y\| = dist(x, \hat{y}) = 1\right\} = k(l_2^p),$$

which completes the proof. ■

Lemma 4.2. *The functions*

$$g(z) = |z - \lambda|^p + \lambda_2|z|^p + \lambda_3 z$$

and

$$g'(z) = p|z - \lambda|^{p-2}(z - \lambda) + \lambda_2 p|z|^{p-2}z + \lambda_3$$

have at most two common real zeros, whenever $p > 2$, $\lambda \neq 0$ and $\lambda_2 \leq 0$.

Proof. Suppose that g and g' are equal to zero at some points $z_1 < z_2 < z_3$. Then one can apply Rolle's theorem to conclude that the first derivative $g'(z)$ has (at least) five distinct real zeros, and that the second derivative

$$g''(z) = p(p-1)\left(|z - \lambda|^{p-2} + \lambda_2|z|^{p-2}\right)$$

has four distinct zeros t_k. Since $\lambda \neq 0$ and $\lambda_2 \leq 0$, we have $t_k \neq 0$ and

$$\left|\frac{t_k - \lambda}{t_k}\right| = (-\lambda_2)^{1/(p-2)} \qquad (k = 1, 2, 3, 4) .$$

This contradicts the fact that the function $t \to |t-\lambda|/|t|$ has exactly three intervals of the (strict) monotonicity. ■

Lemma 4.3. *The identity*

$$k(l_n^p) = k(l_2^p)$$

holds for all $n \geq 2$ and $p > 2$.

Proof. By Lemma 4.1 we have

$$k(l_n^p) \geq k(l_2^p).$$

To prove the reversed inequality, denote

$$s_n = 1/\big(k(l_n^p)\big)^p$$

and use Thele's formula (3.3) to get

$$s_n = \min\,\{dist^p(y,\hat{x}) : x, y \in l_n^p,\ \|x\| = \|y\| = dist(x,\hat{y}) = 1\}. \tag{4.1}$$

The proof will be completed if we show that $s_n \geq s_2$. For this purpose, suppose that $s_{n-1} \geq s_2$ and $n > 2$. Without loss of generality, we may only take the minimum in (4.1) over all vectors $x = (x_1, \ldots, x_n)$ such that $x_i \neq 0$ for $i = 1, 2, \ldots, n$. Indeed, if the minimum is attained for a vector x with a coordinate x_i equal to zero, then the minimal value of

$$dist^p(y,\hat{x}) = |y_i|^p + \inf_{\lambda\in\mathbb{R}} \sum_{k\neq i} |y_k - \lambda x_k|^p$$

is attained whenever $y_i = 0$. To verify this assertion, one can suppose that $|y_i| \neq 1$ and take $\overline{y} = (y - y_i e_i)/\|y - y_i e_i\|$, where e_i is the ith unit vector. Then we have $\|\overline{y}\| = 1$, $x \perp y$, $x \perp y_i e_i$, $x \perp \overline{y}$, and $dist(\overline{y}, \hat{x}) \leq dist(y, \hat{x})$, which yields our assertion. Thus $s_n = s_{n-1}$ in this case, which finishes our inductive proof. Since $dist(x,\hat{y}) = \|x\| = 1$, it follows that 0 is the best approximation in $\hat{y}$ to x. Consequently, by the characterization [9] of best approximations in l_n^p, the condition $dist(x,\hat{y}) = 1$ in (4.1) is equivalent to

$$\sum_{k=1}^{n} |x_k|^{p-2} x_k y_k = 0. \tag{4.2}$$

Hence (4.1) can be rewritten in the equivalent form

$$s_n = \min_{\lambda\in\mathbb{R}} s_n(\lambda) \tag{4.3}$$

with

$$s_n(\lambda) = \min \sum_{k=1}^{n} \alpha_k |z_k - \lambda|^p, \tag{4.4}$$

where the minimum is taken over all real numbers $z_k = y_k/x_k$ and $\alpha_k = |x_k|^p > 0$ $(k = 1, \ldots, n)$ which satisfy the following conditions:

$$\sum_{k=1}^{n} \alpha_k = 1, \ \sum_{k=1}^{n} \alpha_k |z_k|^p = 1, \ \sum_{k=1}^{n} \alpha_k z_k = 0. \tag{4.5}$$

Now denote by λ, $\alpha = (\alpha_1, \ldots, \alpha_n)$, and $z = (z_1, \ldots, z_n)$ a solution of minimization problem (4.3)-(4.5), and consider the function

$$F(\lambda, \alpha, z) = \sum_{k=1}^{n} \alpha_k |z_k - \lambda|^p + \lambda_1 \Big(\sum_{k=1}^{n} \alpha_k - 1\Big) + \lambda_2 \Big(\sum_{k=1}^{n} \alpha_k |z_k|^p - 1\Big) + \lambda_3 \sum_{k=1}^{n} \alpha_k z_k.$$

Then the Euler equations:

$$\sum_{k=1}^{n} \alpha_k |z_k - \lambda|^{p-2}(z_k - \lambda) = 0, \tag{4.6}$$

$$|z_k - \lambda|^p + \lambda_1 + \lambda_2 |z_k|^p + \lambda_3 z_k = 0, \tag{4.7}$$

$$p\alpha_k |z_k - \lambda|^{p-2}(z_k - \lambda) + \lambda_2 p \alpha_k |z_k|^{p-2} z_k + \lambda_3 \alpha_k = 0 \tag{4.8}$$

hold for $k = 1, \ldots, n$, whenever (λ, α, z) is the solution of (4.3)-(4.5). If we multiply equations (4.7) ((4.8)) by α_k (z_k, resp.) and take the sum of them over k, then we can use (4.3)-(4.5) to get

$$s_n + \lambda_1 + \lambda_2 = 0$$

and

$$p \sum_{k=1}^{n} \alpha_k |z_k - \lambda|^{p-2}(z_k - \lambda)[(z_k - \lambda) + \lambda] + \lambda_2 p = p s_n + \lambda_2 p = 0.$$

Hence $\lambda_2 = -s_n \le 0$ and $\lambda_1 = 0$. This in conjunction with (4.7)-(4.8) and the fact that $\alpha_k > 0$ yields

$$|z_k - \lambda|^p + \lambda_2 |z_k|^p + \lambda_3 z_k = 0$$

and

$$p|z_k - \lambda|^{p-2}(z_k - \lambda) + \lambda_2 p |z_k|^{p-2} z_k + \lambda_3 = 0$$

for $k = 1, \ldots, n$. Since $n > 2$, it follows from Lemma 4.2 that either $\lambda = 0$ or $z_k = z_j$ for some $k \ne j$. In the first case, we have $s_n = s_n(\lambda) = 1$ and $k(l_n^p) = 1$, which leads to the contradiction with $k(l_n^p) > 1$. In the second case, identities (4.4)-(4.5) yield $s_n(\lambda) = s_{n-1}(\lambda)$. By the induction hypothesis, it follows that $s_n = s_{n-1} \ge s_2$, which completes the proof. ■

Lemma 4.4. *If $p > 2$ then*

$$k(l_2^p) = \max_{0 \le t \le 1} \left[t^{p-1} + (1-t)^{p-1}\right]^{1/p} \left[t^{1/(p-1)} + (1-t)^{1/(p-1)}\right]^{(p-1)/p}.$$

Proof. By (4.1) we have

$$k(l_2^p) = \max \frac{1}{dist(y,\hat{x})} = \max \frac{1}{\|y-\lambda x\|} \tag{4.9}$$

for some uniquely determined λ, where the maxima are taken over all $x = (z, s)$ and $y = (u, v)$ in l_2^p with $\|x\| = \|y\| = dist(x, \hat{y}) = 1$ and $z,\ s \neq 0$. It follows from (4.2) that the restriction $dist(x, \hat{y}) = 1$ is equivalent to

$$|z|^{p-2}zu + |s|^{p-2}sv = 0. \tag{4.10}$$

Additionally, by the characterization of best approximations in l_2^p [9], the number λ in (4.9) is the unique solution of the equation

$$z|u-\lambda z|^{p-2}(u-\lambda z) + s|v-\lambda s|^{p-2}(v-\lambda s) = 0. \tag{4.11}$$

Since $z,\ s \neq 0$ and $\|y\| = 1$, it follows from (4.10) that $u,\ v \neq 0$. Further, if $x,\ y$ and λ satisfy (4.10) and (4.11), then the same is true for $a = (-z, \mp s),\ b = (-u, \mp v)$ and λ. Moreover, we have $dist(b, \hat{a}) = \|y - \lambda x\|$. Therefore, we can assume that $z,\ s > 0$. Hence (4.10) yields $uv < 0$. By the symmetry, we can assume that $u < 0$ and $v > 0$. This in conjunction with (4.10) and the identity $\|y\| = 1$ yields

$$\left(\frac{z}{s}\right)^{p-1} = -\frac{v}{u} = -\left[1-(-u)^p\right]^{1/p}/u$$

and

$$u = -\left[1+\left(\frac{z}{s}\right)^{p(p-1)}\right]^{-1/p}.$$

Hence one can use the identity $\|x\| = 1$ to obtain

$$\begin{aligned} zv - su &= u\left(z\frac{v}{u} - s\right) = u\left(-\frac{z^p}{s^{p-1}} - s\right) = -u/s^{p-1} \\ &= \left(z^{p(p-1)} + s^{p(p-1)}\right)^{-1/p}. \end{aligned} \tag{4.12}$$

Since $\|y - \lambda x\| > 0$, it follows from (4.11) that

$$(u-\lambda z)(v-\lambda s) < 0$$

and

$$\frac{u-\lambda z}{\lambda s - v} = r \quad \text{with } r = \left(\frac{s}{z}\right)^{1/(p-1)}.$$

Hence we get

$$\lambda = \frac{u+rv}{z+rs}$$

and

$$dist^p(y,\hat{x}) = |u-\lambda z|^p + |\lambda s - v|^p = (1+r^p)|\lambda s - v|^p$$

$$= (1+r^p)\left|\frac{zv-su}{z+rs}\right|^p = \left(z^{p/(p-1)} + s^{p/(p-1)}\right)^{1-p}|zv-su|^p.$$

This together with (4.12) and the identity $s^p = 1 - z^p := 1 - t$ gives

$$\frac{1}{dist(y,\hat{x})} = \left[t^{p-1} + (1-t)^{p-1}\right]^{1/p}\left[t^{1/(p-1)} + (1-t)^{1/(p-1)}\right]^{(p-1)/p},$$

which completes the proof. ■

Proof of Theorem 4.1. By the Franchetti formula $k(X^*) = k(X)$ [5] and the fact that $k(L^2) = 1$ for the Hilbert space L^2, we can assume that $p > 2$. Therefore, in view of Lemmas 4.1 and 4.3, we have

$$k(l_2^p) = k(l_n^p) \leq k(L^p) \tag{4.13}$$

for every integer $n \geq 2$. For the proof of reversed inequality, suppose that $\epsilon > 0$ and $x,\ y \in L^p = L^p(\Omega, \Sigma, \mu)$. Since the subspace of all simple functions in L^p is dense in L^p, there exist simple functions x_ϵ and y_ϵ such that

$$\|x - x_\epsilon\| \leq \epsilon \quad \text{and} \quad \|y - y_\epsilon\| \leq \epsilon. \tag{4.14}$$

Moreover, we can write these simple functions in the form

$$x_\epsilon = \sum_{k=1}^{n} x_k \chi_{A_k} \quad \text{and} \quad y_\epsilon = \sum_{k=1}^{n} y_k \chi_{A_k},$$

for some integer n, where $x_k,\ y_k \in \mathbb{R}$ and χ_{A_k} are characteristic functions of pairwise disjoint measurable subsets A_k $(k = 1, \dots, n)$ of Ω. Hence x_ϵ and y_ϵ can be identified in the usual way with the elements $\big(x_k \mu(A_k)\big)$ and $\big(y_k \mu(A_k)\big)$ of the space l_n^p. Consequently, we obtain

$$\|Rx_\epsilon - Ry_\epsilon\| \leq k(l_n^p)\|x_\epsilon - y_\epsilon\|.$$

This in conjunction with (4.13)-(4.14) and inequality $k(X) \leq 2$ yields

$$\|Rx - Ry\| \leq \|Rx - Rx_\epsilon\| + k(l_n^p)\|x_\epsilon - y_\epsilon\| + \|Ry_\epsilon - Ry\|$$

$$\leq 4\epsilon + k(l_n^p)(\|x_\epsilon - x\| + \|x - y\| + \|y - y_\epsilon\|)$$

$$\leq 8\epsilon + k(l_2^p)\|x - y\|.$$

Letting $\epsilon \to 0$, we get

$$k(L^p) \leq k(l_2^p).$$

Hence one can apply (4.13) and Lemma 4.4 to finish the proof. ■

An exact computation of the maximal value of the function

$$h_p(t) = \left[t^{p-1} + (1-t)^{p-1}\right]^{1/p}\left[t^{1/(p-1)} + (1-t)^{1/(p-1)}\right]^{(p-1)/p},\ 0 \leq t \leq 1,$$

occurring in Theorem 4.1, seems to be a hard problem except for a few values of p. More precisely, it is easy to compute that $k(L^2) = 1$ and

$$k(L^{3/2}) = k(L^3) = \frac{1}{3}(7\sqrt{7}+17)^{1/3}.$$

For example, if $p = 3/2$ then the function

$$h_{3/2}(t) = \left\{\left[1+2\sqrt{t(1-t)}\right]\left[1-2t(1-t)\right]\right\}^{1/3}, \quad 0 \le t \le 1,$$

attains its maximum at the point

$$t_{3/2} = \frac{3-\sqrt{1+2\sqrt{7}}}{6}.$$

Corollary 4.1. *If* $1 < p < \infty$ *then*

$$h_p(t_p) \le k(L^p) \le 2^{|p-2|/p},$$

where

$$t_p = \begin{cases} 0.08345\left[1-(2-p)^{5.83}\right], & \text{if } 1 < p \le 2 \\ t_{p/(p-1)}, & \text{otherwise.} \end{cases}$$

Proof. Since $t_p \in (0,1)$, the lower estimate is a direct consequence of Theorem 4.1. Further, if $1 < p \le 2$ then maximal values of the functions

$$f(t) = t^{p-1} + (1-t)^{p-1} \quad \text{and} \quad g(t) = t^{1/(p-1)} + (1-t)^{1/(p-1)}, \ 0 \le t \le 1,$$

are attained at the points $t = 1/2$ and $t = 0$, respectively. In the case $p > 2$, the same is true for the points $t = 0$ and $t = 1/2$. Hence by Theorem 4.1 we get

$$k(L^p) \le \max\left\{f^{1/p}(1/2)\,g^{(p-1)/p}(0),\ f^{1/p}(0)\,g^{(p-1)/p}(1/2)\right\} = 2^{|p-2|/p},$$

which completes the proof. ■

Note that estimates given in Corollary 4.1 are exact in the case $p = 2$, and that they are asymptotically sharp as $p \to 1$ and $p \to \infty$. Moreover, the lower estimate $h_p(t_p)$ is much more exact than the upper estimate $2^{|p-2|/p}$. In fact, the numerical experiments show that

$$|h_p(t_p) - k(L_p)| \le 4 * 10^{-6}. \tag{4.15}$$

For example, if $p = 3/2$ then $h_{3/2}(t_{3/2}) = 1.0957314\cdots$ and $k(L^{3/2}) = 1.0957314\cdots$. Moreover, the upper estimate $2^{|p-2|/p}$ is better than the estimate $(2/p)max\ \{p-1, 1\}$ from Corollary 3.1. In Fig.1, we present the graphs of these estimates in the case $1 \le p \le 2$. By (4.15) the graphs of $h_p(t_p)$ and $k(L^p)$ can not be distinguished at the picture.

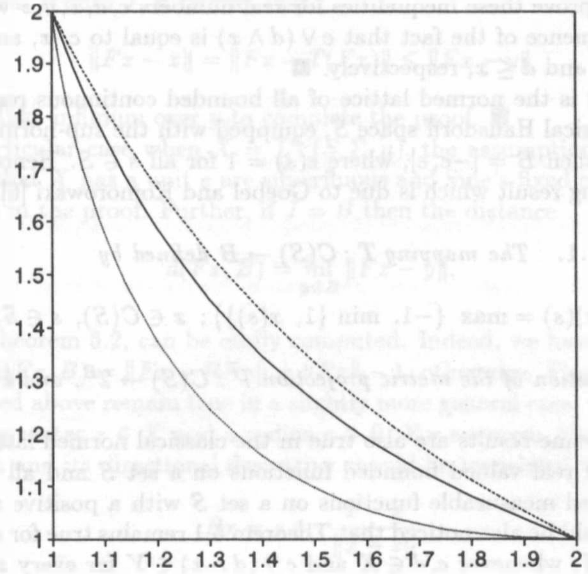

Fig. 1. Estimates $h_p(t_p)$ (dotted), $2^{(2-p)/p}$ (solid) and $2/p$ (dashed) of $k(L^p)$.

5. Optimal selections . Let X be a normed lattice with an order $\leq$ and lattice operatons $\vee$ and $\wedge$, and let $|x| = (x \vee 0) + (x \wedge 0)$ denote the absolute value in X [11]. Moreover, let

$$J = [c, d] := \{x \in X : c \leq x \leq d\}$$

be an *order interval* with endpoints $c, d \in X$ such that $c \leq d$. Replacing the unit ball B of X by J, we define the metric projection $\mathcal{P} : X \to 2^J$, the best Lipschitz constant $k_P(X)$ of a selection P of $\mathcal{P}$, the optimal Lipschitz constant $k_o(X)$, and the optimal selection T of $\mathcal{P}$ as in Section 1.

Theorem 5.1. *Let $J = [c, d]$ be an order interval in a normed lattice X. Then the mapping $T : X \to J$ defined by*

$$Tx = c \vee (d \wedge x), \ \ x \in X,$$

is an optimal selection of the metric projection $\mathcal{P} : X \to 2^J$. Moreover, we have $k_T(X) = k_o(X) = 1$.

Proof. In a Banach lattice X, we have $\|x\| \leq \|y\|$ whenever $|x| \leq |y|$. Hence we have to show that

$$|x - c \vee (d \wedge x)| \leq |x - z|$$

and

$$|c \vee (d \wedge x) - c \vee (d \wedge y)| \leq |x - y|$$

for all $x, y \in X$ and $z \in J$. By Yudin's principle of invariance of relations [8, p.279], it is sufficient to prove these inequalities for real numbers c, d, x, y, z with $c \leq z \leq d$, which is a consequence of the fact that $c \vee (d \wedge x)$ is equal to c, x, and d, whenever $x \leq c$, $c \leq x \leq d$, and $d \leq x$, respectively. ■

If $X = C(S)$ is the normed lattice of all bounded continuous real valued functions on a topological Hausdorff space S, equipped with the sup-norm and the usual pointwise order, then $B = [-e, e]$, where $e(s) = 1$ for all $s \in S$. Hence Theorem 5.1 yields the following result which is due to Goebel and Komorowski [6].

Corollary 5.1. *The mapping $T : C(S) \to B$ defined by*

$$(Tx)(s) = \max \{-1, \min \{1, x(s)\}\}; \ x \in C(S), \ s \in S,$$

is an optimal selection of the metric projection $\mathcal{P} : C(S) \to 2^B$, and $k_T(C(S)) = 1$.

Clearly, the same results are also true in the classical normed lattices $B(S)$ and $L^\infty(S, \Sigma, \mu)$ of all real valued bounded functions on a set S and all real valued μ-essentially bounded measurable functions on a set S with a positive measure μ, respectively. It should be also noticed that Theorem 5.1 remains true for each sublattice Y of the lattice X, whenever $c, d \in X$ and $c \vee (d \wedge x) \in Y$ for every $x \in Y$. For example, let $C_o(S)$ be the sublattice of $C(S)$, which consists of all $x \in C(S)$ such that the inequality $|x(s)| \leq \varepsilon$ holds for each $\varepsilon > 0$ and for all s outside a compact subset $Q \subset S$ dependent on x and ε. Then we get

Corollary 5.2. *The mapping $T : C_o(S) \to B$ defined by*

$$(Tx)(s) = \max \{-1, \min \{1, x(s)\}\}; \ x \in C_o(S), \ s \in S,$$

is an optimal selection of the metric projection $\mathcal{P}$ from $C_o(S)$ into its unit ball B, and $k_T(C_o(S)) = 1$.

As a final application of Theorem 5.1, we prove Ky Fan's *approximation principle* [3] for nonexpansive mappings F defined on an order interval J in $L^\infty(S, \Sigma, \mu)$ (see [10] for related results and related references). For this purpose, recall that a mapping $F : J \to X$ is said to be nonexpansive if $\|Fx - Fy\| \leq \|x - y\|$ for all $x, y \in J$.

Theorem 5.2. *Let J be an order complete order interval in an abstract M-space X with a unit e, and let $F : J \to X$ be a nonexpansive mapping. Then there exists an element $x \in J$ such that*

$$\|Fx - x\| = \inf_{y \in J} \|Fx - y\|.$$

Proof. Since X is order isometric to $C(Q)$ for some compact Hausdorff space Q [11, p.16], it follows from Theorem 5.1 that there exists an optimal selection T of the metric projection $\mathcal{P} : X \to 2^J$ with $k_T(X) = 1$. Hence the mapping $TF : J \to J$ is nonexpansive. Therefore, one can apply Borwein-Sims's fixed point theorem [1, Theorem 7.1] to get a point $x \in J$ such that $TFx = x$. Since T is a selection of the metric projection $\mathcal{P}$ onto J, it follows that

$$\|Fx - x\| = \|Fx - T(Fx)\| \le \|Fx - y\|$$

for all $y \in J$. Take infimum over y to complete the proof. ■

In the particular case when $X = L^\infty(S, \Sigma, \mu)$, the assumptions that J is order complete and that X has a unit e are superfluous and Sine's fixed point theorem [14] can be applied in the proof. Further, if $J = B$ then the distance

$$d(Fx, B) = \inf_{y \in B} \|Fx - y\|,$$

occurring in Theorem 5.2, can be easily computed. Indeed, we have $d(Fx, B) = 0$, if $Fx \in B$, and $d(Fx, B) = \|Fx - RFx\| = \|Fx\| - 1$, otherwise. Finally, note that the results presented above remain true in a slightly more general case, when $B = B(z, r)$ is a ball with a center $z \in X$ and a radius $r > 0$. For example, the formulae for the radial selection and its directional derivative should be translated to

$$Rx = z + r\frac{x - z}{\|x - z\|}$$

and

$$R'(x)h = \frac{rh - \tau(x - z, h)(Rx - z)}{\|x - z\|},$$

whenever $x \notin B$ and $h \in X$.

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