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An Extension of Typically Real Functions

Abstract. For a fixed $\lambda>0$ let $T_R(\lambda)$ stand for the class of functions f defined by the formula $f(z)=\int_{-1}^{1} z(1-2xz+z^2)^{-\lambda}\,d\mu(x)$, where μ is a probability measure on $[-1,1]$.

Obviously $T_R(1)$ coincides with the class of typically real functions. Some convolution and coefficient results previously established for $T_R(1)$ are extended to the class $T_R(\lambda)$.

1. Introduction

Let $A_1(D)$ denote the class of holomorphic functions

$$f(z) = z + a_2 z^2 + \cdots \ , \tag{1}$$

in the unit disk $D = \{z : |z| < 1\}$.

By $T_R(\lambda)$, $\lambda \geq 0$ we denote the subclass of $A_1(D)$ consisting of functions f which have the integral representation

$$f(z) = \int_{-1}^{1} \frac{z}{(1-2xz+z^2)^\lambda}\, d\mu(x)\ , \tag{2}$$

where μ is a probability measure on the interval $[-1,1]$.

If $S_R^*(\alpha)$, $-\infty < \alpha \leq 1$, is the family of holomorphic functions of the form (1) which are starlike of order α in D and have real coefficients, then we see that the function

$$s_\lambda(z,x) := \frac{z}{(1-2xz+z^2)^\lambda}\ ,\ x \in [-1,1]\ ,\ z \in D\ , \tag{3}$$

is in $S_R^*(1-\lambda)$ because

$$\operatorname{Re}\frac{zs'_\lambda(z,x)}{s_\lambda(z,x)} = 1-2\lambda+2\lambda\operatorname{Re}\frac{1-xz}{1-2xz+z^2} \geq 1-\lambda\ ,\ z\in D\ .$$

This fact implies that $T_R(\lambda_1)\subset T_R(\lambda_2)$ for $\lambda_1<\lambda_2$. Because $T_R(0)=\{z\}$ in what follow we assume $\lambda>0$.

Let us observe that $T_R(1)=T_R$ is the well-known class of typically-real functions [1], [6], [11]. Moreover, the class $T_R(\lambda)$ is a convex set in the space $A_1(D)$ which is a locally convex linear topological space with the respect to the topology given by uniform convergence on compact subsets of D. So by Krein-Milman theorem every convex functional on $T_R(\lambda)$ attains its extremal values on the extreme points of $T_R(\lambda)$ [6]. It has been proved by Hallenbeck [2] that

$$(4)\quad T_R(\lambda)=\overline{\text{co}}S_R^*(1-\lambda)\ ,\quad \text{ext}T_R(\lambda)=\{s_\lambda(z,x): x\in[-1,1]\}\ .$$

The following two results are known for typically-real functions:

Theorem A (Robertson [8]). *If* $f(z)=z+\sum_{n=2}^\infty a_n z_n\in T_R$ *and* $g(z)=z+\sum_{n=2}^\infty b_n z^n\in T_R$, *then*

$$(f*_1 g)(z):=z+\sum_{n=2}^\infty \frac{a_n b_n}{n}\in T_R\ .$$

Theorem B (Leeman [4]). *If* $f\in T_R$, *then*

$$n-a_n\leq\frac{1}{6}n(n^2-1)(2-a_2)\ ,\quad n=3,4,\ldots\ .$$

Alternative proofs of Theorem A and Theorem B were presented by Krzyż and Złotkiewicz in [3] and by Ruscheweyh in [9] and [10].

In this note we extend in an appropriate way Theorem A and Theorem B to the class $T_R(\lambda)$. We will use convolution results of

Ruscheweyh [9] and Lewis [5] and the properties of Gegenbauer polynomials $C_n^{(\lambda)}(x)$, $\lambda > 0$, $x \in [-1,1]$, $n = 0,1,\dots$, which are defined by the generating function

(5) $$\frac{z}{(1-2xz+z^2)^\lambda} = z\sum_{n=0}^{\infty} C_n^{(\lambda)}(x)z^n,\ z \in D,\ \lambda > 0 .$$

2. Statements of results

In what follow we will use the following notations:

$$(\alpha)_n := \alpha(\alpha+1)\dots(\alpha+n-1)\ ,\ n = 1,2,\dots\ ,\ (\alpha)_0 = 1\ ,\ \alpha \neq 0\ ,$$

(6) $$s_\lambda(z,1) = \frac{z}{(1-z)^{2\lambda}} = \sum_{n=1}^{\infty} A_n(\lambda)z^n\ ,\ A_n(\lambda) = \frac{(2\lambda)_{n-1}}{(n-1)!}\ .$$

Theorem 1. *If* $f \in T_R(\lambda)$, *then*

(7) $$|a_n| \le \frac{(2\lambda)_{n-1}}{(n-1)!}\ ,\ n = 1,2,\dots\ .$$

Inequality (7) *is sharp and the extremal function has the form* (6).

Theorem 2. *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in T_R(\lambda)$ *and* $g(z) = z + \sum_{n+2}^{\infty} b_n z^n \in T_R(\lambda)$, *then*

(8) $$(f *_\lambda g)(z) := \sum_{n=1}^{\infty} \frac{a_n b_n}{A_n(\lambda)} \in T_R(\lambda)\ .$$

Corollary 1. *If* $\lambda = 1$ *then we have Robertson's result* [8] (Theorem A).

Corollary 2. *If* $\lambda = 1/2$, *then we have the result that the class* $T_R(1/2) = \overline{\text{co}}S_R^*(1/2)$ *is closed under Hadamard product.*

Theorem 3. *If* $f \in T_R(\lambda)$, *then the following sharp estimate holds*

$$\frac{(2\lambda)_{n-1}}{(n-1)!} - a_n \le \frac{(2\lambda+2)_{n-2}}{(n-2)!}(2\lambda - a_2) \ , \ n = 3,4,\dots \ . \tag{9}$$

For the function $f(z) = s_\lambda(z,x)$ *we have*

$$\lim_{z\to 1^-} \frac{\frac{(2\lambda)_{n-1}}{(n-1)!} - a_n}{2\lambda - a_2} = \frac{(2\lambda+2)_{n-2}}{(n-2)!} .$$

Corollary 3. *If* $f \in S^*_R(1-\lambda)$, $\lambda > 0$, *then the sharp estimate* (9) *holds.*

Corollary 4. *If* $C_n^{(\lambda)}(x)$, $n = 1,2,\dots$, $\lambda > 0$, *is a Gegenbauer polynomial, then*

$$\frac{C_n^{(\lambda)}(1) - C_n^{(\lambda)}(x)}{C_1^{(\lambda)}(1) - C_1^{(\lambda)}(x)} \le \frac{(2\lambda+2)_{n-1}}{(n-1)!} \quad \textit{for } x \in [-1,1] \ .$$

3. Lemmas

For the proof of Theorem 2 we need the following two lemmas.

Lemma 1 [9]. *Let* $V \subset A_1(D)$ *with* $W = \overline{\text{co}}\ V$ *compact. Assume there is a function h in* $A_1(D)$ *such that for all* $f, g \in V$ *we have*

$$h *_{1/2} f *_{1/2} g \in W \ . \tag{10}$$

Then (10) *holds for all* $f, g \in W$.

Lemma 2 [5]. *Let* $S^*(\alpha)$, $-\infty < \alpha \le 1$ *denote the class of* α*-starlike functions in* D . *If* $f(z) = z + \sum_{n=2}^\infty a_n z^n \in S^*(\alpha)$, $g(z) = z + \sum_{n=2}^\infty b_n z^n \in S^*(\alpha)$ *then*

$$(f *_{1-\alpha} g)(z) = \sum_{n=1}^\infty \frac{a_n b_n}{A_n(1-\alpha)} z^n \in S^*(\alpha) \ . \tag{11}$$

Lemma 3. *Let*

$$s_n := \sum_{k=0}^{n} \frac{(2\lambda)_k}{k!}, \quad n = 0, 1, \dots,$$

$$\sigma_n := \sum_{k=1}^{n} k\frac{(2\lambda)_k}{k!}, \quad n = 1, 2, \dots,$$

$$\tau_n := \sum_{j=1}^{n} \frac{(j-1)!}{(2\lambda)_j}\kappa_j,$$

$$\kappa_j := \sum_{k=1}^{j} \left(\frac{\lambda + k - 1}{\lambda}\right)\frac{(2\lambda)_{k-1}}{(k-1)!}, \quad j = 1, 2, \dots, \quad n = 1, 2, \dots, .$$

Then the following identities hold

$$s_n = \frac{(2\lambda+1)_n}{n!}, \quad n = 0, 1, \dots,$$

$$\sigma_n = 2\lambda\frac{(2\lambda+2)_{n-1}}{(n-1)!}, \quad n = 1, 2, \dots, \tag{12}$$

$$\tau_n = \frac{n(2\lambda+n)}{2\lambda(2\lambda+1)}, \quad n = 1, 2, \dots .$$

Proof. The proof of all identities (12) is based on induction argument. We will prove the third equality of (12). Formula (12) for τ_n is true for $n = 1$ and let us assume that it is true for $(n-1)$. Then we have

$$\tau_n = \tau_{n-1} + \frac{(n-1)!}{(2\lambda)_n}\kappa_n = \frac{(n-1)(2\lambda+n-1)}{2\lambda(2\lambda+1)}$$

$$+ \frac{(n-1)!}{(2\lambda)_n}\sum_{k=1}^{n}(1 + \frac{k-1}{\lambda})\frac{(2\lambda)_{k-1}}{(k-1)!}$$

$$= \frac{(n-1)(2\lambda+n-1)}{2\lambda(2\lambda+1)} + \frac{(n-1)!}{(2\lambda)_n}$$

$$\times \left\{1 + (1+\frac{1}{\lambda})\frac{(2\lambda)_1}{1!} + \dots + (1+\frac{n-1}{\lambda})\frac{(2\lambda)_{n-1}}{(n-1)!}\right\}$$

$$= \frac{(n-1)(2\lambda+n-1)}{2\lambda(2\lambda+1)} + \frac{(n-1)!}{(2\lambda)_n}\left\{s_{n-1} + \frac{1}{\lambda}\sigma_{n-1}\right\}$$
$$= \frac{(n-1)(2\lambda+n-1)}{2\lambda(2\lambda+1)} + \frac{(n-1)!}{(2\lambda)_n}\left\{\frac{(2\lambda)_{n-1}}{(n-1)!} + 2\frac{(2\lambda+2)_{n-2}}{(n-2)!}\right\}$$
$$= \frac{n(2\lambda+n)}{2\lambda(2\lambda+1)},$$

which ends the proof.

4. Proofs of theorems

Proof of Theorem 1. From the integral representation (2) and (5) we find that

$$|a_n| \le \max_{-1\le x\le 1} |C_{n-1}^{(\lambda)}(x)| .$$

Using the integral formula for Gegenbauer polynomials [7]

$$C_n^{(\lambda)}(x) = \frac{(2\lambda)_n \Gamma(\lambda+\frac{1}{2})}{n!\Gamma(\frac{1}{2})\Gamma(\lambda)} \times \int_0^{\pi} \left[x + \sqrt{x^2-1}\cos\varphi\right]^n \sin^{2\lambda-1}\varphi \, d\varphi , \quad n = 0, 1, \dots ,$$

we get after some manipulation with Euler Gamma function that

$$(13) \qquad |C_n^{(\lambda)}(x)| \le \frac{(2\lambda)_n}{n!} \qquad \text{for } x \in [-1,1] ,$$

which implies (7).

Proof of Theorem 2. Let $f, g \in T_R(\lambda)$. We will apply Lemma 1 and 2. In our case by (2) and (4) we have

$$V = \left\{ s_\lambda(z,x) : s_\lambda(z,x) = \frac{z}{(1-2xz+z^2)^\lambda} , \ x \in [-1,1] \right\} ,$$
$$W = \overline{\text{co}}\, V = T_R(\lambda) .$$

Let us put

$$h(z)=\sum_{k=1}^{\infty}A_n^{-1}(\lambda)z^n\ ,\quad A_n(1/2)=1\ .$$

Then we have

$$(f *_\lambda g)(z)=\sum_{n=1}^{\infty}\frac{a_n b_n}{A_n(\lambda)}z^n=(h *_{1/2} f *_{1/2} g)(z)\ .$$

If f and g are in V, then they are starlike of order $(1-\lambda)$ and by Lemma 2 so does $f *_\lambda g$, which implies $(h *_{1/2} f *_{1/2} g)\in W$. Applying Lemma 1 we end the proof.

Proof of Theorem 3. For $f(z)=z+\sum_{n=2}^{\infty}a_n z^n\in T_R(\lambda)$ we define the coefficients B_n , $n=1,2,\dots$, by the relation

$$\begin{aligned} nB_{n-1}&=na_{n+1}-2(\lambda+n-1)a_n+(2\lambda+n-2)a_{n-1}\ ,\\ a_1&=1\ ,\qquad a_0=0\ , \end{aligned}\tag{14}$$

From (2) we know that

$$a_n=\int_{-1}^{1}C_{n-1}^{(\lambda)}(x)\,d\mu(x)\ ,\ n=1,2,\dots\ .\tag{15'}$$

Using the recurrence formula for Gegenbauer polynomials [7]

$$\begin{aligned} &nC_n^{(\lambda)}(x)-2x(\lambda+n-1)C_{n-1}^{(\lambda)}(x)\\ &\qquad+(2\lambda+n-2)C_{n-2}^{(\lambda)}(x)=0\ ,\ n=2,3,\dots\ ,\\ &C_0^{(\lambda)}=1\ ,\quad C_1^{(\lambda)}(x)=2\lambda x\ ,\quad C_n^{(\lambda)}(1)=2\lambda_n/n!\ , \end{aligned}\tag{16}$$

we find from (15′) and (16) that

$$\begin{aligned} nB_{n-1}&=\int_{-1}^{1}\big[nC_n^{(\lambda)}(x)-2(\lambda+n-1)C_{n-1}^{(\lambda)}(x)\\ &\quad+(2\lambda+n-2)C_{n-2}^{(\lambda)}(x)\big]\,d\mu(x)\\ &=-2\int_{-1}^{1}(\lambda+n-1)(1-x)C_{n-1}^{(\lambda)}(x)\,d\mu(x)\\ &=-\frac{\lambda+n-1}{\lambda}\int_{-1}^{1}C_{n-1}^{(\lambda)}(x)(2\lambda-2\lambda x)\,d\mu(x)\ . \end{aligned}$$

By (13) we can write

$$(17)\quad nB_{n-1}=\Big(1+\frac{n-1}{\lambda}\Big)\frac{(2\lambda)_{n-1}}{(n-1)!}[a_2-2\lambda]\gamma_n\,,\ n=2,\dots,\gamma_1=1$$

where $\gamma_n\in[-1,1]$. From (14) we find

$$(18)\qquad \sum_{k=1}^{n} kB_{k-1}=na_{n+1}-(2\lambda+n-1)a_n\,,\ n=1,2,\dots\,.$$

After some calculations from (18) and (17) together with Lemma 3 one can get the following identity ($n\geq 2$)

$$a_n-\frac{(2\lambda)_{n-1}}{(n-1)!}=\frac{(2\lambda)_{n-1}}{(n-1)!}(a_2-2\lambda)\times\left[\frac{1}{(2\lambda)_1}S_1+\frac{1!}{(2\lambda)_2}S_2+\dots+\frac{(n-2)!}{(2\lambda)_{n-1}}S_{n-1}\right],$$

where $S_n=\sum_{k=1}^{n}(1+\frac{k-1}{\lambda})\frac{(2\lambda)_{k-1}}{(k-1)!}\gamma_k$. Taking into account that $|\gamma_n|\leq 1$, $n=1,2,\dots$, we have from the above relations

$$\frac{(2\lambda)_{n-1}}{(n-1)!}-a_n\leq\frac{(2\lambda)_{n-1}}{(n-1)!}(2\lambda-a_2)\tau_{n-1}\,.$$

Applying Lemma 3 we find (9).

REFERENCES

[1] Duren, P.L., *Univalent functions*, A Series of Comprehensive Studies in Mathematics 259, Springer Verlag 1983.

[2] Hallenbeck, D.J., *Convex hulls and extreme points of families of starlike and close-to-convex mappings*, Pacific J. Math. 57 (1975), 167–176.

[3] Krzyż, J.G. and E. Złotkiewicz, *Two remarks on typically-real functions*, Ann. Univ. Mariae Curie-Skłodowska Sect. A 30 (1976), 57–61.

[4] Leeman, G.B., *A local estimate for typically–real functions*, Pacific J. Math. 52 (1974), 481-484.

[5] Lewis, J.L., *Convolutions of starlike functions*, Indians Univ. Math. J. 27 (1978), 671–688.

[6] Pommerenke, Ch., *Univalent functions*, Vandenhoeck and Ruprecht, Göttingen 1975.

[7] Rainville, E., *Special functions*, Mac Millan Company, New York 1965.

[8] Robertson, M.S., *Applications of a lemma of Fejér to typically-real functions* , Proc. Amer. Math. Soc. 1 (1950), 555–561.

[9] Ruscheweyh, St., *Convolutions in geometric function theory*, Les Presses de l'Université de Montréal 1982.

[10] Ruscheweyh, St., *Nichtlineare Extremalprobleme für holomorphe Stieltjesintegrale*, Math. Z. 142 (1975), 19–23.

[11] Schober, G., *Univalent Functions– Selected Topics*, Lecture Notes in Mat., No.478, Springer–Verlag 1975.

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