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Computing the Distribution of the Poisson - Kac Process

Abstract. The paper presents a new method of computing the distribution of the Poisson - Kac process, based on comparing analytic and probabilistic formulas for the solutions of the two-dimensional Dirac equation. Both density (obtained previously by A. Janssen and E. Siebert [22]) and characteristic function are computed, which in turn leads to a new probabilistic formula for the solutions to the telegraph equation, and gives some information concerning asymptotic behaviour of the Poisson - Kac process.

Introduction. Let $N_a(t)$, $t \geq 0$ be the Poisson process with the mean value $EN_a(t) = at$, where $a > 0$ is a given positive number. The Poisson - Kac process is defined as

$$\mathfrak{g}_a(t) = \left(\int_0^t (-1)^{N_a(s)} \, ds \, , \, (-1)^{N_a(t)} \right) .$$

This process is a homogeneous Markov process when considered as a process with values in the group $\mathcal{G} = R \times \{-1, 1\}$ with the multiplication rule $(\xi, k) \cdot (\eta, l) = (\xi l + \eta, kl)$. The former means that (see [26]), for every non-negative t and h,

(*) the random variable $\mathfrak{g}_a(t+h)\, \mathfrak{g}_a^{-1}(t)$ is independent of the σ-field $\sigma\{\mathfrak{g}_a(s)\, ,\ 0 \leq s \leq t\}$ and

(**) the random variables $\mathfrak{g}_a(t+h)\, \mathfrak{g}_a^{-1}(t)$ and $\mathfrak{g}_a(h)$ have the same distribution.

The system of equations:

$$\begin{aligned} \frac{\partial \tilde{u}(t,x)}{\partial t} &= \nu \frac{\partial \tilde{u}(t,x)}{\partial x} + a\tilde{v}(t,x) - a\tilde{u}(t,x) \, , \\ \frac{\partial \tilde{v}(t,x)}{\partial t} &= -\nu \frac{\partial \tilde{v}(t,x)}{\partial x} + a\tilde{u}(t,x) - a\tilde{v}(t,x) \, , \end{aligned} \tag{0.1}$$

was used by M. Kac [23-24] to prove the connection of the Poisson - Kac process with the equation:

$$\frac{\partial^2 u(t,x)}{\partial t^2} + 2a \, \frac{\partial u(t,x)}{\partial t} = \nu^2 \frac{\partial^2 u(t,x)}{\partial x^2} \, . \tag{0.2}$$

Note that, for $\nu = 1$, the system constitutes the Kolmogorov backward equation of the considered process ([16], [22], [26]).

Recall the association of the equation (0.2) with the system of differential equations

$$(0.2')\qquad \begin{aligned} &\frac{\partial U(t,x)}{\partial x} + L\,\frac{\partial J(t,x)}{\partial t} + R\,J(t,x) = 0\,, \\ &\frac{\partial J(t,x)}{\partial x} + C\,\frac{\partial U(t,x)}{\partial t} + G\,U(t,x) = 0\,, \end{aligned}$$

where L, R, C, G are non-negative constants, $LC \neq 0$; if the functions U, J satisfy (0.2′) then

$$u(t,x) = \exp\left[\frac{G}{C}t\right] U(t,x) \quad (\text{and} \quad u(t,x) = \exp\left[\frac{G}{C}t\right] J(t,x))$$

satisfies (0.2) with $\nu = (LC)^{-1/2}$ and $a = 1/2[(R/L) - (G/C)]$. Both (0.2) and (0.2′) are often called telegraph equation or equation of transmission line. The solution to (0.2) with the initial conditions:

$$(0.2'')\qquad \frac{\partial u(0,x)}{\partial t} = u_1(x)\,, \quad u(0,x) = u_0(x)\,,$$

is

$$(0.3)\quad u(t,x) = \frac{1}{2}\,Eu_0[x + \nu\xi_a(t)] + \frac{1}{2}\,Eu_0[x - \nu\xi_a(t)] + \frac{1}{2}\,E\int_{-\xi_a(t)}^{\xi_a(t)} u_1(x+\nu s)\,ds\,,$$

where

$$\xi_a(t) = \int_0^t (-1)^{N_a(s)}\,ds\,.$$

Note that the above formula was proved by M. Kac [23-24] in the case when $u_1 \equiv 0$, the grup $\mathcal{G}$ and the term depending on u_1 were introduced by J. Kisyński [26]. See also S.M. Ermakov et al. [6], p.171, H. Heyer [16], p.13, S. N. Ethier and T. G. Kurtz [5] pp.468-471, where references are given to R. J. Griego and R. Hersh [13-15], and M. A. Pinsky [31]. Compare S. Goldstein [12], M. Pinsky [30] and S. Kaplan [25].

Some generalizations of this result were given by A. Janssen and E. Siebert in [22], where furthermore the distributions of the variables $\xi_a(t)$, $t \geq 0$ are computed via the use of the convolution semigroup of measures on $\mathcal{G}$ related to (0.1). The construction of the measures is based on the Phillips perturbation theorem [29]. It is shown there that the probability that $\xi_a(t) = t$ equals e^{-at} and that apart from this point the random variable $\xi_a(t)$ has the density

$$(0.4)\qquad e^{-at}\,\frac{a}{2}\,I_0\left(a\sqrt{t^2-s^2}\right) + e^{-at}\,\frac{a}{2}\,\frac{t+s}{\sqrt{t^2-s^2}}\,I_1\left(a\sqrt{t^2-s^2}\right)\,.$$

On the other hand, R. Gaveau et al. [11] suggested that the probabilistic approach involving the Poisson - Kac process should also work for so-called two-dimensional Dirac equation

$$(0.5)\qquad \begin{aligned} &\frac{\partial u(t,x)}{\partial t} = \frac{\partial u(t,x)}{\partial x} + z\,v(t,x)\,, \\ &\frac{\partial v(t,x)}{\partial t} = -\frac{\partial v(t,x)}{\partial x} + z\,u(t,x)\,, \\ &u(0,x) = u_0(x)\,,\ v(0,x) = v_0(x)\,, \end{aligned}$$

(where z is a complex number). The problem was solved by Ph. Blanchard, Ph. Combe, M. Sirugue, M. Sirugue - Colin (see [1], where an abundant bibliography is given). Compare also T. Ichinose [18-20] and A. Bobrowski [2]. The paper [35], by T. Zastawniak is devoted to the same problem. Additionally, the autor presents a rigorous mathematical sense to the idea of analytic continuation between the solution of the telegraph and the two-dimensional Dirac equation, suggested in [11].

In this paper we compute the density (0.4) directly, solving the equation (0.5) (compare note on the page 523 in [22]). Observe that the equation (0.1), with $\nu = 1$ and a replaced by a complex number z, is equivalent to (0.5) in the sense that

$$\tilde{u}(t,x) = e^{-zt}u(t,x)\,, \quad \tilde{v}(t,x) = e^{-zt}v(t,x)\,, \tag{0.6}$$

where u, v and $\tilde{u}, \tilde{v}$ are the solutions of (0.5) and (0.1), respectively. As a corollary we prove a probabilistic formula (new but similar to (0.3)) for the solutions of the telegraph equation.

In a variety of context and assumptions the authors of [7], [27-28], [32-33], [36-38] have been involved in the question of convergence, when $a = \nu^2$ tends to infinity, of the solutions of the equations of the type (0.2). Equations in Hilbert and Banach spaces, the convergence of solutions and of their derivatives, the rate of the convergence, the boundary layer, several kinds of initial and boundary conditions were considered. The main result however, in our particular case, is that the solution of the problem (0.2), (0.2″) tends to the solution of

$$\frac{\partial u(t,x)}{\partial t} = \frac{1}{2}\,\frac{\partial^2 u(t,x)}{\partial x^2}\,, \quad u(0,x) = u_0(x)\,.$$

In view of the formula (0.3) (with $u_1 \equiv 0$) we have then

$$\begin{aligned}\lim_{a\to\infty}&\left\{\frac{1}{2}\,Eu_0[x+\sqrt{a}\xi_a(t)] + \frac{1}{2}\,Eu_0[x-\sqrt{a}\xi_a(t)]\right\}\\ &= \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\infty}\exp\left[-\frac{y^2}{2t}\right]\,u_0(x+y)\,dy = E\,u_0[x+w(t)]\end{aligned} \tag{0.7}$$

where $w(t),\ t \ge 0$ is the standard Brownian motion. The above result is equivalent to the fact that the symmetrized random variable $\tilde{\xi}_a(t),\ t \ge 0$

$$P(\tilde{\xi}_a(t) \in \mathbb{B}) = \frac{1}{2}P(\xi_a(t) \in \mathbb{B}) + \frac{1}{2}P(\xi_a(t) \in -\mathbb{B})$$

multiplied by the term $\sqrt{a}$ converges in distribution to $w(t),\ t \ge 0$.

Hence, the theorem that was expressed and proved in the language of functional analysis or partial differential equations can also be considered as a theorem on convergence of stochastic processes. Therefore the problem arises of whether it is possible to prove it by using standard probability methods. In order to solve the problem we find the characteristic function of the variable $\sqrt{a}\xi_a(t)$ and then prove easily that it is pointwise convergent to the characteristicc function of the Gaussian distribution. Our proof states that $\sqrt{a}\xi_a(t),\ t \ge 0$ (without symmetrization) converges in distribution to the Brownian motion, too. Our result, thus, is somewhat stronger than (0.7).

Note that M. Zlamal in his early works [36-38] used the Fourier analysis to estimate the difference between solutions of heat and telegraph equations. His results are, on one hand, more general than ours is: for example instead of fixed coefficients he considered time and x dependent ones. On the other hand however, he was seemingly unaware of S. Goldstein's and M. Kac's works and accordingly of issuing probabilistic implications. Thus, whereas the method remains the same, the points of view, approaches to the problem and goals to reach are different. Different are also details of calculations.

1. Computing the distribution of the Poisson - Kac process. Let $z \neq 0$ be a complex number. Let us recall that the Bessel functions I_0, I_1 are defined as follows

$$I_0(z) = J_0(iz) = \sum_{k=0}^{\infty} \frac{1}{k!k!} \left(\frac{z}{2}\right)^{2k},$$

$$I_1(z) = J_1(iz) = \sum_{k=0}^{\infty} \frac{1}{k!(k+1)!} \left(\frac{z}{2}\right)^{2k+1}.$$

Proposition 1. *Let $u_0(x)$, $v_0(x)$ be functions of the class $C^1(R)$. The unique, vector valued, class $C^1(R^+ \times R)$ solution to the Cauchy's problem* (0.5) *is:*

$$u(t,x) = u_0(x+t) + \int_{-t}^{t} \frac{z}{2}\, I_0\Big(z\sqrt{t^2-s^2}\Big)\, v_0(x-s)\, ds$$
$$+ \int_{-t}^{t} \frac{z}{2}\, \frac{t+s}{\sqrt{t^2-s^2}}\, I_1\left(z\sqrt{t^2-s^2}\right) u_0(x+s)\, ds\,,$$

$$v(t,x) = v_0(x-t) + \int_{-t}^{t} \frac{z}{2}\, I_0\left(z\sqrt{t^2-s^2}\right) u_0(x+s)\, ds$$
$$+ \int_{-t}^{t} \frac{z}{2}\, \frac{t+s}{\sqrt{t^2-s^2}}\, I_1\left(z\sqrt{t^2-s^2}\right) v_0(x-s)\, ds\,.$$

Proof. We are going to prove that there exist functions f, g of class $C^1(D)$, where $D \stackrel{\text{def}}{=} \{(s,t) \in R^2 : |s| \le t\}$, such that for every $u_0(x)$, $v_0(x)$ of the class $C^1(R)$ the formula

$$u(t,x) = u_0(x+t) + \int_{-t}^{t} f(t,s)v_0(x-s)\, ds + \int_{-t}^{t} g(t,s)u_0(x+s)\, ds\,,$$
(1.1)
$$v(t,x) = v_0(x-t) + \int_{-t}^{t} f(t,s)u_0(x+s)\, ds + \int_{-t}^{t} g(t,s)v_0(x-s)\, ds\,,$$

defines the class C^1 solution to the problem (0.5).

Let us find the conditions for f, g. Setting $u_0 \neq 0$, (1.1) gives

$$u(t,x) = \int_{-t}^{t} f(t,s)v_0(x-s)\, ds\,,$$
(*)
$$v(t,x) = v_0(x-t) + \int_{-t}^{t} g(t,s)v_0(x-s)\, ds\,.$$

If this is to be the solution of (0.5), by the first equation of it, we need

$$f(t,t)v_0(x-t)+f(t,-t)v_0(x+t)+\int_{-t}^{t}\frac{\partial}{\partial t}f(t,s)v_0(x-s)\,ds$$
$$=\int_{-t}^{t}f(t,s)v_0'(x-s)\,ds+zv_0(x-t)+z\int_{-t}^{t}g(t,s)v_0(x-s)\,ds\,.$$

Integration by parts in the first term of the right-hand side of the above equality yields

$$[2f(t,t)-z]v_0(x-t)+\int_{-t}^{t}\left[\frac{\partial f(t,s)}{\partial t}-\frac{\partial f(t,s)}{\partial s}-zg(t,s)\right]v_0(x-s)\,ds=0\,.$$

The function v_0 being arbitrary, we need

$$f(t,t)=\frac{z}{2}\,,\quad \frac{\partial f(t,s)}{\partial t}=\frac{\partial f(t,s)}{\partial s}+zg(t,s)\,. \tag{1.2}$$

Analogously, by the second equation of (0.5), we need

$$g(t,-t)=0\,,\quad \frac{\partial g(t,s)}{\partial t}=-\frac{\partial g(t,s)}{\partial s}+zf(t,s)\,. \tag{1.3}$$

Furthermore, it is evident that (1.2) - (1.3) constitute not only necessary but also sufficient conditions for the formula (*) to define the solution of (0.5), provided $v_0(x)$ is of class $C^1(R)$.

On the other hand, observe that setting $v_0\equiv 0$ we obtain (1.2) and (1.3) once again. Thus, an simple argument based on linearity of (0.5) shows that if the functions f,g of class $C^1(D)$ enjoy (1.2) - (1.3) then the formula (1.1) defines the solution of the Dirac equation, provided $u_0(x)$, $v_0(x)$ are of class $C^1(R)$.

Now, the boundary conditions $f(t,t)=z/2$, $g(t,-t)=0$, and the (common) domain D of f,g suggest the change of variables; $p=t+s$, $q=t-s$. The functions f,g are the solutions of (1.2) - (1.3) iff the functions $F,G\colon R^+\times R^+\mapsto C$,

$$F(p,q)=f\left(\frac{p+q}{2},\frac{p-q}{2}\right)\,,\quad G(p,q)=g\left(\frac{p+q}{2},\frac{p-q}{2}\right)$$

are the solutions of

$$F(p,0)=\frac{z}{2}\,,\quad \frac{\partial F(p,q)}{\partial q}=\frac{z}{2}G(p,q)\,,$$
$$G(0,q)=0\,,\quad \frac{\partial G(p,q)}{\partial p}=\frac{z}{2}F(p,q)\,.$$

It is easy to solve the above equation. For

$$\frac{\partial F}{\partial q}(0,q)=0\,,\ F(p,0)=\frac{z}{2}$$

and

$$\frac{\partial^2 F(p,q)}{\partial p \partial q} = \frac{z^2}{4} F(p,q)$$

so that,

$$F(p,q) = \frac{z^2}{4} \int_0^p \int_0^q F(r_1, r_2)\, dr_1 dr_2 + \frac{z}{2},$$

and succesive approximation (with $F_0(p,q) = z/2$) gives

$$F(p,q) = \frac{z}{2} \sum_{k=0}^{\infty} \frac{1}{k!k!} \left(\frac{z^2 pq}{4} \right)^k ,$$

$$G(p,q) = \frac{2}{z} \frac{\partial F(p,q)}{\partial q} = \frac{z^2 p}{4} \sum_{k=0}^{\infty} \frac{1}{k!(k+1)!} \left(\frac{z^2 pq}{4} \right)^k .$$

Thus, remembering the definition of the Bessel functions I_0, I_1 we conclude that the functions

$$f(t,s) = \frac{z}{2} \sum_{k=0}^{\infty} \frac{1}{k!k!} \left(\frac{z^2(t^2 - s^2)}{4} \right)^k = \frac{z}{2} I_0(z\sqrt{t^2 - s^2}),$$

(1.4)
$$g(t,s) = \frac{z^2(t+s)}{4} \sum_{k=0}^{\infty} \frac{1}{k!(k+1)!} \left(\frac{z^2(t^2 - s^2)}{4} \right)^k = \frac{z}{2} \frac{t+s}{\sqrt{t^2 - s^2}} I_1(z\sqrt{t^2 - s^2}),$$

with the convention $g(t,-t) = 0$, $g(t,t) = z^2 t/2$, are a solution to (1.2) - (1.3). Thus, by above remarks, (1.1) defines a solution to the Dirac equation.

The uniqueness of the solution is proved by the general theory of the hyperbolic systems; for the simple proof one can consult [35].

The role played by the functions f, g defined by (1.4) is explained in the succeeding two corollaries, which are expressed in the language of probability theory and generalized functions, respectively.

Corollary 1. *Let $a > 0$ be a real number and $N_a(t)$, $t \geq 0$ be the Poisson process with the mean value $EN_a(t) = at$. Set*

$$A(t) = \{\omega : N_a(\omega) = 1, 3, 5, \dots\},\ B(t) = \{\omega : N_a(\omega) = 2, 4, 6, \dots\}.$$

Then for any continuous function $w : R \to R$ we have

(1.5)
$$E\{1_{A(t)} w[x + \xi_a(t)]\} = e^{-at} \int_{-t}^{t} \frac{a}{2} I_0\left(a\sqrt{t^2 - s^2}\right) w(x+s)\, ds ,$$

$$E\{1_{B(t)} w[x + \xi_a(t)]\} = \int_{-t}^{t} e^{-at} \frac{a}{2} \frac{t+s}{\sqrt{t^2 - s^2}} I_1\left(a\sqrt{t^2 - s^2}\right) w(x+s)\, ds .$$

and, consequently, for any Borel set S,

$$Prob\Big(\xi_a(t)\in S\Big)=e^{-at}1_S(t)+\int_{[-t,t]\cap S} e^{-at}\,\frac{a}{2}\,I_0(a\sqrt{t^2-s^2})\,ds$$
(1.6)
$$+\int_{[-t,t]\cap S} e^{-at}\,\frac{a}{2}\,\frac{t+s}{\sqrt{t^2-s^2}}\,I_1(a\sqrt{t^2-s^2})\,ds\,.$$

Proof. By the Lemma 1 of [26] the solution of (0.1) (with $\nu=1$) is

$$\tilde u(t,x)=e^{-at}u_0(x+t)+E\{1_{A(t)}v_o[x-\xi_a(t)]\}+E\{1_{B(t)}u_0[x+\xi_a(t)]\}\,,$$
$$\tilde v(t,x)=e^{-at}v_0(x-t)+E\{1_{A(t)}u_o[x-\xi_a(t)]\}+E\{1_{B(t)}v_0[x-\xi_a(t)]\}\,.$$

On the other hand, by (0.6) and Proposition 1, the *unique* solution to this equation is

$$\tilde u(t,x)=e^{-at}u_0(x+t)+\int_{-t}^{t} e^{-at}\frac{a}{2}\,I_0(a\sqrt{t^2-s^2})v_0(x-s)\,ds$$
$$+\int_{-t}^{t} e^{-at}\frac{a}{2}\frac{t+s}{\sqrt{t^2-s^2}}I_1(a\sqrt{t^2-s^2})u_0(x+s)\,ds\,,$$
$$\tilde v(t,x)=e^{-at}v_0(x-t)+\int_{-t}^{t} e^{-at}\frac{a}{2}\,I_0(a\sqrt{t^2-s^2})u_0(x+s)\,ds$$
$$+\int_{-t}^{t} e^{-at}\frac{a}{2}\frac{t+s}{\sqrt{t^2-s^2}}\,I_1(a\sqrt{t^2-s^2})v_0(x-s)\,ds\,.$$

By comparing the two above formulas we obtain (1.5) and (1.6) as well.

The formula (1.6) was originally obtained by A. Janssen and E. Siebert in [22], p.522. The formulas (1.5) may be easily derived from their results, too. (See [22] p.522, the definition of the functions $h(x,+1)$, $h(x,-1)$.) They were proved also directly from the probabilistic definition of the process (without the use of telegraph equation) by F. W. Steutel [34].

Now, let S_+, S_- be two distributions on R^2 defined by

$$<S_+,\varphi>=\int_0^\infty \varphi(t,t)\,dt\ ,\quad <S_-,\varphi>=\int_0^\infty \varphi(t,-t)\,dt\ ,$$

for all $\varphi\in C_0^\infty(R^2)$.

Corollary 2. *The two-dimensional Dirac equation has a unique, vector valued, fundamental solution* $(f+S_-,\ g+S_+)$ *with support in the forward cone D, where* $|s|\le t$.

Proof. In the sequel $\frac{\partial f}{\partial s}$ and $\frac{\partial}{\partial s}f$ denote the derivative in the usual sense and in the sense of distributions, respectively.

Observe first that, for $\varphi \in C_0^\infty(R^2)$,

$$< \frac{\partial}{\partial t} S_+, \varphi > + < \frac{\partial}{\partial s} S_+, \varphi > = - < S_+, \frac{\partial \varphi}{\partial t} > - < S_+, \frac{\partial \varphi}{\partial s} >$$

$$= - \int_0^\infty \frac{d}{dt} \varphi(t,t)\, dt = \varphi(0,0) ,$$

which shows that

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s} \right) S_+ = \delta_{(0,0)} . \tag{1.7}$$

Analogously, we prove

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial s} \right) S_- = \delta_{(0,0)} . \tag{1.8}$$

Furthermore,

$$< (\frac{\partial}{\partial t} - \frac{\partial}{\partial s}) f - zg, \varphi > = \iint_{|s| \le t} \left[f(-\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial s}) - zg\varphi \right] ds\, dt$$

$$= \int_0^\infty \int_{-t}^t f \frac{\partial \varphi}{\partial s}\, ds\, dt - \int_{-\infty}^\infty \int_{|s|}^\infty f \frac{\partial \varphi}{\partial t}\, dt\, ds - \iint_{|s| \le t} zg\varphi\, ds\, dt$$

$$= \int_0^\infty f\varphi]_{-t}^t\, dt - \int_{-\infty}^\infty f\varphi]_{|s|}^\infty\, ds + \iint_{|s|<t} \left[\frac{\partial f}{\partial t} - \frac{\partial f}{\partial s} - zg \right] \varphi\, ds\, dt .$$

Using $f(t,t) = f(t,-t) = z/2$ and (1.2),

$$< (\frac{\partial}{\partial t} - \frac{\partial}{\partial s}) f - zg, \varphi > = z < S_+, \varphi > .$$

Yet, remembering (1.8), this is

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial s} \right) (f + S_-) - z(g + S_+) = \delta_{(0,0)} .$$

Analogously we show that

$$< (\frac{\partial}{\partial t} + \frac{\partial}{\partial s}) g - zf, \varphi > = z < S_-, \varphi > ,$$

which, together with (1.7) shows that

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s} \right) (g + S_+) - z(f + S_-) = \delta_{(0,0)} ,$$

as desired.

In order to complete the proof observe that the map

$$D \times D \mapsto R^2\ ,\ (t,x) \times (s,y) \mapsto (s+t, x+y)$$

is proper and, consequently, the convolution of any two distributions with the supports in D is well-defined (for the definition we use and a proof of the above statement see [17] p.104). Thus the uniqueness of the solution one proves as in Th. 6.2.3., p.141 of [17].

It is well-known (see [3] ch.VI. 8 and VI. 12, compare [7] p.167) that the unique solution of the telegraph equation (0.2) - (0.2″) is

$$u(t,x) = \left(a + \frac{1}{2}\frac{\partial}{\partial t}\right)\left(e^{-at}\int_{-t}^{t} I_0(a\sqrt{t^2-s^2})u_0(x+\nu s)\,ds\right) \tag{1.9}$$
$$+\frac{1}{2}e^{-at}\int_{-t}^{t} I_0(a\sqrt{t^2-s^2})u_1(x+\nu s)\,ds\ .$$

Using Corollary 1 and the above formula one can prove a result which is similar to (0.3).

Corollary 3. *The solution of the telegraph equation* (0.2) *is given by*

$$u(t,x) = \frac{1}{2}\,Eu_0[x+\nu\xi_a(t)] + \frac{1}{2}\,Eu_0[x-\nu\xi_a(t)]$$
$$+\frac{1}{a}\,E\{1_{A(t)}u_1[x+\nu\xi_a(t)]\}\ ,$$

where $A(t) = \{\omega : N(\omega) = 1,3,5,\ldots\}$.

Proof. By the linearity of the telegraph equation, the result of M.Kac (i.e. (0.3) with $u_1 \equiv 0$), and the formula (1.9) it is enough to prove that

$$\frac{1}{a}\,E\{1_{A(t)}u_1[x+\nu\xi_a(t)]\} = \frac{1}{2}\,e^{-at}\int_{-t}^{t} I_0(a\sqrt{t^2-s^2})u_1(x+\nu s)\,ds\ .$$

Since the above equation is equivalent to the first equation in (1.5) the proof is completed.

Let us show additionally that Corollary 1 enables us to prove the M. Kac's formula, as well. Let us set

$$SQ = a\sqrt{t^2-s^2}\ .$$

Using (1.9), with $u_1 \equiv 0$,

$$u(t,x) = \frac{a}{2}\,e^{-at}\int_{-t}^{t} I_0(SQ)u_0(x+\nu s)\,ds + \frac{1}{2}\,e^{-at}u_0(x+\nu t)$$
$$+\frac{a}{2}\,e^{-at}\int_{-t}^{t}\frac{t}{\sqrt{t^2-s^2}}\,I_1(SQ)u_0(x+\nu s)\,ds + \frac{1}{2}\,e^{-at}u_0(x-\nu t)\ .$$

Now, the equation

$$\frac{a}{2}e^{-at}\int_{-t}^{t}\frac{t}{\sqrt{t^2-s^2}}I_1(SQ)u_0(x+\nu s)\,ds$$

$$=\frac{1}{2}\left[\frac{a}{2}e^{-at}\int_{-t}^{t}\frac{t+s}{\sqrt{t^2-s^2}}I_1(SQ)u_0(x+\nu s)\,ds\right.$$

$$\left.+\frac{a}{2}e^{-at}\int_{-t}^{t}\frac{t-s}{\sqrt{t^2-s^2}}I_1(SQ)u_0(x+\nu s)\,ds\right]$$

results in

$$u(t,x)=\frac{1}{2}\left[\frac{a}{2}e^{-at}\int_{-t}^{t}I_0(SQ)u_0(x+\nu s)\,ds\right.$$

$$+\frac{a}{2}e^{-at}\int_{-t}^{t}\frac{t+s}{\sqrt{t^2-s^2}}I_1(SQ)u_0(x+\nu s)\,ds$$

$$+\frac{a}{2}e^{-at}\int_{-t}^{t}I_0(SQ)u_0(x+\nu s)\,ds$$

$$+\frac{a}{2}e^{-at}\int_{-t}^{t}\frac{t-s}{\sqrt{t^2-s^2}}I_1(SQ)u_0(x+\nu s)\,ds$$

$$\left.+\frac{1}{2}e^{-at}u_0(x+\nu s)+\frac{1}{2}e^{-at}u_0(x-\nu t)\right].$$

Performing the substitution $s\to -s$ in the third and fourth integral, by (1.5), (compare [22] p.530),

$$u(t,x)=\frac{1}{2}\left[E\{1_{A(t)}u_0[x+\nu\xi_a(t)]\}+E\{1_{B(t)}u_0[x+\nu\xi_a(t)]\}+\frac{1}{2}e^{-at}u_0(x+\nu t)\right.$$

$$\left.+E\{1_{A(t)}u_0[x-\nu\xi_a(t)]\}+E\{1_{B(t)}u_0[x-\nu\xi_a(t)]\}+\frac{1}{2}e^{-at}u_0(x-\nu t)\right].$$

Taking into account that $\xi_a(t)=t$ iff $N_a(t)=0$, we obtain the desired result.

As we have mentioned before, the probabilistic formula for the solutions of the telegraph equation was proved for the first time by M. Kac [23-24], in the case when $u_1\equiv 0$. The result of J. Kisyński (0.3) ([26]), which extends the probabilistic approach to the general case, differs from Corollary 3 only in the term that depends on u_1. The form of this term introduced above seems to be interesting. For example, we see that, when $u_0\equiv 0$, the solution $u(t,x)$ is, at the moment t, independent of the events ω with the property that $N_a(t)(\omega)$ is even. Furthermore, from our formula it is evident that, when $a\to\infty$, the term that depends on u_1 vanishes. This is, obviously, well-known (see Introduction) but it is not seen from (0.3).

Since the asymptotic behaviour of the solutions of the telegraph equation was examined by a number of authors, who employed various, often advanced methods, it may be found somewhat surprising that simple calculations presented in this paper may so easily lead us to results, that are in essence parallel to those proved before (see also Corollary 4). Note however that Corollary 3 was obtained by comparing the result that was proved using a strictly probabilistic approach (Lemma 1, [26]) with the analytic formulas such like the equation (1.9) and formulas appearing in the proof of Proposition 1. Thus, as a matter of fact, in the proof of Corollary 3 (and Corollary 4) both probabilistic approach presented in [26] and analytic methods employed in [3] were involved.

On the other hand, in some cases the formula (0.3) (with $u_0 \equiv 0$) is much more convenient in applying, than Corollary 3 is. For example, it is obvious that $u(t,x)$ should be twice differentiable with respect to t, provided u_1 is of the class C^1. But, since the set $A(t)$ varies together with t, the attempt of differentiation of the term appearing in Corollary 3 faces serious difficulties, even if $u_1 \in C_0^\infty$. Such difficulties do not occur when we employ (0.3).

2. The Poisson - Kac process as a modified jump process on $\mathcal{G}$. We would like to show a close relation between some modification of the jump process considered by W. Feller ([8] ch.10) and the Poisson - Kac process. For, let $N_a(t)$, $t \geq 0$ be a Poisson process and let K be a stochastic kernel $K(\cdot,\cdot)$ on R (i.e., for $x \in R, K(x,\cdot)$ is a probability measure on R). Let us consider a process which can be described as follows: we demand that the point $x \in R$ jumps according to the law $K(x,\cdot)$ when $N_a(t)$ jumps and that it moves with velocity $+1$ between jumps. W. Feller noted that such a process exists under the assumption that $K(x,\cdot)$ has a density $k(x,\cdot)$. Then the probability that the process which starts at the point x will be, after a time $t > 0$, at the point $x + t$ is equal e^{-at} (this is probability that $N_a(t) = 0$); apart from this point it has a density $h(t,x)$. Furthermore,

$$\frac{\partial h(t,x)}{\partial t} = -\frac{\partial h(t,x)}{\partial x} - ah(t,x) + a\int_R h(t,y)k(y,x)\,dy\,, \tag{2.1}$$

provided $h(t,x)$ is differentiable with respect to x.

In order to see the analogy between the above considered process and the Poisson - Kac process, note that, by (1.2) and (1.3), the functions $\tilde{f} = e^{-at}f$, $\tilde{g} = e^{-at}$ are solutions of (0.1) and that this equation may be viewed as

$$\frac{\partial F(t,\mathfrak{g})}{\partial t} = -(XF)(t,\mathfrak{g}) - aF(t,\mathfrak{g}) + a\int_{\mathcal{G}} F(t,\mathfrak{g}_1)k(\mathfrak{g}_1,\mathfrak{g})\,d\mathfrak{g}_1\,, \tag{2.2}$$

where :

- $\mathcal{G}$ is the group defined in Introduction,
- X is the element of the one-dimensional Lie algebra corresponding to $\mathcal{G}$; to be more specific: $X(\mathfrak{g})$ is the tangent vector to the curve $t \to (t,1)\mathfrak{g}$,
- $k(\mathfrak{g},\mathfrak{g}_1)$ is the "density" of the probabilistic measure concentrated at the point $\overline{g}$, (where by the definition $\overline{(x,1)} = (x,-1)$), i.e.

$$\int_{\mathcal{G}} F(t,\mathfrak{g}_1)k(\mathfrak{g}_1,\mathfrak{g})\,d\mathfrak{g}_1 = \int_{\mathcal{G}} F(t,\mathfrak{g}_1)k(g,g_1)\,d\mathfrak{g}_1 = F(t,\overline{\mathfrak{g}})\,.$$

(The fact that in our case the kernel $K(\mathfrak{g},\cdot)$ is deterministic constitutes the only difference between the two considered examples - but the existence of the Poisson - Kac process is obvious).

– $F = (\tilde{f}, \tilde{g})$, i. e. $F(x,-1) = \tilde{f}(x)$, $F(x,+1) = \tilde{g}(x)$, is the density of $g_a(t)$, $t \geq 0$.

Indeed, in order to prove (2.2) it sufficies to note that (see Introduction)

$$(2.3) \qquad (t,1)\cdot(s,l) = \begin{cases} (t+s,+1)\ , & l = +1\ , \\ (t-s,-1)\ , & l = -1\ . \end{cases}$$

Thus, the operator X assigns to the function $F : \mathcal{G} \mapsto R$, $F = (f, g)$ the function $XF = (-df/dx, dg/dx)$, as desired.

Hence, in the analogy to the process that was considered by W. Feller, the Poisson - Kac process can be thought as a process on $\mathcal{G}$ that jumps from the point (x,k) to $(x,-k)$ when the Poisson process jumps; it moves along the integral curves (2.3) between jumps. The probability that the process that starts at the point $g \in \mathcal{G}$ will be, after the time $t > 0$, at the point $(t,1)g$ is equal to e^{-at} , apart from this point it has the density (given by (1.4)) which satisfies (2.2).

3. The characteristic function of $\xi_a(t)$. Now we will show the following proposition.

Proposition 2. *The characteristic function $\varphi_a(t,\zeta)$ of $\xi_a(t)$ equals*

$$e^{-at}\left[\frac{1}{2} + \frac{i\zeta + a}{2\sqrt{a^2-\zeta^2}}\right] e^{\sqrt{a^2-\zeta^2}t} + e^{-at}\left[\frac{1}{2} - \frac{i\zeta + a}{2\sqrt{a^2-\zeta^2}}\right] e^{-\sqrt{a^2-\zeta^2}t}$$

for all $\zeta^2 < a^2$, and

$$e^{-at}\left[\cos\sqrt{\zeta^2 - a^2}t + \frac{i\zeta + a}{\sqrt{\zeta^2-a^2}} \sin\sqrt{\zeta^2-a^2}t\right]$$

for all $\zeta^2 > a^2$, and $\varphi_a(t, \pm a) = e^{-at}(1 \pm iat + at)$.

Proof. Using (1.4) with $z = a$, and (1.6) we have:

$$\varphi_a(t,\zeta) = e^{-at}\ e^{it\zeta} + e^{-at}\int_{-t}^{t} f(t,s)\ e^{is\zeta}\ ds + e^{-at}\int_{-t}^{t} g(t,s)\ e^{is\zeta}\ ds\ .$$

Setting

$$F(t,\zeta) = \int_{-t}^{t} f(t,s)\ e^{is\zeta}\ ds\ ,\quad G(t,\zeta) = \int_{-t}^{t} g(t,s)\ e^{is\zeta}\ ds\ ,$$

we get

$$(3.1) \qquad \varphi_a(t,\zeta) = e^{-at}\ e^{it\zeta} + e^{-at}F(t,\zeta) + e^{-at}G(t,\zeta)\ .$$

We would like to compute F and G. By the very definition and (1.2) with z replaced by a , (existence of the derivatives calculated below follows from (1.4) and the definition of F, G)

$$\begin{aligned}
\frac{\partial F(t,\zeta)}{\partial t} &= e^{it\zeta} f(t,t) + e^{-it\zeta} f(t,-t) + \int_{-t}^{t} \frac{\partial f(t,s)}{\partial t} e^{is\zeta}\, ds \\
&= e^{it\zeta} f(t,t) + e^{-it\zeta} f(t,-t) + \int_{-t}^{t} \frac{\partial f(t,s)}{\partial s} e^{is\zeta}\, ds + a\int_{-t}^{t} g(t,s) e^{is\zeta}\, ds \\
&= e^{it\zeta} f(t,t) + e^{-it\zeta} f(t,-t) + e^{is\zeta} f(t,s)\Big]_{-t}^{t} \\
&- i\zeta \int_{-t}^{t} f(t,s) e^{is\zeta}\, ds + a\int_{-t}^{t} g(t,s) e^{is\zeta}\, ds \\
&= 2e^{it\zeta} f(t,t) - i\zeta F(t,\zeta) + aG(t,\zeta) .
\end{aligned}$$

Since $f(t,t) = \frac{a}{2}$, we obtain ultimately,

$$\frac{\partial F(t,\zeta)}{\partial t} = -i\zeta F(t,\zeta) + aG(t,\zeta) + ae^{it\zeta} . \tag{3.2}$$

Analogously, using (1.3) and $g(t,-t) = 0$,

$$\frac{\partial G(t,\zeta)}{\partial t} = i\zeta G(t,\zeta) + aF(t,\zeta) . \tag{3.3}$$

Observe that in the equations (3.2), (3.3) the only derivative that appears is the derivative with respect to t. Therefore, they may be viewed as a system of ordinary differential equations with parameters a, ζ . Thus, by (3.2),

$$G = \frac{1}{a}[F' + i\zeta F] - e^{it\zeta} . \tag{3.4}$$

Performing substitution into (3.3),

$$F'' + (\zeta^2 - a^2)F = 0 .$$

A general solution of this equation is

$$F(t,\zeta) = \begin{cases} C_1 e^{\sqrt{a^2-\zeta^2}t} + C_2 e^{-\sqrt{a^2-\zeta^2}t} & , \zeta^2 < a^2 , \\ C_3 \cos\sqrt{\zeta^2 - a^2}t + C_4 \sin\sqrt{\zeta^2 - a^2}t & , \zeta^2 > a^2 , \\ C_5 t + C_6 & , \zeta^2 = a^2 . \end{cases}$$

Since $Prob\{N(0) > 0\} = 0$, we see that $0 = F(0,\zeta) = G(0,\zeta)$. Thus also, by (3.2), $F'(0,\zeta) = a$. Hence

$$\begin{aligned}
0 &= C_1 + C_2 , \\
a &= [C_1 - C_2]\sqrt{a^2 - \zeta^2} , \\
0 &= C_3 = C_6 , \\
a &= C_4\sqrt{\zeta^2 - a^2} = C_5 .
\end{aligned}$$

Thus, for $\zeta^2 < a^2$

$$F(t,\zeta) = \frac{a}{2\sqrt{a^2-\zeta^2}}\, e^{\sqrt{a^2-\zeta^2}t} - \frac{a}{2\sqrt{a^2-\zeta^2}}\, e^{-\sqrt{a^2-\zeta^2}t},$$

and, consequently, by (3.4),

$$G(t,\zeta) = \left[\frac{1}{2} + \frac{i\zeta}{2\sqrt{a^2-\zeta^2}}\right] e^{\sqrt{a^2-\zeta^2}t} + \left[\frac{1}{2} - \frac{i\zeta}{2\sqrt{a^2-\zeta^2}}\right] e^{-\sqrt{a^2-\zeta^2}t} - e^{it\zeta}$$

so that,

$$\varphi_a(t,\zeta) = e^{-at}\left[\frac{1}{2} + \frac{i\zeta + a}{2\sqrt{a^2-\zeta^2}}\right] e^{\sqrt{a^2-\zeta^2}t} + e^{-at}\left[\frac{1}{2} - \frac{i\zeta + a}{2\sqrt{a^2-\zeta^2}}\right] e^{-\sqrt{a^2-\zeta^2}t}.$$

In the case $\zeta^2 > a^2$ we proceed similarly:

$$F(t,\zeta) = \frac{a}{\sqrt{\zeta^2-a^2}} \sin\sqrt{\zeta^2-a^2}\; t,$$

$$G(t,\zeta) = \cos\sqrt{\zeta^2-a^2}\; t + \frac{i\zeta}{\sqrt{\zeta^2-a^2}} \sin\sqrt{\zeta^2-a^2}\; t - e^{it\zeta}.$$

As a result

$$\varphi_a(t,\zeta) = e^{-at}\left[\cos\sqrt{\zeta^2-a^2}\; t + \frac{i\zeta + a}{\sqrt{\zeta^2-a^2}} \sin\sqrt{\zeta^2-a^2}\; t\right],$$

as desired. The value of $\varphi_a(t,\pm a)$ may be obtained in the same manner or, since φ_a is a continuous function of ζ, as a limit $\lim_{\zeta\to\pm a}\varphi_a(t,\zeta)$. The proposition is proved.

The following corollary is a special case of the general results obtained by M. Fukushima - M. Hitsuda [9], M. Hitsuds - A. Shimizu [10] and R. Ellis [4].

Corollary 4. *When a increases to infinity, then the process $\sqrt{a}\xi_a(t)$, $t \ge 0$ converges in distribution to the standard Brownian motion.*

Proof. It suffices to prove that, for every $t \ge 0$,

$$\lim_{a\to\infty} \varphi_a(t,\sqrt{a}\zeta) = \exp\left[-\frac{t\zeta^2}{2}\right],$$

(pointwise with respect to ζ).

Given $\zeta \in R$, $t \geq 0$, for a large enough, (to be more specific: for $a > \zeta^2$) we have,

$$\varphi_a(t, \sqrt{a}\zeta) = e^{-at}\left[\frac{1}{2} + \frac{\sqrt{a}i\zeta + a}{2\sqrt{a^2 - a\zeta^2}}\right] e^{\sqrt{a^2 - a\zeta^2}\, t}$$

$$+ e^{-at}\left[\frac{1}{2} - \frac{\sqrt{a}i\zeta + a}{2\sqrt{a^2 - a\zeta^2}}\right] e^{-\sqrt{a^2 - a\zeta^2}\, t}$$

$$= \exp\left\{at\left[\sqrt{1 - \zeta^2/a} - 1\right]\right\}\left[\frac{1}{2} + \frac{i\zeta/\sqrt{a} + 1}{2\sqrt{1 - \zeta^2/\sqrt{a}}}\right]$$

$$+ \exp\left\{at\left[-\sqrt{1 - \zeta^2/a} - 1\right]\right\}\left[\frac{1}{2} - \frac{i\zeta/\sqrt{a} + 1}{2\sqrt{1 - \zeta^2/\sqrt{a}}}\right].$$

Observe that the expressions in square brackets tend to 1 and 0 , respectively. Next,

$$0 < \exp\left\{at\left[-\sqrt{1 - \zeta^2/a} - 1\right]\right\} \leq 1 \,,$$

and

$$\lim_{a\to\infty} at\left[\sqrt{1 - \zeta^2/a} - 1\right] = \lim_{a\to\infty} \frac{at\left[1 - \zeta^2/a - 1\right]}{\sqrt{1 - \zeta^2/a} + 1} = \frac{-\zeta^2 t}{2}.$$

Thus

$$\lim_{a\to\infty} \varphi_a(t, \sqrt{a}\zeta) = \exp\left(-t\zeta^2/2\right),$$

as desired.

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Remark. After completion of the paper I was informed that the characteristic function of the process $(\xi_a(t),\ t \leq 0)$ was obtained, in a different way, by M. Pinsky in the monograph "Lecture on Random Evolutions" (World Scientific Press, 1991).

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