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On monotonity-preserving transformations

O przekształceniach zachowujących monotoniczność

О преобрязованиях сохраняющих монотонность

0. Introduction. Many results have been obtained about conditions which are to be satisfied by a transformation of sequence or function if a certain property should be invariant under the transformation considered. The theorem of Toeplitz about sequence-to-sequence transformations preserving the limit and the related theorems on series-to-sequence and sequence-to-function transformations are well known examples for conditions of this kind.

This paper deals with linear transformations (sequence-to-function, function-to-function) preserving the monotonity. E. g. given the linear function-to-function transformation defined by a kernel: $\Phi(t)=\int_0^{+\infty} K(t,s)\varphi(s)ds$, what conditions are to be satisfied by the kernel $K(t,s)$, if the transform $\Phi(t)$ of any increasing function for which the transformation applies, should be an increasing function, too. We obtain here necessary and sufficient conditions which are proved by using some lemmas on series and infinite integrals that seem to be new and may be of interest. A theorem close connected with these lemmas, being a generalisation of a result due to Ch.-J. de la Vallée Poussin and its integral analogue are also proved. Some applications of the obtained results to the theory of summability are given.

The term «monotonic function» means in the sequel that $t_1 < t_2$ implies $\varphi(t_1) \leq \varphi(t_2)$ (increasing function), or $t_1 < t_2$ implies $\varphi(t_1) \geq \varphi(t_2)$ (decreasing function). If the sign of equality is excluded the function will be called «strictly monotonic» (strictly increasing resp. strictly decreasing). Monotonic (strictly monotonic) sequences are defined analogically. All sequences, functions, matrices considered are supposed to be real.

1. Let $\varphi = \{\varphi_n(t)\}$ be a sequence of real functions of a real variable t defined for $t \in D$. The set D can be a continuum or a denumerable or even a finite set but, anyhow, we suppose that D contains at least two numbers. The sequence $\{\varphi_n(t)\}$ defines on a certain set X_φ of the space of all sequences a sequence-to-function transformation φ. It means that for each $x = \{\xi_n\} \in X_\varphi$ the series $\sum_k \varphi_k(t)\xi_k$ is convergent for any $t \in D$. The sum $\sum_{k=1}^{\infty} \varphi_k(t)\xi_k$ shall be denoted $\Phi(t, x)$, or sometimes $\Phi(t)$. In the sequel X_φ is understood to be the greatest set of this kind and shall be called the field of applicability of the transformation φ. X_φ is clearly non-empty and its range depends on D, generally a diminution of D implies an enlargement of X_φ.

We say that the transformation φ preserves the monotonity if for each monotonic sequence $x = \{\xi_n\} \in X_\varphi$ its transform $\Phi(t, x)$ is a function of t monotonic in D, and if $\Phi(t, x)$ and $\{\xi_n\}$ both increase, or both decrease.

We shall now obtain necessary conditions that φ be a transformation preserving the monotonity.

The sequence $x^{(1)} = \{1, 0, 0, \ldots\}$ is a decreasing sequence which belongs to X_φ. Hence $\Phi(t, x^{(1)}) = \varphi_1(t)$ decreases for $t \in D$. Similarly the sequences $x^{(r)} = \{\underbrace{1, 1, \ldots, 1}_{r}, 0, 0, \ldots\}$ decrease and thus $\Phi(t, x^{(r)}) = \sum_{k=1}^{r} \varphi_k(t)$ decreases for $t \in D$ and fixed r, $r = 1, 2, \ldots$:

$$\sum_{k=1}^{r} \varphi_k(t_1) \geqslant \sum_{k=1}^{r} \varphi_k(t_2) \quad \text{for any } t_1 < t_2,\ t_1, t_2 \in D \text{ and } r = 1, 2, \ldots \tag{1.11}$$

The condition (1.11) is also sufficient for preserving of monotonity if every monotonic sequence contained in X_φ is a null sequence (i. e. has zero as a limit).

Proof. Let $\{\xi_n\} \in X_\varphi$ be a decreasing sequence; $\xi_n \to 0$. Put

$$\Phi_k(t) = \varphi_1(t) + \varphi_2(t) + \ldots + \varphi_k(t);$$

$$\Phi(t) = \lim_{k \to +\infty} [\varphi_1(t)\xi_1 + \ldots + \varphi_k(t)\xi_k] =$$

$$= \lim_{k \to +\infty} \{\Phi_1(t)\xi_1 + [\Phi_2(t) - \Phi_1(t)]\xi_2 + \ldots + [\Phi_k(t) - \Phi_{k-1}(t)]\xi_k\} =$$

$$= \lim_{k \to +\infty} [\Phi_1(t)(\xi_1 - \xi_2) + \ldots + \Phi_{k-1}(t)(\xi_{k-1} - \xi_k) + \Phi_k(t)\xi_k].$$

Therefore

$$\Phi(t_2) - \Phi(t_1) = \lim_{k \to +\infty} \{[\Phi_1(t_2) - \Phi_1(t_1)](\xi_1 - \xi_2) +$$
$$+ \ldots + [\Phi_{k-1}(t_2) - \Phi_{k-1}(t_1)](\xi_{k-1} - \xi_k) + [\Phi_k(t_2) - \Phi_k(t_1)]\xi_k\} \leqslant 0$$

if $t_1 < t_2$, since $\xi_{k-1} - \xi_k \geqslant 0$, $\xi_k \geqslant 0$, and $\Phi_k(t_2) - \Phi_k(t_1) \leqslant 0$ by (1.11). This means that $\Phi(t)$ also decreases for $t \in D$.

We have proved following

Theorem 1.1. *In order that the transformation φ preserve the monotonity of all monotonic sequence of X_φ, if each monotonic sequence in X_φ is a null sequence, it is necessary and sufficient that*

$$(1.11) = (3.11) \qquad \sum_{k=1}^{r} \varphi_k(t_1) \geqslant \sum_{k=1}^{r} \varphi_k(t_2) \text{ for } t_1, t_2 \in D,\ t_1 < t_2 \text{ and } r = 1, 2, \ldots$$

Let us now suppose that X_φ contains at least one monotonic sequence which does not converge to zero. From the lemma 2.1, or by using the well known Dirichlet's test of convergence *), it follows that the series $\sum_k \varphi_k(t)$ are convergent for any $t \in D$. This means that the sequence $x^{(\infty)} = \{1, 1, 1, \ldots\}$ also belongs to X_φ and we have $\Phi(t, x^{(\infty)}) = \sum_{k=1}^{\infty} \varphi_k(t)$. Since $x^{(\infty)}$ is simultaneously decreasing and increasing, therefore $t_1 < t_2$ implies $\sum_{k=1}^{\infty} \varphi_k(t_1) \geqslant \sum_{k=1}^{\infty} \varphi_k(t_2)$ and $\sum_{k=1}^{\infty} \varphi_k(t_1) \leqslant \sum_{k=1}^{\infty} \varphi_k(t_2)$, i. e.

$$(3.12) \qquad \sum_{k=1}^{\infty} \varphi_k(t) = a = \text{const. for } t \in D.$$

The conditions (3.11) and (3.12) are necessary for preserving of monotonity in sequence-to-function transformations if the set X_φ contains at least one monotonic non-null sequence. These conditions are also sufficient. In order to prove the sufficiency, we need two lemmas on series. In the next section we shall prove them and this helps us to establish the sufficiency of conditions obtained above. Besides, we prove a generalisation of a result due to Ch.-J. de la Vallée Poussin.

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*) If $\sum_k \varphi_k \xi_k$ converges and $\{\xi_n\}$ is monotonic and does not tend to zero, then $\left\{\frac{1}{\xi_n} - \lim_{n\to+\infty} \frac{1}{\xi_n}\right\}$ tends monotonically to zero. Therefore $\sum_k \varphi_k \xi_k \left(\frac{1}{\xi_k} - \lim_{n\to+\infty} \frac{1}{\xi_n}\right)$ is convergent and thus also $\sum_k \varphi_k$ is convergent.

2. Lemma 2.1. *If* $\sum_k b_k U_k$ *converges and* $\{U_n\}$ *is a monotonic sequence which has not zero as a limit, then also* $\sum_k b_k$ *converges and* $\lim_{n\to\infty} U_n(B-B_n)=0$, *where* $B_n=\sum_{k=1}^{n} b_k$, $B=\sum_{k=1}^{\infty} b_k$.

Proof. Put $R_n = b_{n+1}U_{n+1}+b_{n+2}U_{n+2}\ldots$; let us suppose that $\{U_n\}$ is an increasing sequence and that $U_n>0$ for $n\geqslant N$. Then, for $n\geqslant N$

$$b_{n+1}=\frac{R_n-R_{n+1}}{U_{n+1}},\quad \sum_{k=n+1}^{n+p} b_k=\frac{R_n-R_{n+1}}{U_{n+1}}+\frac{R_{n+1}-R_{n+2}}{U_{n+2}}+$$

$$+\ldots+\frac{R_{n+p-1}-R_{n+p}}{U_{n+p}}=R_n\frac{1}{U_{n+1}}+R_{n+1}\left(\frac{1}{U_{n+2}}-\frac{1}{U_{n+1}}\right)+$$

$$+\ldots+R_{n+p-1}\left(\frac{1}{U_{n+p}}-\frac{1}{U_{n+p-1}}\right)+R_{n+p}\cdot\frac{-1}{U_{n+p}},$$

hence

$$\left|\sum_{k=n+1}^{n+p} b_k\right|\leqslant \sup_{p=0,1,\ldots}|R_{n+p}|\cdot\left[\frac{1}{U_{n+1}}+\left(\frac{1}{U_{n+1}}-\frac{1}{U_{n+2}}\right)+\ldots\right.$$

$$\left.\ldots+\left(\frac{1}{U_{n+p-1}}-\frac{1}{U_{n+p}}\right)+\frac{1}{U_{n+p}}\right]=\frac{2}{U_{n+1}}\sup_{p=0,1,\ldots}|R_{n+p}|.$$

This inequality implies, in view of $\lim_{n\to+\infty}\left(\sup_{p=0,1,\ldots}|R_{n+p}|\right)=0$, the convergence of $\sum b_k$.

We have also, making $p\to+\infty$,

$$U_n|B-B_n|\leqslant U_{n+1}|B-B_n|\leqslant 2\sup_{p=0,1,\ldots}|R_{n+p}|$$

which proves the lemma in this case.

If $\{U_n\}$ decreases and $\lim_{n\to+\infty} U_n=U>0$, then

$$\left|\sum_{k=n+1}^{n+p} b_k\right|\leqslant|R_n|\frac{1}{U_{n+1}}+|R_{n+1}|\left(\frac{1}{U_{n+2}}-\frac{1}{U_{n+1}}\right)+\ldots$$

$$\ldots+|R_{n+p-1}|\left(\frac{1}{U_{n+p}}-\frac{1}{U_{n+p-1}}\right)+|R_{n+p}|\frac{1}{U_{n+p}}\leqslant$$

$$\leqslant\frac{2}{U_{n+p}}\sup_{p=0,1,\ldots}|R_{n+p}|\leqslant\frac{2}{U}\sup_{p=0,1,\ldots}|R_{n+p}|.$$

Hence we deduce easily the validity of lemma.

The case $U_n < 0$ for all sufficiently great integers n can be settled by a simple change of sign.

From the lemma 2.1 we get at once the

Corollary. *If* $\sum_k b_k U_k$ *and* $\sum_k b_k$ *both converge and* $\{U_n\}$ *is a monotonic sequence, then*

$$\lim_{n \to +\infty} U_n (B - B_n) = 0, \quad \left(B_n = \sum_{k=1}^{n} b_k, \; B = \sum_{k=1}^{\infty} b_k\right).$$

Using this corollary, we obtain easily the

Lemma 2.2. *If* $\sum_k b_k U_k$ *and* $\sum_k b_k$ *both converge and* $\{U_n\}$ *is a monotonic sequence,* $u_1 = U_1$, $u_{k+1} = U_{k+1} - U_k$, *then also* $\sum_k u_k (B - B_{k-1})$ *converges and*

$$\sum_{k=1}^{\infty} b_k U_k = \sum_{k=1}^{\infty} u_k (B - B_{k-1}),$$

(*where* $B_0 = 0$, $\quad B_n = \sum_{k=1}^{n} b_k$ *for* $n = 1, 2, \ldots,$ $\quad B = \sum_{k=1}^{\infty} b_k$).

Proof.

$$\sum_{k=1}^{n} b_k U_k = b_1 u_1 + b_2 (u_1 + u_2) + \ldots + b_n (u_1 + u_2 + \ldots + u_n) =$$

$$u_1 B_n + u_2 (B_n - B_1) + \ldots + u_n (B_n - B_{n-1}) = B_n U_n - u_2 B_1 - u_3 B_2 - \ldots$$

$$\ldots - u_n B_{n-1} = B_n U_n - B U_n + B (u_1 + u_2 + \ldots + u_n) - u_2 B_1 - \ldots - u_n B_{n-1} =$$

$$= U_n (B_n - B) + u_1 (B - B_0) + u_2 (B - B_1) + \ldots + u_n (B - B_{n-1}).$$

Therefore

$$\sum_{k=1}^{n} b_k U_k + U_n (B - B_n) = \sum_{k=1}^{n} u_k (B - B_{k-1})$$

and this proves, by corollary, the lemma.

The lemma 2.1 means that, if $\{U_n\}$ is a monotonic non-null sequence, then the convergence of $\sum_k b_k U_k$ implies

$$\lim_{n \to +\infty} U_n (b_{n+1} + b_{n+2} + \ldots) = 0.$$

We can prove quite similarly an analogous

Lemma 2.3. *If $\{U_n\}$ is a monotonic null sequence then the convergence of $\sum_k b_k U_k$ implies*

$$\lim_{n\to+\infty} U_n(b_1+b_2+\ldots+b_n)=0.$$

Proof. We can suppose without loss of generality that $U_n>0$ for the case $U_n=0$ is trivial. Using the same notations as above, we have

$$b_n=\frac{R_n-R_{n+1}}{U_n},\quad |b_n+b_{n+1}+\ldots+b_{n+p}|\leqslant|R_n|\frac{1}{U_n}+$$

$$+|R_{n+1}|\left(\frac{1}{U_{n+1}}-\frac{1}{U_n}\right)+\ldots+|R_{n+p}|\left(\frac{1}{U_{n+p}}-\frac{1}{U_{n+p-1}}\right)+$$

$$+|R_{n+p+1}|\cdot\frac{1}{U_{n+p}}\leqslant\frac{2}{U_{n+p}}\sup_{k=0,1,\ldots}|R_{n+k}|.$$

Hence

$$|b_n+b_{n+1}+\ldots+b_{n+p}|\cdot U_{n+p}\leqslant 2\sup_{k=0,1,\ldots}|R_{n+k}|.$$

Choose now N such that $2\sup\limits_{k=0,1,\ldots}|R_{N+k}|<\varepsilon/2$, $\varepsilon>0$ arbitrary, (which is possible by convergence of $\sum_k b_k U_k$) and, N being fixed, choose k_0 so that

$$|b_1+b_2+\ldots+b_{N-1}|\cdot U_{N+k}<\varepsilon/2 \quad\text{for each}\quad k>k_0$$

(which is possible,since $U_n\to 0$). Then

$$|b_1+b_2\ldots+b_{N+k}|\cdot U_{N+k}\leqslant(|b_1+b_2+\ldots+b_{N-1}|+$$

$$+|b_N+\ldots+b_{N+k}|)\cdot U_{N+k}<\varepsilon \quad\text{for each}\quad k>k_0$$

and this proves our statement, being a generalisation of a theorem due to de la Vallée Poussin (see [6], p. 416, ex. 10). Putting $b_n=\pm 1$, we obtain a result due to E. Lasker (see Pólya-Szegö [3] *), p. 25). Besides, it is easy to see that the lemma 2.3 is equivalent to a well known result due to Kronecker.

3. We prove now the sufficiency of conditions (3.11) and (3.12). Let $u=\{U_n\}\in X_\varphi$. We can suppose that $\{U_n\}$ increases. Then, in view of lemma 2.2

*) Numbers in square brackets refer to the list at the end of this paper.

$$\Phi(t,u)=\sum_{k=1}^{\infty}\varphi_k(t)U_k=\sum_{k=1}^{\infty}\varphi_k(t)(u_1+u_2+\ldots+u_k)=$$

$$=\sum_{k=1}^{\infty}u_k\left[\sum_{n=1}^{\infty}\varphi_n(t)-\sum_{n=1}^{n-1}\varphi_n(t)\right]=\sum_{k=1}^{\infty}u_k\left[a-\sum_{n=1}^{k-1}\varphi_n(t)\right].$$

Therefore, if $t_1<t_2$, $t_1,t_2\in D$,

$$\Phi(t_2)-\Phi(t_1)=\sum_{k=1}^{\infty}u_k\left[a-\sum_{n=1}^{k-1}\varphi_n(t_2)\right]-\sum_{k=1}^{\infty}u_k\left[a-\sum_{n=1}^{k-1}\varphi_n(t_1)\right]=$$

$$=\sum_{k=2}^{\infty}u_k\left[\sum_{n=1}^{k-1}\varphi_n(t_1)-\sum_{n=1}^{k-1}\varphi_n(t_2)\right]\geqslant 0,$$

since $u_k\geqslant 0$ for $k\geqslant 2$, and by (3.12) (the sum $\sum_{n=1}^{k-1}\varphi_n(t)$ means 0 if $k=1$).

Thus we obtain

Theorem 3.1. *In order that the transformation φ preserve the monotonity of all monotonic sequences in X_φ, if X_φ contains at least one monotonic non-null sequence, it is necessary and sufficient that*

(3.11) $$\sum_{k=1}^{r}\varphi_k(t_1)\geqslant\sum_{k=1}^{r}\varphi_k(t_2) \text{ for } t_1<t_2,\ t_1,t_2\in D \text{ and } r=1,2,\ldots, \text{ and}$$

(3.12) $$\sum_{k=1}^{\infty}\varphi_k(t)=a=\text{const. for each } t\in D.$$

We dealt above with sequence-to-function transformations. The case of linear sequence-to-sequence transformation defined by an infinite matrix is evidently contained in the foregoing considerations. It suffices only to take as the set D the set of all integers.

If we should deal with transformations preserving strict monotonity, we would easily obtain that the condition (3.11) with the omitted sing of equality and condition (3.12) are sufficient for preserving of strict monotonity but the so modified condition (3.11) is by no means necessary. The necessity of (3.12) can be obtained easily by considering of transforms of the strictly monotonic sequences $\{1\pm\varepsilon^n\}$, where $0<\varepsilon<1$. Then

$$\sum_{k=1}^{\infty} \varphi_k(t_1)(1+\varepsilon^k) > \sum_{k=1}^{\infty} \varphi_k(t_2)(1+\varepsilon^k),$$

$$\sum_{k=1}^{\infty} \varphi_k(t_1)(1-\varepsilon^k) < \sum_{k=1}^{\infty} \varphi_k(t_2)(1-\varepsilon^k) \quad (t_1 < t_2).$$

Making $\varepsilon \to 0$, we obtain necessity of (3.12) by continuity of power series. It follows then from the equality

$$\Phi(t_2) - \Phi(t_1) = \sum_{k=2}^{\infty} u_k \left[\sum_{n=1}^{k-1} \varphi_n(t_1) - \sum_{n=1}^{k-1} \varphi_n(t_2) \right]$$

that the condition (3.11) shall be replaced by a more complicated one:

if $t_1, t_2 \in D$, $t_1 < t_2$, then

(3.13) $$\sum_{n=1}^{r} \varphi_n(t_1) \geqslant \sum_{n=1}^{r} \varphi_n(t_2) \text{ for } r = 1, 2, \dots \text{ and}$$

$$\sum_{n=1}^{r_0} \varphi_n(t_1) > \sum_{n=1}^{r_0} \varphi_n(t_2) \text{ for one at least } r_0 = r_0(t_1, t_2).$$

The conditions (3.12) and (3.13) are necessary and sufficient for preserving of strict monotonity if X_φ contains monotonic sequences which do not tend to zero.

We can quite similarly modify the theorem 1.1.

The condition (3.13) is necessary and sufficient for preserving of strict monotonity if each monotonic sequence in X_φ is a null sequence.

4. We shall now deal with conditions for preserving of monotonity in linear function-to-function transformations. Let $K(t,s)$ be a real function of two real variables t and s defined for $t \in D$, (D being an arbitrary set of reals, containing at least two numbers), and for $0 \leqslant s < +\infty$. Besides, we suppose that $K(t,s)$ is summable in s over any finite interval $[0, A]$ for each fixed $t \in D$. The kernel $K(t,s)$ defines on a set X_K of functions summable over any finite interval $[0, A]$ a function-to-function transformation K. This means that for each $\varphi(t) \in X_K$ the integral $\Phi(t) = \int_0^{+\infty} K(t,s)\varphi(s)\,ds$ exists for each $t \in D$. In the sequel the symbol $\int_0^{+\infty}$

means the limit $\lim\limits_{A \to +\infty} \int\limits_0^A$ so that we do not assume the absolute intergrability over the infinite interval of function considered. For absolutely integrable functions our considerations hold naturally, too. The non-empty set of all functions for which the transformation applies shall be denoted X_K.

We say that transformation K preserves the monotonity if for each function $\varphi(s) \epsilon X_K$ monotonic in the interval $[0, +\infty)$ its transform $\Phi(t)$ is monotonic in D and if $\varphi(s)$ and $\Phi(t)$ are both increasing, or both decreasing.

Conditions for preserving of monotonity can be obtained quite analogically as before. We define the function

$$\varphi_A(s) \begin{matrix} = 1 \text{ for } 0 \leqslant s \leqslant A \\ = 0 \text{ for } A < s < +\infty \end{matrix}.$$

$\varphi_A(s)$ is a decreasing function of s which belongs to X_K. Its transform

$$\int\limits_0^{+\infty} K(t,s)\varphi_A(s)\,ds = \int\limits_0^A K(t,s)\,ds$$

shall be a decreasing function of $t \epsilon D$ if A is fixed. Thus we obtain as a necessary condition for preserving of monotonity:

$$\int\limits_0^A K(t_1,s)\,ds \geqslant \int\limits_0^A K(t_2,s)\,ds \tag{4.11}$$

for $t_1, t_2 \epsilon D$, $t_1 < t_2$ and for arbitrary $A > 0$.

If each monotonic function in X_K tends to zero as $s \longrightarrow +\infty$, then the ondition (4.11) is also sufficient.

Proof. Let $\varphi(s) \epsilon X_K$. We can suppose that $\varphi(s)$ decreases. Then, in view of $\lim\limits_{s \to +\infty} \varphi(s) = 0$, there is $\varphi(s) \geqslant 0$. We have

$$\Phi(t) = \lim_{A \to +\infty} \int\limits_0^A K(t,s)\varphi(s)\,ds.$$

Let $t_1, t_2 \epsilon D$, $t_1 < t_2$.

Then

$$\Phi(t_2) - \Phi(t_1) = \lim_{A \to +\infty} \int\limits_0^A [K(t_2,s) - K(t_1,s)]\,\varphi(s)\,ds.$$

It suffices to prove that for any $A > 0$ the integral

$$\int_0^A [K(t_2, s) - K(t_1, s)]\, \varphi(s)\, ds$$

is non-positive. The function $\varphi(s)$ is of bounded variation in $[0, A]$ and $K(t_2, s) - K(t_1, s)$ is summable over this interval, so we can apply the theorem on integration by parts for Lebesgue integrals (see e. g. Saks [4], p. 298)

$$\int_0^A [K(t_2, s) - K(t_1, s)]\, \varphi(s)\, ds = \left\{\varphi(s) \int_0^s [K(t_2, \sigma) - K(t_1, \sigma)]\, d\sigma\right\}_{s=0}^{s=A} -$$

$$-(RS)\int_0^A \left\{\int_0^s [K(t_2, \sigma) - K(t_1, \sigma)]\, d\sigma\right\} d\varphi(s) = \varphi(A) \int_0^A [K(t_2, s) - K(t_1, s)]\, ds +$$

$$+ (RS)\int_0^A \left\{\int_0^s [K(t_2, \sigma) - K(t_1, \sigma)]\, d\sigma\right\} d\,[-\varphi(s)]$$

Since $[-\varphi(s)]$ increases and $\varphi(A) \geqslant 0$, both terms on the right side are non-positive by (4.11). The symbol $(RS)\int_a^b$ denotes the Riemann-Stieltjes integral.

Thus we have proved

Theorem 4.1. *Necessary and sufficient condition that the function-to-function transformation K*

$$\Phi(t) = \int_0^{+\infty} K(t, s)\, \varphi(s)\, ds$$

preserve the monotonity of all monotonic functions in X_K, if each of them tends to zero, is

$$\int_0^A K(t_1, s)\, ds \geqslant \int_0^A K(t_2, s)\, ds \tag{4.11}$$

for $t_1 < t_2$, $t_1, t_2 \in D$ and for any $A > 0$.

This theorem is an analogue of the theorem 1.1 and the latter may be considered as a particular case of the former one. However, the proof of the theorem 4.1 is based on the «non-elementary» theorem on integration by parts for Lebesgue integrals.

5. We shall now prove the integral analogues of the lemmas of section 2.

Lemma 5.1. *If the limit* $\lim\limits_{A\to+\infty}\int\limits_0^A b(x)U(x)\,dx$ *exists and* $U(x)$ *is a monotonic function which does not tend to zero as* $x\to+\infty$ *and, besides,* $b(x)$ *is summable over each finite interval* $[0,A]$, *then also* $\lim\limits_{A\to+\infty}\int\limits_0^A b(x)\,dx=\int\limits_0^{+\infty} b(x)\,dx$ *exists and moreover* $U(x)\int\limits_x^{+\infty} b(t)\,dt\to 0$ *as* $x\to+\infty$.

Proof. We can suppose that for $x\geqslant K$ there is $U(x)>0$. Then, $b(x)=-\dfrac{\eta'(x)}{U(x)}$ for almost every $x\geqslant K$, where $\eta(x)=\int\limits_x^{+\infty} b(x)U(x)\,dx$. Integrating by parts, we have for $K\leqslant\alpha<\beta$

$$\int\limits_\alpha^\beta b(x)\,dx=\int\limits_\alpha^\beta -\frac{\eta'(x)}{U(x)}\,dx=\left[-\frac{\eta(x)}{U(x)}\right]_{x=\alpha}^{x=\beta}+(RS)\int\limits_\alpha^\beta \eta(x)\,d\left[\frac{1}{U(x)}\right].$$

We have

$$\left|(RS)\int\limits_\alpha^\beta \eta(x)\,d\left[\frac{1}{U(x)}\right]\right|\leqslant\max_{\alpha\leqslant x\leqslant\beta}|\eta(x)|\cdot V_\alpha^\beta\left[\frac{1}{U(x)}\right]=\left|\frac{1}{U(\alpha)}-\frac{1}{U(\beta)}\right|\cdot\max_{\alpha\leqslant x\leqslant\beta}|\eta(x)|.$$

$V_\alpha^\beta[\varphi(x)]$ denotes the total variation of $\varphi(x)$ over $[\alpha,\beta]$. Therefore

$$\left|\int\limits_\alpha^\beta b(x)\,dx\right|\leqslant\frac{|\eta(\alpha)|}{U(\alpha)}+\frac{|\eta(\beta)|}{U(\beta)}+\left|\frac{1}{U(\alpha)}-\frac{1}{U(\beta)}\right|\cdot\max_{\alpha\leqslant x\leqslant\beta}|\eta(x)|.$$

If $\alpha\to+\infty,\beta\to+\infty$, all terms on the right side tend to zero and thus $\int\limits_0^{+\infty} b(x)\,dx$ exists.

In order to prove the second part of lemma it suffices to consider the non-trivial case $U(x)\to+\infty$. Making $\beta\to+\infty$, we obtain

$$\left|\int\limits_\alpha^{+\infty} b(x)\,dx\right|\leqslant\frac{|\eta(\alpha)|}{U(\alpha)}+\frac{1}{U(\alpha)}\sup_{x\geqslant\alpha}|\eta(x)|,\ \text{i. e.}$$

$$U(\alpha)\left|\int\limits_\alpha^{+\infty} b(x)\,dx\right|\leqslant 2\sup_{x\geqslant\alpha}|\eta(x)|.$$

Since $\eta(x)\to 0$ as $x\to+\infty$, the lemma is proved.

Corollary. *If the integrals* $\int_0^{+\infty} b(x)U(x)\,dx$ and $\int_0^{+\infty} b(x)\,dx$ *both exist and* $U(x)$ *is a monotonic function, then*

$$\lim_{x\to+\infty} U(x)\int_x^{+\infty} b(t)dt = 0.$$

Lemma 5.2. *If the infinite integrals* $\int_0^{+\infty} b(x)U(x)\,dx$ *and* $\int_0^{+\infty} b(x)\,dx = a$ *both exist and* $U(x)$ *is a monotonic function, then*

$$\int_0^{+\infty} b(x)U(x)\,dx = -(RS)\int_0^{+\infty}\left[\int_0^x b(t)\,dt\right]dU(x).$$

P r o o f. The function $b(x)$ has $\int_0^x b(t)\,dt = a - \int_x^{+\infty} b(t)\,dt$ as an indefinite integral. Integrating by parts, we have

$$\int_0^A b(x)U(x)\,dx = \left\{U(x)\left[a - \int_x^{+\infty} b(t)\,dt\right]\right\}_{x=0}^{x=A} - (RS)\int_0^A\left[\int_0^x b(t)\cdot dt\right]dU(x) =$$

$$= -U(A)\int_A^{+\infty} b(t)\,dt - (RS)\int_0^A\left[\int_0^x b(t)\,dt\right]dU(x).$$

Making $A\to+\infty$, we obtain, in view of corollary, our lemma.

We can prove quite similarly an integral lemma analogical to that of d e l a V a l l é e P o u s s i n.

Lemma 5.3. *Suppose that the limit* $\lim_{A\to+\infty}\int_0^A b(x)U(x)\,dx$ *exists and that* $U(x)$ *tends to zero monotonically as* $x\to+\infty$ *and, besides, that* $b(x)$ *is summable over any finite interval* $[0, A]$. *Then*

$$\lim_{x\to+\infty} U(x)\int_0^x b(t)\,dt = 0.$$

P r o o f. It is sufficient to suppose that $U(x) > 0$ because the case $U(x) = 0$ for great values of x is trivial. Put $\eta(x) = \int_x^{+\infty} b(t)\,U(t)\,dt$. There is almost everywhere $b(x) = -\dfrac{\eta'(x)}{U(x)}$. We have

$$\int_a^x b(t)\,dt = \int_a^x -\frac{\eta'(t)}{U(t)}dt = -\frac{\eta(x)}{U(x)} + \frac{\eta(a)}{U(a)} + (RS)\int_a^x \eta(t)\,d\left[\frac{1}{U(t)}\right],\ (0<a<x).$$

Hence

$$U(x)\left|\int_a^x b(t)\,dt\right| \leqslant |\eta(x)| + \frac{|\eta(a)|}{U(a)}U(x) +$$

$$+ U(x)\left[\frac{1}{U(x)} - \frac{1}{U(a)}\right]\cdot \sup_{x \geqslant a}|\eta(x)| \leqslant 3 \sup_{x\geqslant a}|\eta(x)|.$$

Choose now a_0 such that $3\sup_{x\geqslant a_0}|\eta(x)| < \varepsilon/2$ (which is possible in view of $\lim_{x\to+\infty}\eta(x)=0$) and, a_0 being fixed, choose $x_0 > a_0$ such that $U(x)\left|\int_0^{a_0} b(t)dt\right| < \varepsilon/2$ for each $x > x_0$ (which is possible, since $U(x)\to 0$). Then $U(x)\left|\int_0^x b(t)\,dt\right| < \varepsilon$ for each $x > x_0$ and this proves the lemma.

6. **Theorem 6.1.** *Necessary and sufficient conditions that the function-to-function transformation* K: $\Phi(t) = \int_0^{+\infty} K(t,s)\varphi(s)\,ds$ *preserve the monotonity of all monotonic functions in* X_K, *if* X_K *contains at least one monotonic function which does not tend to zero as* $s \to +\infty$, *are*

(6.11) $\displaystyle\int_0^A K(t_1,s)\,ds \geqslant \int_0^A K(t_2,s)\,ds$ *for* $t_1 < t_2$, $t_1, t_1 \in D$ *and for any* $A > 0$;

(6.12) $$\int_0^{+\infty} K(t,s)\,ds = a = \text{const} \quad \textit{for each } t \in D.$$

Proof. The necessity of (6.11) has been proved in sec. 4. The lemma 5.1 implies the existence of $\int_0^{+\infty} K(t,s)\,ds$ for any $t\in D$. The function $\varphi(s)\equiv 1$ is simultaneously decreasing and increasing and this implies the necessity of (6.12).

Sufficiency. Let $\varphi(s)$ be a decreasing function, $\varphi(s)\in X_K$. Then, in view of lemma 5.2.

$$\Phi(t) = \int_0^{+\infty} K(t,s)\varphi(s)\,ds = -(RS)\int_0^{+\infty}\left[\int_0^s K(t,\sigma)\,d\sigma\right] d\varphi(s);\quad \Phi(t_2) - \Phi(t_1) =$$

$$= -(RS)\int_0^{+\infty}\left[\int_0^s K(t_2,\sigma)\,d\sigma\right]d\varphi(s) + (RS)\int_0^{+\infty}\left[\int_0^s K(t_1,\sigma)\,d\sigma\right]d\varphi(s) =$$

$$= (RS)\int_0^{+\infty}\left\{\int_0^s [K(t_1,\sigma) - K(t_2,\sigma)]\,d\sigma\right\}d\varphi(s).$$

The increments of $\varphi(s)$ are non-positive, moreover for $t_1 < t_2$

$$\int_0^s [K(t_1,\sigma) - K(t_2,\sigma)]\,d\sigma \geqslant 0$$

by (6.11) and therefore

$$\Phi(t_2) - \Phi(t_1) \leqslant 0.$$

All our consideration are valid, with evident modifications, in the case when the transformation is defined by a finite integral. Then we obtain

Theorem 6.2. *Necessary and sufficient conditions that the function-to-function transformation* K: $\Phi(t) = \int_a^b K(t,s)\varphi(s)\,ds$ *preserve the monotonity of all function monotonic in* $[a,b]$ *are*

$$\int_a^A K(t_1,s)\,ds \geqslant \int_a^A K(t_2,s)\,ds \tag{6.21}$$

for $t_1 < t_2$, $t_1, t_2 \in D$ *and any* $A \in [a,b]$ *and*

$$\int_a^b K(t,s)\,ds = \text{const.} \quad \textit{for each } t \in D. \tag{6.22}$$

Conditions for preserving of strict monotonity can be also obtained easily. An analogue of the theorem 6.1 is the

Theorem 6.3. *Necessary and sufficient conditions that the function-to-function transformation* K: $\Phi(t) = \int_0^{+\infty} K(t,s)\varphi(s)\,ds$ *preserve the strict monotonity of all strictly monotonic functions in* X_K, *if* X_K *contains at least one monotonic function which does not tend to zero as* $s \to +\infty$, *are*

$$\int_0^A K(t_1,s)\,ds \geqslant \int_0^A K(t_2,s)\,ds \tag{6.31}$$

for $t_1 < t_2$, $t_1, t_2 \in D$ *and for any* $A > 0$; *besides, for one value* A_0 *at least*

$$\int_0^{A_0} K(t_1, s)\, ds > \int_0^{A_0} K(t_2, s)\, ds;$$

$$(6.32) = (6.12) \qquad \int_0^{+\infty} K(t, s)\, ds = a = \text{const.} \;\; \textit{for each} \;\; t \in D.$$

P r o o f. Necessity. If $0 < \varepsilon < 1$. $t_1 < t_2$, then

$$(6.33) \qquad \begin{aligned} \int_0^{+\infty} K(t_1, s)(1 + \varepsilon^{s+1})\, ds &> \int_0^{+\infty} K(t_2, s)(1 + \varepsilon^{s+1})\, ds, \\ \int_0^{+\infty} K(t_1, s)(1 - \varepsilon^{s+1})\, ds &< \int_0^{+\infty} K(t_2, s)(1 - \varepsilon^{s+1})\, ds. \end{aligned}$$

From the second mean-value theorem it follows that

$$\left| \int_0^A K(t, s)\, \varepsilon^{s+1}\, ds \right| \leqslant \varepsilon \cdot \sup_{\xi} \left| \int_0^{\xi} K(t, s)\, ds \right|$$

and therefore

$$\lim_{\varepsilon \to 0} \int_0^{+\infty} K(t, s)\, \varepsilon^{s+1}\, ds = 0.$$

In view of (6.33) this implies the necessity of (6.32). Considering the transform of the strictly decreasing function $\varphi_A(s) + \varepsilon^{s+1}$, we have

$$\int_0^A K(t_1, s)\, ds + \int_0^{+\infty} K(t_1, s)\, \varepsilon^{s+1}\, ds > \int_0^A K(t_2, s)\, ds + \int_0^{+\infty} K(t_2, s)\, \varepsilon^{s+1}\, ds.$$

This implies the necessity of (6.11). Then we have

$$\Phi(t_2) - \Phi(t_1) = (RS) \int_0^{+\infty} K(s)\, d\varphi(s),$$

where

$$K(s) = \int_0^s [K(t_1, \sigma) - K(t_2, \sigma)]\, d\sigma \geqslant 0$$

by (6.11). If $\varphi(s)$ increases strictly, then $\Phi(t_2) - \Phi(t_1) > 0$ and therefore $K(A_0) > 0$ for one A_0 at least. This and (6.11) imply (6.31).

Sufficiency. If (6.31) and (6.32) are fulfilled, we have

$$\Phi(t_2) - \Phi(t_1) = (RS)\int_0^{+\infty} K(s)\, d\varphi(s),$$

where $K(s) \geqslant 0$, $K(A_0) > 0$ and $\varphi(s)$ is strictly increasing. Since $K(s)$ is continuous, there is $K(s) > h > 0$ for $A_0 - \delta < s < A_0 + \delta$ and we have

$$\Phi(t_2) - \Phi(t_1) \geqslant (RS)\int_{A_0-\delta}^{A_0+\delta} K(s)\, d\varphi(s) > h\,|\varphi(A_0+\delta) - \varphi(A_0-\delta)| > 0.$$

We can prove also easily that the condition (6.31) is necessary and sufficient for preserving of strict montonity if X_K contains only monotonic null functions.

7. Applications. The above obtained theorems can be applied to the transformations used in the theory of summability. Since every regular (i. e. fulfilling the well known Silverman-Toeplitz regularity conditions, see e. g. Banach [1], pp. 90-91) method of summability evaluates some monotonic non-null sequences, we obtain easily the

Theorem 7.1. *Let $A = (a_{ik})$ be a regular method of summability. Necessary and sufficient conditions that (a_{ik}) preserve the monotonity of all $x = \{\xi_n\} \in X_A$ are*

(7.11) $$\sum_{k=1}^{\infty} a_{ik} = 1 \text{ for } i = 1, 2, \ldots;$$

(7.12) $$\sum_{k=1}^{r} a_{ik} \downarrow 0 \text{ if } i \to +\infty \text{ and } r \text{ is fixed } (r = 1, 2, \ldots).$$

(The symbol $\downarrow$ means monotonic decreasing and convergence).

Proof. Necessity. (a_{ik}) fulfil the well known Silverman-Toeplitz conditions:

(7.13) $$\sum_{k=1}^{\infty} |a_{ik}| \leqslant M \quad \text{for } i = 1, 2, \ldots;$$

(7.14) $$\lim_{i\to+\infty} a_{ik} = 0 \quad \text{for } k = 1, 2, \ldots;$$

(7.15) $$\lim_{i\to+\infty} \sum_{k=1}^{\infty} a_{ik} = 1.$$

By (7.15) and necessity of (3.12) (the set D is then the set of all integers) we obtain (7.11) · (3.11) and (7.14) imply (7.12).

Sufficiency is obvious by theorem 3.1.

Nörlund-Woronoi means are defined by the triangular matrix:

$$\begin{matrix} 1, & 0, & 0, & 0, & \dots \\ \frac{q_1}{Q_1}, & \frac{q_0}{Q_1}, & 0, & 0, & \dots \\ \frac{q_2}{Q_2}, & \frac{q_1}{Q_2}, & \frac{q_0}{Q_2}, & 0, & \dots \\ \dots & \dots & \dots & \dots & \dots \end{matrix}$$

where $Q_n = q_0 + q_1 + \dots + q_n$, $q_0 > 0$, $q_n \geqslant 0$, $\frac{q_n}{Q_n} \to 0$.

The condition (7.11) is fulfilled at any rate. The condition (7.12) takes the form

$$\frac{Q_k - Q_{k-n}}{Q_k} \downarrow 0, \text{ i. e.} \tag{7.16}$$

$$\frac{Q_{k-n}}{Q_k} \uparrow 1 \tag{7.17}$$

for $k \to +\infty$ and every fixed n, ($Q_{k-n} = 0$ if $k < n$).

Thus *(7.17) is the necessary and sufficient condition that the* Nörlund-Woronoi *means always increase if the transformed sequence increases.*

This helps us to verify that *Cesàro means* of *any positive order* r *preserve the monotonity.* We have

$$c_{ik}^{(r)} = \frac{\binom{i+r-k-1}{r-1}}{\binom{i+r}{r}}$$

for $k \leqslant i$, and $c_{ik}^{(r)} = 0$ for $k > i$ ($i, k = 0, 1, 2, \dots, r = 1, 2, \dots$). Obviously the Cesàro means are of Nörlund-Woronoi type with the generating sequence:

$$q_n^{(r)} = \binom{n+r-1}{r-1}, \; Q_n^{(r)} = \binom{n+r}{r}.$$

We have

$$\frac{Q^{(r)}_{k-n}}{Q^{(r)}_k}=\frac{\binom{k-n+r}{r}}{\binom{k+r}{r}}=\frac{(k-n+r)!\,k!}{(k+r)!\,(k-n)!}=\frac{(k-n+1)(k-n+2)\dots(k-n+r)}{(k+1)(k+2)\dots(k+r)}=$$

$$=\left(1-\frac{n}{k+1}\right)\left(1-\frac{n}{k+2}\right)\dots\left(1-\frac{n}{k+r}\right)$$

for $k-n\geqslant 0$ and this increases to 1 if $k\to+\infty$ and n, r are fixed. Thus we have proved that the Cesàro means preserve the monotonity. It is obvious that they preserve strict monotonity, too.

It follows easily from the theorem 6.1 that a row finite matrix (a_{ik}) transforms each decreasing sequence into increasing one and viceversa if, and only if,

(7.18) $$\sum_{k=1}^{m} a_{ik} \text{ increases with } i,\ m \text{ being fixed;}$$

(7.19) $$\sum_{k=1}^{\infty} a_{ik}=a \text{ for } i=1,2,\dots$$

To show it observe that the matrix $(-a_{ik})$ preserves then the monotonity.

This remark helps us to prove that, if $\{a_n\}$ *is an increasing (resp. decreasing) sequence, then its* Cesàro *means of successively increasing orders decrease (increase)* *).

Proof. The matrix defining the transformation considered has the form

$$\frac{1}{n+1},\quad \frac{1}{n+1},\ \dots,\ \frac{1}{n+1},\ 0,\ 0,\dots$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots,\ 0,\ 0,\dots$$

$$\frac{\binom{n+i-1}{i-1}}{\binom{n+i}{i}},\quad \frac{\binom{n+i-2}{i-1}}{\binom{n+i}{i}},\ \dots,\ \frac{\binom{i-1}{i-1}}{\binom{n+i}{i}},\ 0,\ 0,\dots$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots,\ 0,\ 0,\dots$$

*) The author was suggested by Prof. M. Biernacki to prove this statement.

The condition (7.19) is obviously fulfilled. It suffices to prove that, n and m being fixed, $m \leqslant n+1$, the expression

$$\frac{1}{\binom{n+i}{i}}\left[\binom{n+i-1}{i-1}+\binom{n+i-2}{i-1}+\ldots+\binom{n+i-m}{i-1}\right]$$

increases with i. We have

$$\frac{1}{\binom{n+i}{i}}\left\{\left[\binom{n+i-1}{i-1}+\binom{n+i-2}{i-1}+\ldots+\binom{n+i-m}{i-1}\right]+\right.$$

$$\left.+\left[\binom{n+i-m-1}{i-1}+\ldots+\binom{i-1}{i-1}\right]-\left[\binom{n+i-m-1}{i-1}+\ldots+\binom{i-1}{i-1}\right]\right\}=$$

$$=\frac{1}{\binom{n+i}{i}}\left[\binom{n+i}{i}-\binom{n+i-m}{i}\right]=1-\frac{\binom{n-m+i}{i}}{\binom{n+i}{i}}=1-\frac{(n-m+i)!\,n!}{(n-m)!\,(n+i)!}$$

and this increases to 1 as $i\to+\infty$, since

$$\frac{(n-m+i)!}{(n+i)!}=\frac{1}{(n+i-m+1)(n+i-m+2)\ldots(n+i)}$$

decreases to zero as $i\to+\infty$, n, m being fixed. This proves our statement.

The integral transformation corresponding to Cesàro means of the first order, is defined by the kernel

$$K(t,s)\begin{array}{l}=\dfrac{1}{t}\ \text{for}\ 0\leqslant s\leqslant t,\\[2mm] =1\ \text{for}\ t<s.\end{array}$$

The set D consists of all positive reals.

The conditions (6.11) and (6.12) are obviously fulfilled so that this transformation preserves the monotonity.

A function $F(x)$ is said to be convex if the curve $y=F(x)$ between x_1 and x_2 always lies below the chord joining the points $(x_1, F(x_1))$ and $(x_2, F(x_2))$.

A necessary and sufficient condition for convexity of $F(x)$ in $(-\delta, 1+\delta)$, $\delta>0$, is that $F(x)=\int_0^x f(t)\,dt+F(0)$, where $f(x)$ is a bounded increasing function of $x\in(-\delta,\ 1+\delta)$ (see Titchmarsh [5], p. 372, ex. 8). We shall prove following

Theorem 7.2. *If $F(x)$ is convex in the interval $(-\delta, 1+\delta)$ and $\{\mu_n\}$ is the sequence of moments of $F(x)$: $\mu_n = (RS)\int_0^1 t^n dF(t)$, then the sequence $\{(n+1)\mu_n\}$ increases.*

Proof. Put $a_n = (n+1)\mu_n \cdot a_n = (RS)\int_0^1 (n+1)t^n dF(t) = \int_0^1 (n+1)\, t^n f(t)\, dt$, $f(t)$ being an increasing function. We can now apply the theorem 6.2 with $K(s,t) = (s+1)\, t^s$ and D being the set of all non-negative integers. We have $\int_0^1 K(s,t)\, dt = [t^{s+1}]_{t=0}^{t=1} = 1$, $\int_0^A K(s,t)\, dt = A^{s+1}$ decreases for $s \to +\infty$ and $A \in [0,1]$ and this proves the theorem.

The lemmas on series proved in section 3. admit also of various applications. As a matter of example we shall prove without using the notion of absolute continuity two well known lemmas concerning the theory of Lebesgue integrals.

Let $f(x)$ be a non-negative function summable over the set E and $\{U_n\}$ an increasing sequence of positive numbers such that $\lim_{n \to +\infty} U_n = +\infty$. If

$$E_n = \underset{x}{E}\{f(x) \geqslant U_n;\ x \in E\}, \text{ then}$$

(1°) $$\lim_{n \to +\infty} U_n m(E_n) = 0,$$

(2°) the series $\sum_n (U_n - U_{n-1})\, m(E_n)$ converges.

Proof. Obviously $E_1 \supset E_2 \supset \ldots$. The integrability of $f(x)$ implies the convergence of the series

$$\sum_n U_n [m(E_n) - m(E_{n+1})] \text{ and, besides, } \lim_{n \to +\infty} m(E_n) = 0$$

Putting $b_n = m(E_n) - m(E_{n+1})$ we obtain $b_{n+1} + b_{n+2} + \ldots = m(E_{n+1})$ and in view of lemma 2.1 $\lim_{n \to +\infty} U_n m(E_{n+1}) = 0$, hence by convergence of

$$\sum_n U_n [m(E_n) - m(E_{n+1})] \text{ follows (1°).}$$

The lemma 2.2 implies (2°) immediately. (For the case $U_n = n$ see Titchmarsh [5], p. 342, ex. VII and Halmos [2], p. 115, ex. 4).

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Streszczenie

W pracy tej podaję warunki konieczne oraz dostateczne na to, by przekształcenia liniowe ciągów i funkcji zachowywały monotoniczność wraz z jej kierunkiem.

Dla wykazania dostateczności potrzebnych było kilka lemmatów, które same w sobie mogą być interesujące. Metoda dowodu tych lemmatów pozwala uogólnić pewne twierdzenie de la Vallée Poussina.

Podane są niektóre zastosowania otrzymanych twierdzeń do teorii sumowalności i teorii momentów.

Резюме

В этой работе даю необходимые и достаточные условия для того, чтобы линейные преобразования последовательностей и функций сохраняли их монотонность вместе с направлением монотонности.

Доказательство достаточности опирается на нескольких леммах, которые могут иметь и самостоятельный интерес. Метод доказательства этих лемм позваляет обобщить одну теорему Ш. Ж. де ля Валле Пуссена.

Полученные теоремы допускают приложения к теории суммируемости и к теории моментов.