ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN-POLONIA

VOL. XLVIII, 11 SECTIO A 1994

Maciej SKWARCZYŃSKI (Warszawa)

The Bergman Function, Biholomorphic Invariants and the Laplace Transform

Abstract. In this article the author attempts to present some trends in holomorphic geometry developed during the period 1970–1990.

Contents

1. Preliminary remarks

The pioneering investigations by Stefan Bergman (1895-1977) have affected, inspired and reshaped a vast part of complex analysis. We find it justified and convenient to refer to this area of research as "holomorphic geometry". While the object of the study is quite classical (biholomorphic mappings and their invariants), the methods employed are more recent and as a rule are borrowed from other,more special fields of analysis. This theory originated in 1921 during a seminar conducted in Berlin by E. Schmidt (among participants there were S. Bergman and S. Bochner). We now learn from [SHF 1] that Bergman misunderstood the task and investigated the orthogonal development not in the real interval (as was required of him) but in a complex domain $D \subset \mathbb{C}$. As a result he was led to the kernel function $K_D(z,t)$, $(z,t) \in D \times D$ which became a starting point of further research and is now known as the Bergman function of a domain D.

The fundamental ideas of the Bergman theory extend easily to several complex variables. This fortunate fact played an important role during the pioneering period of multidimensional function theory. It soon turned out that in almost every other aspect the multidimensional theory is radically different from the one–dimensional case.

The ideas of S. Bergman have stimulated many areas of analysis. In functional analysis one should mention the abstract approach of [ARN] and [MES]. In classical potential theory there is an important relation with the Green function $G_D(z,t)$, see [BS]*. Differential geometers became interested in properties of the Bergman metric tensor [KOB 1], [LI]. Some function theoretic aspects were developed in [BRM 1]. For an extensive bibliography of the subject until 1970 the reader is referred to the Bergman monograph [BE 1] (second edition). His methods (like representa tive coordinates, doubly orthogonal systems, comparison domains) still await full exploitation.

In the present article I attempt to report some of the progress in holomorphic geometry which took place during the period 1970-1990. There is no serious claim to completeness or objectivity and the reader is strongly urged to consult other surveys, such as [DIE], [HI], [SHA], [SAI]. Here we try in the first place to present the results which have simple formulation and elementary proofs. In principle we restrict our attention to functional Hilbert spaces of the form $L^2H(D,\mu)$ in which the inner product is defined by integration over the domain $D \subset \mathbb{C}^N$ and $d\mu/dm$ is a continuous positive function. Such "modera te generalization" is sufficient to include Fock–type spaces $L^2H(\mathbb{C}^N,\mu)$ which brings us to the point of contact with interesting physical interpretations. We are less interested in abstract theories [ARN], [HOF] in which holomorphicity and biholomorphic invariants do not play a dominant role. We are particularly interested in the interplay of holomorphic structure with the metric structure. To a large extent the importance and the appeal of complex analysis is due to its relation with Fourier analysis. "Going into complex domain permits to extend Fourier analysis beyond its normal range", see [MAC, p. 309].

2. Bergman spaces and evaluation functionals

The term "Bergman space" (without precise definition) has been circulating since 1970 mainly in papers on functional analysis. In the present article we shall restrict its use to the space $L^2H(D)$ extensively studied by S. Bergman. Our notation is similar to that of E. Hille [HI 2], who writes $L^2H(D)$. The Bergman space consists of all functions which are holomorphic and Lebesgue square integrable in a domain $D \subset \mathbb{C}^N$. It is a closed subspace of $L^2(D)$, hence a separable Hilbert space. For simply connected plane domains this space was studied already in 1914 by L. Bieberbach [BIE].

Slightly more general than $L^2H(D)$ is the weighted Bergman space $L^2H(D,\phi)$, obtained by replacing the Lebesgue measure m by a Borel measure μ with $\phi := d\mu/dm$ continuous and positive. As an example we mention the Fock space $F_\alpha(\mathbb{C}^N)$, $(\alpha > 0)$ obtained when $D = \mathbb{C}^N$, $\phi(z) = (\alpha/\pi)^n \exp(-\alpha z^2)$. It has interesting interpre- tations in quantum physics, with parameter α playing a role similar to the Planck constant. See [JPR]* p.48 and [KS]*. Some of the aspects of Bergman theory can be presented quite effectively in an abstract setting as done in [ARN], [MES]. In such axiomatic approach the Bergman function is replaced by the reproducing kernel in an appropriate Hilbert space. Usually the corresponding inner product is induced from $L^2(D)$ (the Bergman kernel), or from $L^2(\partial D)$ (the Szegö kernel). Further generalizations involve more general Banach spaces of analytic functions in a domain $D \subset \mathbb{C}^N$. Usually the corresponding norm is induced from $L^p(D)$ (e.g. the case of Dzhrbashyan spaces, see [DZH 1, 2]) or from $L^p(\partial D)$ (e.g. the case of Hardy spaces $H^p(D)$, see [HOF], [DUR]).

With the general situation described we now proceed to the holomorphic geometry proper. For $t \in D$ and a polydisc $P(t; r_1, \dots, r_n)$ contained in D one proves an elementary estimate

$$(2.1) \qquad |f(t)|^2 = (\text{vol}P)^{-1} \int_P f(z)\overline{f(z)}\, dm(z) \le \frac{\|f\|^2}{\pi^n (r_1, \dots, r_n)^2}$$

which implies continuity of the evaluation functional $\chi_t^\#$. Moreover, (1) shows that any norm convergent sequence $f_k \in L^2 H(D)$ is locally uniformly convergent to the same limit.

Nontriviality of the Bergman function. The Bergman function for an arbitrary domain $D \subset \mathbb{C}^N$ is defined by the following formula , see [SKW 3], [SHA]

$$(2.2) \qquad K_D(z,t) := <\chi_t, \chi_z> , \quad (z,t) \in D \times D.$$

Here $\chi_t \in L^2 H(D)$ represents $\chi_t^\#$. Since $K_D(z,t)$ is the value of χ_t at $z \in D$, it follows that K_D is holomorphic in n variables $z = (z_1, \dots, z_n)$. Since

$$(2.3) \qquad \overline{K_D(z,t)} = K_D(t,z)$$

it follows that K_D is antiholomorphic in n variables $t = (t_1, \dots, t_n)$. By Hartogs theorem on separate holomorphicity K_D is a holomorphic (hence continuous) function of $2n$ variables $(z, \bar{t})$. As a consequence $K_D(z,z)$, $z \in D$, is a continuous function.

The original definition of K_D was slightly less general. It was formulated under the assumption that $L^2 H(D) \neq \{0\}$, which means precisely that K_D does not vanish identically on $D \times D$. Let h_m, $m = 1, 2, \dots$, be a complete orthonormal system in $L^2 H(D)$. For a fixed $t \in D$ the Fourier series for χ_t with coefficients $<\chi_t, h_m> = \overline{h_m(t)}$ converges in $L^2 H(D)$, hence locally uniformly in D. Therefore $K_D(z,t)$ can be defined by apointwise convergent series

$$(2.4) \qquad K_D(z,t) = \sum_{m=1}^{\infty} h_m(z)\overline{h_m(t)} , \quad (z,t) \in D \times D .$$

This is the original definition given by S. Bergman. Several remarks are useful

1^0 Formula (2.4) shows that its right side does not depend on the choice of a complete orthonormal system in $L^2H(D)$.

2^0 One proves easily that the series (2.4) converges locally uniformly in $D \times D$. Indeed, by the Dini theorem the convergence is locally uniform on the "diagonal" $z = t$. The conclusion follows now from the inequality

(2.5)

$$\left|\sum_{m=k}^{k+s} h_m(z)\overline{h_m(t)}\right| \leq \left(\sum_{m=k}^{k+s} |h_m(z)|^2\right)^{1/2} \left(\sum_{m=k}^{k+s} |hm(t)|^2)\right)^{1/2} .$$

3^0 In general the definition (2.4) is not "constructive". For $n > 1$ the theory of mean square approximation by holomorphic functions is still in infancy, and we do not know how to prove that a given orthogonal system is complete in $L^2H(D)$. (Note however that for particular case of n– circular domain D all square integrable monomials square integrable form a complete orthogonal system in $L^2H(D)$, see [SKW 6]).

4^0 For some "favorable" domains D one may succeed to sum up the series (2.4) and obtain an explicit formula representing $K_D(z, t)$ in a closed form. One of the most impressive computations is due to Zinoviev [ZIN 1]. For $D \subset \mathbb{C}^N$ given by $D = \{|z_1|^{2/p_1} + \cdots + |z_n|^{2/p_n} < 1\}$ where $p_j \in \mathbb{N}$ he proved that

(2.6)

$$K_D(z,t) = (\pi^n p_1, \ldots, p_n)^{-1} \frac{\partial^n}{\partial q_1, \ldots, \partial q_n} \sum \frac{1}{1 - v_1 - \cdots - v_n}$$

where $q := (z_1\bar{t}_1, \ldots, z_n\bar{t}_n)$ and each v_j ranges over all roots of p_j–th degree of $q_j = z_j\bar{t}_j$. (One shows that the expression to be differentiated is actually a rational function of q_j, $j = 1, 2, \ldots, n$.

Note that according to (2.2)

$$(2.7) \qquad f(t) = \chi_t^{\#} f = <f, \chi_t> = \int_D f(z)\overline{K_D(z,t)}\, dm(z)$$

for every $t \in D$ and every $f \in L^2H(D)$. This is known as "the reproducing property" of the Bergman function. Surprisingly perhaps, one can generalize (2.7) to an arbitrary functional $\Phi^{\#} \in \overline{L^2H(D)}$, represented by $\Phi \in L^2H(D)$. The element Φ can be recovered from K_D , namely

$$(2.8) \qquad \Phi(z) = <\Phi, \chi_z> = \overline{\Phi^{\#}\chi_z} = \overline{\Phi^{\#}K_D(\cdot, z)} = \Phi^{\#}\overline{K_D(z, \cdot)}$$

For example, the evaluation at t of the partial derivative with respect to the variable z_m is represented by the function $\Phi(z) := (\partial/\partial\bar{t}_m)K_D(z,t)$ and we see that the latter is square integrable in D. Moreover, the reproducing property plays a role in the study of bounded linear operators $A, B : L^2H(D) \to L^2H(D)$. Introducing $A^t\chi_w$ one obtains easily the following anologue of the usual matrix multiplication formula, see [GUI], [BER 1]

$$(2.9) \qquad <AB\chi_z, \chi_w> = \int_D <A\chi_u, \chi_w><B\chi_z, \chi_u> dm(u)$$

We shall see that an important role in holomorphic geometry is played by the mapping $\chi : D \to L^2H(D)$ where $\chi(t) := \chi_t$. For n=1 we know that χ vanishes identically (i.e. dim $L^2H(D) = 0$) if and only if $\mathbb{C} \setminus D$ is polar. See [CAR], [SKW 8]. If this is the case, one can show [SKW 8] that the family $\{\chi(t),\ t \in D\}$ is linearly independent. Moreover, the evaluations of all derivatives at all points of D are linearly independent, as was lateron shown in [CHO 1]. As a consequence for a plane domain (n=1) dim$L^2H(D)$ is never a finite positive number.

For $n > 1$ the problem is more difficult and the situation is more complicated. First, we are unable so far to characterize domains with dim$L^2H(D) = 0$. Moreover, there exist (unbounded) domains for which χ vanishes at some points without vanishing identically [SKW 6]. Indeed, let us assume $n = 2$ and consider $D := \{|z_1| < 1,\ |z_2| < |z_1|^{-1}\}$. All square integrable monomials (i.e. $z_1^{m_1} \cdot z_2^{m_2}$, $m_1 > m_2$) give rise to a complete orthogonal system in $L^2H(D)$. After normalization all functions in the orthonormal system vanish on the plane $z_1 = 0$ and so does χ. Using the above idea Wiegerinck [WIE] was able to show (for $n > 1$) the existence of non-trivial, finite dimensional Bergman spaces. The same idea was applied by H. Boas in his counterexample to the Lu Qi-keng conjecture, see [BSH].

Transformation rule under biholomorphic mappings. In holomorphic geometry problems and methods are invariant under biholomorphic transformation $g : D \to G$ and the corresponding unitary mapping (canonical isometry) $U_g : L^2H(G) \to L^2H(D)$ given by

$$(U_g h)(z) : h(g(z))\partial g/\partial z \tag{2.10}$$

where $\partial g/\partial z$ denotes the complex Jacobian of g. The proof that U_g preserves inner product uses the well known identity $J_g(z) = |\partial g/\partial z|^2$, which in turn is a consequence of Cauchy-Riemann equations. We see here a natural and deep relation of the notion of the Bergman space with classical complex analysis and in particular with the theory of biholomorphic mappings. On the other hand (2.10) is relevant to a fundamental idea in functional analysis: a finite dimensional non-linear problem (biholomorphic mapping) corresponds to a linear infinitely dimensional problem (canonical isometry).

Applying (2.10) to a complete orthonormal system in $L^2H(G)$ one obtains immediately the following rule of transformation for the Bergman function

$$K_D(z,t) = K_G(g(z),\, g(t))(\partial g/\partial z)\overline{(\partial g/\partial t)}. \tag{2.11}$$

Thus the Bergman function for D is known if it is known for some biholomorphic image of D. Take for the particular case $n = 1$, $g : D \to \Delta$ - the Riemann mapping function onto the unit disc with $g(t) = 0$, $g'(t) > 0$. Immediate calculations show that $K_\Delta(w,s) = \pi^{-1}(1 - w\overline{s})^{-2}$. Hence (2.11) yields

$$\pi K_D(z,t) = g'(z)\, g'(t)\,. \tag{2.12}$$

Computing $g'(t)$ and $g'(z)$ we find the Riemann mapping function of D in terms of its Bergman function

$$g(z) = \int_t^z \frac{\pi K_D(z,t)}{g'(t)}\, dz = \sqrt{\frac{\pi}{K_D(t,t)}} \int_t^z K_D(z,t)\, dz\ . \tag{2.13}$$

In terms of evaluation functionals formula (2.10) can be rewritten as

$$\chi^D(t) = \overline{(\partial g/\partial t)}(U_g \chi^G(s))\ ,\quad s = g(t)\ . \tag{2.14}$$

Hence we see that $\chi^D(t)$, $t \in D$ are linearly independent iff $\chi^G(s)$, $s \in G$, are linearly independent. For a bounded G the latter condition is obviously satisfied since the Lagrange interpolation polynomials belong to $L^2H(G)$. In a particular case $D = G$ another simple consequence of (2.14) was observed by A. Odzijewicz. He noticed that for each fixed point $t = g(t) \in D$ the element $\chi^D(t) \in L^2H(D)$ is an eigenfunction for the unitary operator U_g.

Note that with any sequence of different points $t_n \in D$, $n = 1, 2, \ldots$ one can associate corresponding elements

$$\chi^D(t_n) \in L^2H(D)\ , \quad n = 1, 2, \ldots\ . \tag{2.15}$$

We recall from [SIN 1, 2] three types of independence of elements (2.15), which can be considered as conditions on the sequence t_n :

(i) algebraic independence,
(ii) minimality (no elements belong to the linear closure of all remaining elements),
(iii) basic sequence property (the sequence $\chi^D(t_n)$ defines a basis in the linear closure of all elements (2.15)).

It follows from (2.14) follows that each of these conditions is invariant under biholomophic mappings. Obviously (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). In a bounded domain (i) is always satisfied but the characterization of sequences satisfying (ii) or (iii) is essentially an open problem. (Compare remarks in [WAL 1] chapter 10.)

Some general properties of the Bergman function. H. Bremermann [BRM 1] found an expression for $K_H(z, w;\ t, s)$ in a product domain $H = D \times G$. We know that $K_D(\cdot\,, \cdot\,; t, s)$ can be described as the unique element in $L^2H(D)$ with reproducing property. For $t \in D$, $s \in G$, consider the product $K_D(\cdot\,, t)K_G(\cdot\,, s)$. It belongs to $L^2H(D \times G)$. Its reproducing property at $(t, s) \in D \times G$ follows easily from Fubini theorem in view of the fact that for every function $f \in L^2H(D \times G)$ and every $w \in G$ the function $z \mapsto f(\cdot\,, w)$ belongs to $L^2H(D)$. To see the latter one considers an estimate

(2.16)
$$\int_D |f(z, w)|^2\, dm(z) \le K_G(w, w) \int_D \int_G |f(z, u)|^2\, dm(u)\, dm(z) < \infty\ .$$

This proves the following theorem of H.Bremermann

$$K_{D\times G}((z,w),(t,s)) = K_D(z,t)K_G(w,s) \,. \tag{2.17}$$

E. Ligocka [LIG 1] proved the converse of (17) in the following form. Assume that $H \subset \mathbb{C}^{k\times m}$ is a domain of existence of a real analytic function $K_H((z,w),(z,w)) = \Phi(z)\Phi(w)$. Then H is a product domain $D \times G$ and $\Phi(z)$, $\Phi(w)$ are proportional to $K_D(z,z)$ and $K_G(w,w)$, respectively.

As another instance of the general principle let us consider a plane domain D bounded by finitely many analytic curves. Let $G_D(z,t)$ be the Green function with a pole at $t \in D$. It is harmonic on $(\mathrm{cl}\, D) \times D$ by Schwarz symmetry principle. A more detailed analysis [BS]* shows that

$$K_D(z,t) = -\frac{2}{\pi}\frac{\partial^2}{\partial z \partial \bar{t}} G_D(z,t) \,. \tag{2.18}$$

(The right-hand side singularity is removed in the process of double differentiation). A. Suszczynski [SUS] proved that (2.18) remains valid for an arbitrary plane domain D with non-polar complement. The proof is based on two ideas: 1^0 both sides transform in the same way under biholomorphic mappings, hence (2.18) holds for an arbitrary domain D_m bounded by a finite number of Jordan curves, 2^0 identity (2.18) for D, the union of an increasing sequence of domains D_m, $m = 1,2,\ldots$, follows by passing to the limit in the corresponding identity for D_m. (The limit on the right exists by standard arguments from potential theory, and the limit on the left exists in view of a general theorem due to I. Ramadanov [RAM], [SKW 2,11], see below.) The Schiffer–Suszczynski identity so far was not generalized to $N > 1$, so we can still search for multidimensional complex potential theory adequate for this purpose.

3. The invariant distance and Lu Qi–Keng Domains

Before going into details let us recall a general construction related to a separable Hilbert space $\mathbf{H}$. The projective Hilbert space $P(\mathbf{H})$ consists of all one-dimensional subspaces of $\mathbf{H}$. Equivalently,

$P(\mathbf{H})$ consists of proportionality classes in $\mathbf{H}\setminus\{0\}$, two non-zero vectors $f,g\in\mathbf{H}$ being proportional iff there exists $c\in\mathbb{C}\setminus\{0\}$ such that $f=cg$. For $[f],[h]\in P(\mathbf{H})$ one considers the distance $\rho([f],[h])$ in $\mathbf{H}$ between circles $S\cap[f]$, $S\cap[h]$ on the unit sphere S, see [KOB 1]. One verifies that $P(\mathbf{H})$ becomes a metric space with the distance ρ. For $f_n\in S$ and $h\in S$ it is easy to see that $\lim[fn]=[h]$ iff there exist complex unimodular constants c_n such that c_nf_n converges in $\mathbf{H}$ to h. In particular the projection $f\mapsto[f]$ is continuous. Also it is easy to see that $P(\mathbf{H})$ is a complete metric space. Indeed, consider a sequence $f_n\in S$ such that $[f_n]$ is Cauchy in $P(\mathbf{H})$. We need to show that it has a convergent subsequence and (by a remark above) it suffices to show that f_n has a convergent subsequence. Since $\mathbf{H}$ is complete, the latter follows if for every $\varepsilon>0$ there is a finite 2ε–net for $\{f_1,f_2,\dots\}$. Let m be such that $\rho([f_n],[f_m])<\varepsilon$ for all $n>m$. Then the desired 2ε–net consists of elements f_j, $j=1,2,\dots,m$ and $e^{\varepsilon i}f_m, e^{2\varepsilon i}f_m,\dots,e^{s\varepsilon i}f_m$ (s is chosen to satisfy $s\varepsilon>2\pi$).

We return now to a bounded domain $D\subset\mathbb{C}^N$ and consider $H=L^2H(D)$. The evaluations in D (and in biholomorphic image $G=g(D)$) are linearly independent. Therefore $[\chi^D(t)]$, $t\in D$, is a one–to–one mapping into $P(\mathbf{H})$. As a consequence the distance ρ in $P(\mathbf{H})$ induces a distance in D, see [KOB 1], [SKW 3], [BE 3] (2-nd ed.), [JP 2]*. Note that for $s=g(t)$ we can introduce normalized evaluations $k^D(t):=\chi^D(t)/||\chi^D(t)||$ and $k^G(s):=\chi^G(s)/||\chi^G(s)||$. In view of (14) the isometry U_g maps the circle $Z^G_s:=\{e^{i\alpha}k^G(s):\ \alpha\in\mathbb{R}\}$ onto the circle $Z^D_t:=\{e^{i\beta}k^D(t):\ \beta\in\mathbb{R}\}$. It follows that the quantity

$$\rho_D(z,t)=2^{-1/2}\operatorname{dist}(Z^D_z,Z^D_t)\ ,\quad z,t\in D \tag{3.1}$$

is invariant unter biholomorphic transformations. It is a pleasent surprize [SKW 3] that the invariant distance ρ_D admits a simple expression in terms of the Bergman function. In fact, the normalizing constant in (19) was chosen to simplify the expression ρ_D for the unit disc $D=\Delta$. One finds that

$$\begin{aligned}\rho_D(z,t)&=\left[1-\left(\frac{K_D(z,t)\,K_D(t,z)}{K_D(z,z)\,K_D(t,t)}\right)^{1/2}\right]^{1/2}\\&=\left(1-H_D^{1/2}(z,t)\right)^{1/2}.\end{aligned} \tag{3.2}$$

A useful feature of ρ_D is that it is determined by a symmetric quotient $H_D := (1-\rho_D^2)^2$, a nonnegative function which is $\mathbb{R}$-analytic on $D \times D$. This is important because $\mathbb{R}$-analytic identities are preserved in a process of analytic continuation. The identity $\rho_D^2 = 1 - H_D^{1/2}$ shows that ρ_D^2 is $\mathbb{R}$-analytic herever $K_D(z,t) \neq 0$, moreover

$$\frac{\partial H_D^{1/2}}{\partial z_j} = \frac{1}{2} H_D^{1/2} \left(\frac{\partial/\partial z_j \, K_D(z,t)}{K_D(z,t)} - \frac{\partial/\partial z_j \, K_D(z,z)}{K_D(z,z)} \right) = \frac{1}{2} H_D^{1/2} \frac{\partial}{\partial z_j} \ln \frac{K_D(z,t)}{K_D(z,z)}$$

$$\frac{\partial H_D^{1/2}}{\partial \overline{z}_j} = \frac{1}{2} H_D^{1/2} \left(\frac{\partial/\partial \overline{z}_j \, K_D(t,z)}{K_D(t,z)} - \frac{\partial/\partial \overline{z}_j \, K_D(z,z)}{K_D(z,z)} \right) = \frac{1}{2} H_D^{1/2} \frac{\partial}{\partial \overline{z}_j} \ln \frac{K_D(t,z)}{K_D(z,z)} \, .$$

We see in particular that the above derivatives vanish for $z = t$. We use this information to compute second order derivatives at $z = t$. We find that

$$\left(\frac{\partial^2}{\partial z_j \partial z_k} \right)_{z=t} H^{1/2} = \frac{1}{2} \left(\frac{\partial^2}{\partial z_j \partial z_k} \right)_{z=t} \ln \frac{K_D(z,t)}{K_D(z,z)} = \frac{1}{2} \left(\frac{\partial^2}{\partial z_j \partial z_k} \right)_{z=t} \ln \frac{K_D(z,t)}{K_D(z,t)} = 0$$

$$\left(\frac{\partial^2}{\partial z_j \partial \overline{z}_k} \right)_{z=t} H^{1/2} = \frac{1}{2} \left(\frac{\partial^2}{\partial z_j \partial \overline{z}_k} \right)_{z=t} \ln \frac{K_D(z,t)}{K_D(z,z)} = -\frac{1}{2} \left(\frac{\partial^2}{\partial z_j \partial \overline{z}_k} \right)_{z=t} \ln K_D(z,z) = -\frac{1}{2} \frac{\partial^2}{\partial t_j \partial \overline{t}_k} \ln K_D(t,t) \, .$$

Let us consider the function $z \mapsto \rho^2(z,t)$ and its differentials at $z = t$. This function vanishes at t along with its first differential. The second differential is equal to

$$D^2 \rho^2(t; w) = \frac{1}{2} \sum_{\substack{1 \le j \le N \\ 1 \le k \le N}} T_{j\overline{k}} w_j \overline{w}_k + T_{\overline{j}k} \overline{w}_j w_k \tag{3.3}$$

where $T_{j\bar{k}} = (\overline{T_{\bar{j}k}}) := (\partial^2/\partial t_j \partial t_{\bar{k}}) \ln K_D(t,t)$ are components of the Bergman metric tensor. This result gives a precise explanation to hybrid property $(T_{jk} = 0 = T_{\bar{j}\bar{k}})$ of the Bergman metric tensor. According to the Taylor formula of the 2-nd order

$$(3.4) \qquad |\rho_D^2(t,z) - \frac{1}{2!} D^2 \rho^2(t; z-t)| \leq A\,||z-t||^3\,,$$

where A depends only on partial derivatives of ρ^2 (or $H^{1/2}$) on $[t,z] \subset \{K_D \neq 0\}$. Therefore, if $z,\ t$ stay away from ∂D and are close to each other, we may take A = const. This in view ot the fact that the Bergman tensor is positive definite [KOB 1] yields an estimate

$$(3.5) \qquad \left|\rho_D(t,z) - \left[\frac{1}{2} D^2 \rho^2(t; z-w)\right]^{1/2}\right| \leq A_0\,||z-w||\,.$$

This shows that $l_\rho(\gamma)$ (ρ_D–length of C^1 curve $\gamma : [0,1] \to D$) can be expressed by the Bergman metric tensor. Indeed, let $1 = s_o < s_1 < \cdots < s_n = 1$ be an ε–partition of $[0,1]$ and denote $v^{(\alpha)} := \gamma(s_\alpha)$. Then the sum

$$(3.6) \qquad \sum_{\alpha=0}^{n-1} \left[D^2 \rho^2(v^{(\alpha)}\,;\, v^{(\alpha+1)} - v^{(\alpha)})\right]^{1/2}$$

converges to $2^{1/2} l_\rho(\gamma)$, as $\varepsilon \to 0$. On the other hand, for the integral which expresses Bergman length of γ

$$(3.7) \qquad 2^{-1/2} l_d(\gamma) := \int_{[0,1]} \Bigg(\sum_{\substack{1\leq j\leq N \\ 1\leq k\leq N}} T_{j,\bar{k}}\, \gamma_j'(s)\, \overline{\gamma_k'(s)} \Bigg)^{1/2} dm(s)$$

one can write a Riemann sum (corresponding to the same partition) as follows

$$(3.8) \qquad \sum_{\alpha=0}^{n-1} \left[D^2 \rho^2(v^{(\alpha)}; \gamma_*(d/ds)_{s_\alpha}\right]^{1/2}\,.$$

If $\varepsilon \to 0$, then the difference between the sums (3.8) and (3.6) approaches zero, hence the identity $2^{-1/2}l_d = 2^{1/2}l_\rho$ and finally

$$l_d = 2l_\rho \; . \tag{3.9}$$

For $p, q \in D$ taking infimum on both sides of (3.9) over all piecewise C^1 curves in D which join p to q yields

$$d_D(p,q) \geq 2\rho_D(p,q) \; . \tag{3.10}$$

More details concerning (3.10) can be found in [MPS]*. Note that an explicit computation for the unit disc Δ reveals a relation with the Blaschke factor , namely

$$\rho_\Delta(z,t) = \left| \frac{z-t}{1-z\bar{t}} \right| \tag{3.11}$$

and $\gamma(s) := re^{is}$ supplies an easily verifiable example of (3.9).

In the above context we would like to mention two basic problems

Problem 1 (Lu Qi-keng domains). From (3.2) it follows by Schwarz inequality that $\rho_D(z,t)$ is never greater than 1 and is equal to 1 if and only if $K_D(z,t) = 0$. We call D a Lu Qi-keng domain if K_D does not attain the zero value in $D \times D$. By the transformation rule this property is invariant under biholomorphic mappings . Since $K_\Delta(z,t) = \pi^{-1}(1-z\bar{t})^{-2}$, the unit disc Δ is a Lu Qi-keng domain, and so is every bounded simply connected plane domain. Speculations about more general results , see [LUK], [SKW 3] were referred to as the Lu Qi-keng conjecture. Meanwhile an elementary example of a non Lu Qi-keng ring was given in [SKW 3], and classical properties of elliptic functions were used in [ROS] to show that every ring is not Lu Qi-keng. N.Suita and A.Yamada [SY]* proved that every multiply connected plane domain D bounded by a finite number of analytic curves is not Lu Qi-keng. (They used the identity (2.18) together with the Riemann–Roch theorem on a compact Riemann surface defined as the Schottky double of D). Finally for n=2 H.Boas [BSH 1] constructed a bounded non–Lu Qi-keng domain which is strictly pseudo–convex and topologically trivial.

Problem 2 (ρ_D–topology). One would like to understand the relation between ρ_D–topology and euclidean topology in D. The euclidean topology is always stronger (formula (3.2) and continuity of K_D), but it is not known if both topologies are equal in all domains for which evaluation functionals $\chi_D(t)$, $t \in D$, are linearly independent. It is easy to see [SKW 6] that both topologies are equal for all bounded domains (more generally for $L^2H(D)$ containing the functions $1, z_1, \ldots, z_n$). An (invariant) assumption that for every $t \in D$ there is $\varepsilon > 0$ (possibly small) such that $B(t, \varepsilon)$ is relatively compact in the euclidean topology of D also implies that both topologies in D are equal.

4. Representative coordinates and biholomorphic equivalence

Holomorphic geometry offers a natural approach to the biholomorphic equivalence problem. Much awaits exploration here , but within limits of this article we attempt to describe the basic idea. For domains D, $G \subset \mathbb{C}^N$ we ask about the existence of biholomorphic mappping $g : D \to G$. For simplicity we assume that both domains are bounded and ρ–complete. (In particular D, G are domains of holomorphy). We may restrict our attention to the slightly more special problem : given $p \in D$ and $q \in G$ we look for biholomorphic mapping g which satisfies $g(p) = q$. In sufficiently small neighbourhood of (p,p) we consider the function $\Phi(z,t) := \ln K_D(z,t)$. By a result of Bergman [BE 1], [KOB 1] $\Phi(t,t)$, $t \in D$, is a Kaehler potential for the invariant metric. In particular the (Bergman) metric tensor with components $T_{r\bar{s}} := (\partial^2/\partial t_r \partial \bar{t}_s)\Phi$ is positive definite. It implies that near p the mapping $z \mapsto \mu^D(z,p)$, where

$$(4.1) \qquad \mu_s^D(z,t) := \left(\frac{\partial}{\partial \bar{t}_s}\right) \ln \frac{K_D(z,t}{K_D(t,t)}\,, \quad s = 1, 2, \ldots, n\,,$$

defines local coordinates (so called representative coordinates). The interest in this notion is due to the following fact: In terms of representative coordinates near $t \in D$ and near $s \in G$ the mapping g is given as a <u>linear</u> mapping. Indeed, in view of the identity

$$(4.2) \qquad (\partial/\partial \bar{t}_s) H_D(z,t) = H_D(z,t)\mu_s^D(z,t)$$

one can first differentiate both sides of

$$H_G(g(z), g(t)) = H_D(z,t) \tag{4.3}$$

and then divide by $H_D(z,t)$ to obtain

$$\mu_s^D(z,t) = \sum_{r=1}^{N} \mu_r^g(g(z), g(t))\overline{(\partial g_r/\partial t_s)} \, . \tag{4.4}$$

We can now make the following observation: *suffices to find biholomorphic mapping* g *of a small connected neighbourhood of* p *onto a neighbourhood of* q *, such that* $g(p) = q$ *and* g *satisfies the rule of transformation:* $K_D(z,t) = K_G(g(z), g(t))g'(z)\overline{g'(t)}$. Indeed, we may continue g analytically along a curve in D and (4.3) shows that it is always possible, provided $g(z)$ stays in G. But the latter condition follows from the assumption that G is complete. Moreover, the continuation is path–independent (even without assuming simple connectivity of D!). To see this consider two paths in D, which join p to p' and two corresponding continuations of g ; then we continue analytically (4.3), from $(p,p) \to (p',p')$ in such a way that z moves along the first path in D while t moves along the second path in D. Finally we claim that the image $g(D)$ is the whole domain G, since by the previous reasoning (with D and G interchanged) the mapping g^{-1} with values in D can be continued arbitrarily in G.

The meaning of the above result becomes more evident if the Bergman function near $p \in D$ (or $q \in G$) is from the outset expressed in some special local coordinates. Indeed, we can take $\mathbb{C}$–linear modification of representative coordinates for which the components of the Bergman metric tensor at p satisfies $T_{r\bar{s}}(0) = \delta_{rs}$. In terms of such new coordinates z', w' the local biholomorphic mapping $w' = g(z')$ is $\mathbb{C}$-linear, hence $w' = Uz'$, where the matrix U is unitary. Denoting by K'_D, K'_G the corresponding expressions for Bergman functions we obtain the transformation rule in the form $K'_D(z', t') = K'_G(g(z'), g(t'))$. There is no loss of generality in considering this equality only for $z' = t'$. Then it takes the form

$$K'_D(z',z') = K'_G(Uz',Uz') \, , \quad U - \text{unitary} \, . \tag{4.5}$$

Both sides in (4.5) are real analytic and positive. We see that the germs of K'_D and K'_G at $0 \in \mathbb{C}^N$ contain all information about biholomorphic equivalence. Moreover, both domains are equivalent iff both functions have congruent graphs; more precisely if the graph of K'_D can be obtained from the graph of K'_G by $\mathbb{C}$–linear isometry of the space $\mathbb{C}^N = \mathbb{R}^{2N}$.Thus in a sense the problem of biholomorphic equivalence is reduced to an apparently more algebraic problem.

5. More on representative coordinates

The importance of the notion of representative coordinates is well recognized [KR 2]. Therefore we shall add some more remarks concerning this mathematical idea including some unpublished computations from [SKW 2].

First of all let us mention the way in which S.Bergman used to introduce contravariant representative coordinates near a point $p \in D$. (We assume N=2 for simplicity). Among all functions $f \in L^2H(D)$ which satisfy $f(p) = 1$ there is a unique one with minimal norm ; it is denoted by M^1. Analogously M^{010} (respectively M^{001}) is determined by the conditions $f(p) = 0$, $(\partial f/\partial z_1)(p) = 1$, $(\partial f/\partial z_2)(p) = 0$ (respectively by $f(p) = 0$, $(\partial f/\partial z_1)(p) = 0$, $(\partial f/\partial z_2)(p) = 1$). The (contravariant) representative coordinates are defined by the formula

$$\nu_1(z) := \frac{M^{010}(z)}{M^1(z)} \quad \nu_2(z) := \frac{M^{001}(z)}{M^1(z)} \ . \tag{5.1}$$

It is easy to see that $M^1(z) = K_D(z,p)/K_D(p,p)$. To determine say M^{010} note that evaluations at p of a function f and its partial derivatives are linearly independent functionals on $L^2H(D)$, represented respectively by

$$\begin{aligned} g_1(z) &= K_D(z,p), \\ g_2(z) &= (\partial/\partial\bar{t}_1)_p K_D(z,t), \\ g_3(z) &= (\partial/\partial\bar{t}_2)_p K_D(z,t) \ . \end{aligned} \tag{5.2}$$

The admissible variety for, say M^{010} is non–void, closed and convex, hence contains the unique element with minimal norm. This admissible variety is mapped into itself by orthogonal projection onto the subspace spanned by g_1, g_2, g_3 . Hence there exist complex constants a_1, a_2, a_3 such that

$$a_1 g_1(z) + a_2 g_2(z) + a_3 g_3(z) = M^{010}(z). \tag{5.3}$$

After taking inner product with representing elements g_i and solving resulting Cramer system one finds that

$$a_i = \frac{0\,G_{1i} + 1\,G_{2i} + 0\,G_{3i}}{\det G}, \quad i = 1,2,3 \tag{5.4}$$

where G stands for the corresponding Gram matrix and G_{ij} are algebraic complements of G. Now substituting (5.4) into (5.3) yields

$$M^{010}(z) = \frac{G_{21}g_1(z) + G_{22}g_2(z) + G_{23}g_3(z)}{\det G} = \frac{\begin{vmatrix} K & K_{\overline{p}_1} & K_{\overline{p}_2} \\ K(z,p) & K_{\overline{p}_1}(z,p) & K_{\overline{p}_2}(z,p) \\ K_{p_2} & K_{p_1\overline{p}_2} & K_{p_2\overline{p}_2} \end{vmatrix}}{\begin{vmatrix} K & K_{\overline{p}_1} & K_{\overline{p}_2} \\ K_{p_1} & K_{p_1\overline{p}_1} & K_{p_1\overline{p}_2} \\ K_{p_2} & K_{p_2\overline{p}_1} & K_{p_2\overline{p}_2} \end{vmatrix}}. \tag{5.5}$$

We transform the denominator using the obvious identity

$$K_{p_i\overline{p}_j} - \frac{1}{K}K_{p_i}K_{\overline{p}_j} = KT_{i\overline{j}}\,. \tag{5.6}$$

To this end we subtract from the second column the first column multiplied by $K_{\overline{p}_1}/K$. Also from the third column we subtract the first column multiplied by $K_{\overline{p}_2}/K$. It follows easily that in (5.5) the value of the denominator is $K^3\det(T_{i\overline{j}})$. In the numerator we factor out $K(z,p)$ from the second row and K from each other row. Then operating with the first row we clear the first column. This yields covariant representative coordinates in the second row. Development with respect to this row yields

$$\begin{aligned} M^{010}(z) &= \frac{K^2K(z,p)}{K^3\det(T_{i\overline{j}})}\begin{vmatrix} 1 & K_{\overline{p}_1}/K & K_{\overline{p}_2}/K \\ 0 & \mu_1(z) & \mu_2(z) \\ 0 & T_{2\overline{1}} & T_{2\overline{2}} \end{vmatrix} \\ &= M^1(z)(T^{\overline{1}1}\mu_1(z) + T^{\overline{2}1}\mu_2(z))\,. \end{aligned} \tag{5.7}$$

Computing similarly $M^{001}(z)$ we arrive at the final formula

$$\nu^\alpha(z) = \sum_\beta T^{\overline{\beta},\alpha}\mu_\beta \tag{5.8}$$

where $T^{\overline{\beta},\alpha}$, β, $\alpha = 1,2,\ldots,N$ denotes the inverse matrix to $T_{\alpha\overline{\beta}}$, α, $\beta = 1,2,\ldots,N$. We see that covariant representative coordinates are related to contra variant representative coordinates via linear transformation. There is also another, natural and interesting approach to covariant representative coordinates. It is based on the Taylor expansion of the antiholomorphic mapping $z \mapsto \chi(z)$ near $p \in D$:

$$\chi(z) = \sum_{k=(k_1,\ldots,k_N)} g_k \overline{(z-p)}^k , \quad (g_k \in L^2H(D)) . \tag{5.9}$$

Denote by P_m, $(m \in \mathbb{N})$ the orthogonal projection onto the subspace $\operatorname{lin}\{g_k : |k| < m\}$. The covariant representative coordinates of order m are defined by the formula

$$\mu_k^m(z) := \frac{(P_m g_k)(z)}{g_0(z)} , \quad (|k| = m) . \tag{5.10}$$

(For $P_m g_k$, $|k| = m$ we propose the name *innovation coefficients* to indicate an analogy with the notion of innovation vector defined in the prediction theory.) Computing (5.10) in particular case $m = 1, N = 2$ for $\mu_1(z) := \mu^1_{(1,0)}(z)$, $\mu_2(z) := \mu^1_{(0,1)}(z)$ yields

$$\begin{aligned} \mu_1(z) &= \left(g_{(1,0)} - \frac{<g_{(1,0)}, g_0> \; g_0}{||g_0||^2} \right) : g_0 \\ &= \frac{K_{\overline{p}_1}(z,p)}{K(z,p)} - \frac{K_{\overline{p}_1}(p,p)}{K(p,p)} \end{aligned} \tag{5.11}$$

and similarly for $\mu_2(z)$.

The Taylor development (5.9) was essential in the proof of the following

Theorem ([MS]*). *Assume that $D \subset \mathbb{C}^N$ is a bounded domain and $g : D \to D$ is a biholomorphic automorphism with a fixed point. Then the set of all eigenfunctions for the unitary operator $U_g f = (f \circ g)\, g'$ is linearly dense in $L^2H(D)$.*

6. Invariant distance and Kobayashi completeness conjecture

For simplicity we shall restrict our attention to a bounded domain $D \subset \mathbb{C}^N$. We have already seen that completeness with respect to the invariant distance plays an essential role in the study of biholomorphic equivalence. We are going to discuss completeness with respect to:
(a) the invariant distance ρ_D and
(b) the geodesic distance d_D induced by the Bergman metric.

This is also a good occasion to restate some points from [MPS]* concerning a very inspiring and important paper [KOB 1]. We recall that by definition ρ_D is equal up to a constant to the distance induced from $PS(L^2H(D))$ via the Bergman imbedding $\chi/||\chi||$. Moreover, $PS(L^2H(D))$ is a complete metric space. Hence D is ρ_D complete iff the set

$$\left\{ \frac{\chi(z)}{||\chi(z)||} : z \in D \right\} \tag{6.1}$$

is closed in $PS(L^2H(D))$. S.Kobayashi gave in [KOB 1] an inte resting condition (K) which reads as follows

Definition 6.1. A bounded domain $D \subset \mathbb{C}^N$ satisfies the condition (K) if for every sequence $p_m \in D$, $m = 1, 2 \ldots$ which converges to some boundary point of D and for every $f \in L^2H(D)$

$$\lim \frac{|f(p_m)|^2}{K_D(p_m, p_m)} = 0 . \tag{6.2}$$

The original formulation was manifold oriented, hence more general, however (as admitted in [KOB 1, p.267]) this gene ralization is not essential. For bounded domains the original Kobayashi condition is obviously equivalent with definition 6.1. The motivation for (K) revealed in [KOB 1] is very important and we quote it in extenso.

"Bremermann [BRM 1] has studied the bounded domains with the following property (P) : the kernel $K_D(z, z)$ goes to infinity at every boundary point. He has shown that a bounded domain with the property (P) is a domain of holomorphy and that the converse is not true. Making use of of this result he has proved that if a bounded domain is complete with respect to the Bergman metric, then it is a domain of holomorphy. Since the kernel $K_D(z,z)$ is not intrinsically defined the property (P) is not intrinsic. We consider therefore condition (K) which is stronger than (P) but which is intrinsic".

It is easy to see that in the language of functional analysis condition (K) can be restated as follows : $p_m \to p \in \partial D$ implies that $k_m := \chi_D(p_m)/||\chi_D(p_m)||$ converges weakly to zero in $L^2H(D)$.

The latter property of k_m easily implies that $c_m k_m$ is convergent in $L^2H(D)$ for any sequence c_m on the unit circle . (Indeed, the $L^2H(D)$–limit of $c_m k_m$ must lie on the unit sphere, while its weak limit is zero.) Hence (K) implies that $[k_m]$ converges in $PS(L^2H(D))$ for no $p_m \to p \in \partial D$, which in turn implies that the Bergman imbedding of D has closed image in $PS(L^2H(D))$. We have thus arrived at

Theorem 6.2. *Condition* (K) *implies* ρ_D*–completeness.*

Using the inequality $\rho_D \le d_D/2$ obtained in section 3 it is easy to see that ρ_D–completeness implies d_D–completeness. This suggests the following remark. The famous conjecture of [KOB 1] that d_D–completeness implies condition (K) (Kobayashi completeness conjecture) is by now 35 years old . It implies two (formally easier) statements which in our opinion should be treated separately:

1^0 d_D–completeness implies ρ_D–completeness

2^0 ρ_D–completeness implies (K) .

The statement 2^0 will be referred to as SCC (small completeness conjecture). So far SCC has not been settled even for N=1 due to difficulties with domains of infinite connectivity.

7. The ideal boundary

From the point of view of holomorphic geometry a theory of boundary behaviour should carefully distinguish between its intrinsic and non–intrinsic components. This postulate is suggested by a well established custom in classical potential theory, where the intrinsic notion of Martin boundary plays such an eminent role [HLM]. As a first step in realizing such a program one has to construct an invariant compactification and to investigate the corresponding ideal boundary. For Lu Qi-keng domains one easily imitates the construction of Martin boundary by replacing the Green function by the Bergman function [SKW 6]. Here we recall a more refined and general construction [SKW 10], based on the Bergman imbedding into the projective Fréchet space $PH(D)$. We denote by $H(D)$ the space of all functions holomorphic in D with the (Fréchet) topology of locally uniform convergence. Let $H^*(D) := H(D) \setminus \{0\}$. Functions $f, g \in H^*(D)$ are called proportional iff there exists complex constant $c \neq 0$ such that $f = cg$. This is obviously an equivalence relation and we denote by $PH(D)$ the set of all equivalence classes. It is considered with the quotient topology (i.e. the largest topology such that the canonical projection $\pi(f) := [f]$ is continuous). One verifies that PH(D) is a separable Hausdorff space and that π is open. A sequence $[f_m] \in PH(D)$, $m = 1, 2, \dots$ converges to $[f] \in PH(D)$ iff there exist complex constants $c_m \neq 0$ such that $c_m f_m \to f$ in $H^*(D)$. Also the eqivalence relation in $H^*(D)$ is closed.

Definition 7.1. A compactification of a domain $D \subset \mathbb{C}^N$ is a homeomorphism $q : D \to X$ onto an open dense subset in a compact Hausdorff space X. With no loss of generality we can additionally require $q = \mathrm{id}$. Two compactifications $q_i : D \to X_i$, $i = 1, 2$ are called equivalent if there exists a homeomorphism $w : X_1 \to X_2$ such that $q_2 = w \circ q_1$.

When $D \subset \mathbb{C}^N$ is bounded there exists the euclidean compactification $\mathrm{id} : D \to \mathrm{cl}\, D$. We say that D is regular if $q : D \to X$ is equivalent to the euclidean compactification. In particular the unit disc Δ is regular with respect to the Carathéodory (prime ends) compactification. Every $h \in \mathrm{Aut}(\Delta)$ extends to a homeomorpfism $h : \mathrm{cl}\, \Delta \to \mathrm{cl}\, \Delta$. Hence , up to equivalence, one can define Carathéodory compactification of simply connected plane domain $D \neq \mathbb{C}$ as $q : D \to \mathrm{cl}\, \Delta$. (Here $q : D \to \Delta$ is a Riemann mapping function.) The following definition is introduced with the aim to generalize the notion of Carathéodory compactification to several complex variables. In the present section we restrict our attention to domains D for which the invariant distance is well defined . (This means that $\chi_D(z)$ and $\chi_D(t)$ are linearly independent provided that $z, t \in D$ and $z \neq t$.)

Definition 7.2. Assume that the mapping $p : D \to PH(D)$ given by $p(t) := [K_D(\,\cdot\,; t)]$ has a relatively compact image and $p : D \to p(D)$ is a homeomorphism onto an open dense subset of $\hat{D} := \operatorname{cl} p(D)$. Then we say that D admits the invariant compactification $p : D \to \hat{D}$ and the compact set $\hat{D} \setminus p(D)$ is called the ideal boundary of D.

In view of trasformation rule of K_D it is easy to verify that the property described in this definition is invariant under biholomorphic mappings. (One obtains a homeomorphism of $H^*(D)$ using multiplication by a zero–free holomorphic functions.) The name "invariant compactification" is explained by the following

Theorem 7.3. *Let $p_1 : D_1 \to \hat{D}_1$, $p_2 : D_2 \to \hat{D}_2$ be invariant compactifications of D_1, D_2. For every biholomorphic mapping $h : D_1 \to D_2$ the homeomorphism*

$$p_2 \circ h \circ p_1^{-1} : p_1(D_1) \to p_2(D_2) \tag{7.1}$$

extends to the unique homeomorphism $w : \hat{D}_1 \to \hat{D}_2$.

If in Theorem 7.3 we are willing to identify D_i with $p_i(D_i)$ for $i = 1, 2$ (why not ?) then $p_i = \mathrm{id}$ and (7.1) can be stated as follows: every biholomorphic mappping $h : D_1 \to D_2$ extends as a homeomorphism to ideal boundaries. In a particular case when both domains are bounded and regular we see that every biholomorphic mapping $h : D_1 \to D_2$ extends to homeomorphism $\hat{h} : \operatorname{cl} D_1 \to \operatorname{cl} D_2$. If in the latter case $D_1 = D = D_2$ we see that every $h \in \operatorname{Aut}(D)$ extends to the unique homeomorphism $h : \operatorname{cl} D \to \operatorname{cl} D$.

The detailed description of domains which admit the invariant compactification is not in sight. Nevertheless one can give a number of examples by applying the following theorems

Theorem 7.4. *A domain D admits the invariant compactification $p : D \to \hat{D}$ iff there exists a compactification $q : D \to X$ such that $p \circ q^{-1}$, (p as in* Definition 7.2*), extends as a one-to-one continuous mapping of X into $PH(D)$. In such a case q is equivalent to the invariant compactification p.*

Corollary 7.5. *A bounded domain $D \subset \mathbb{C}^N$ is regular iff*

$p(t) := [K_D(\cdot\,, t)]$ *extends as a continuous one-to-one mapping of* $\mathrm{cl}\, D$ *into* $PH(D)$.

Using Theorem 7.4 one can describe the behaviour of the invariant compactification with respect to cartesian product, or with respect to $L^2H(D)$–negligible subsets of D. Using Corollary 6.5 one proves the regularity of: 1^0 plane domains bounded by a finite number of Jordan curves, 2^0 complete circular bounded domains $D \subset \mathbb{C}^N$ which satisfy $\mathrm{cl}\, D \subset rD$ for each $r > 1$, 3^0 strictly pseudo–convex domains $D \subset \mathbb{C}^N$ with smooth boundary.
For details see [SKW 10].

8. Alternating projections and invariant angles

As illustrated by our previous considerations the Bergman function K_D plays a significant role in complex analysis. It is therefore important to realize that the simple (but rather non–constructive) definition of K_D in terms of an orthonormal basis can be supplemented with another definition which is both general and constructive. The latter definition is based on a method of alternating projection due to J.v.Neumann and I.Halperin , a fundamental result in functional analysis. It is worth to recall that the alternating method has its origin in the potential theory (H.Schwarz) or even in ancient number theory (euclidean algorithm). It has its manifestations in many areas of contemporary analysis including numerical methods (Kaczmarz), prediction theory (Wiener) and integral geometry (Helgason).

Theorem 8.1 (alternating projections). *Let* $\mathbf{H}$ *be a Hilbert space and for* $i = 1, 2, \ldots, m$ *let* $P_i : \mathbf{H} \to F_i$ *be the orthogonal projection onto a closed linear subspace* F_i. *Then for every* $f \in \mathbf{H}$

$$\lim_{n\to\infty} (P_m P_{m-1} \ldots P_1)^n f = Pf \tag{8.1}$$

where $P : \mathbf{H} \to F$ *denotes the orthogonal projection onto the inter section* F *of all subspaces* F_i.

This theorem was originally proved by I. Halperin. A very elegant proof, given by Amemiya and Ando can be found in Helgason book [HLG]. We shall also need

Theorem 8.2. *Let* $\mathbf{H}$ *be a Hilbert space and let* $P_j : \mathbf{H} \to F_j$, $j = 1, 2, \ldots$ *be a sequence of orthogonal projections onto a decreasing sequence of closed subspaces* F_j. *Then for every* $f \in \mathbf{H}$

$$\lim_{j\to\infty} P_j f = Pf \; . \tag{8.2}$$

Here $P : \mathbf{H} \to F$ *denotes the orthogonal projection onto the intersection* F *of all subspaces* F_j. *(The analogous result for an increasing sequence of subspaces* F_j *follows by conside ring orthogonal complements* $F_j^{\perp}$.*)*

For a proof see [STN]. The orthogonal projection $Q : L^2(D) \to L^2H(D)$ is known as the Bergman projection in D. It determines the Bergman function: If $B \subset D$ is a ball centred at $t \in D$ then $K_D(\cdot\,, t)$ is the Bergman projection of $(\mathrm{vol}B)^{-1}\chi_B$. Conversely $(Qf)(t)$ is given as the scalar product $<f, K_D(\cdot\,,t)>$ (in $L^2(D)$). We shall now discuss the Bergman projection in D using the alternating method. (See [SKW 11,12].)

Assume that D is equal to the union of (more simple) domains $D_1, \ldots, D_m$ such that the Bergman projection Q_j in D_j is known for $j = 1, \ldots, m$.

Denote by F_j the (closed) subspace in $L^2(D)$ consisting of functions with holomorphic restriction to D_j. Then the inter section F of all F_j, $j = 1, \ldots, m$ is the Bergman space $L^2H(D)$.

The orthogonal projection $P_j : L^2(D) \to F_j$ is known since it modifies a function f on D_j according to the (known) Bergman projection $Q_j : L^2(D_j) \to L^2H(D_j)$. More precisely, for $j = 1, \ldots, m$

$$(P_j f)(t) = \begin{cases} (Q_j f_{|D_j})(t), & t \in D_j \\ f(t), & t \in D \subset D_j \end{cases} \tag{8.3}$$

and we see that $Q = P$ where P is given by (8.1) and (8.3). For example domains D_j, $j = 1, 2, \ldots, m$ can be balls in $\mathbb{C}^N$ and Q can be the Bergman projection in a domain which is represented as a finite union of balls. Now any domain $D \subset \mathbb{C}^N$ is a union of an increasing sequence of domains D_j, $j = 1, 2, \ldots$, where each D_j is a finite union of balls . Denote by F_j the subspace in $L^2(D)$ of functions with holomorphic restriction to D_j . This is a decreasing sequence of subspaces with intersection $F = L^2H(D)$. According to Theorem 8.2 the sequence of (known) orthogonal projections $P_j : L^2(D) \to F_j$ converges pointwise to $P : L^2(D) \to F$, the Bergman projection in D. (A generalization of the above argument yields a result due to I. Ramadanov [RAM 1]). We have thus arrived at a general and constructive description of the Bergman projection which (at least in principle) can be used as a basis for numerical computation of $K_D(z,t)$ in an arbitrary domain $D \subset \mathbb{C}^N$. Instead of balls one can use other domains for which the Bergman function is known explicitly. Although the alternating method so far yields no new instances of computing $K_D(z,t)$ in a closed form, we feel that it deserves a closer study. Such a study leads to the notion of endogeneous operators. (The reader undoubtedly will notice an analogy with the Toeplitz operator).

Definition 8.3. Let T be a subdomain in D. An endogeneous operator $P_{TD} : L^2H(T) \to L^2H(D)$ is (by definition) the composition of trivial extension to D with the Bergman projection in D.

For simplicity ($m = 2$) consider D represented as the union of domains A, B and denote by T the intersection of A and B. Assume that all relevant boundaries have plane measure zero. The Bergman projection in D of a piecewise L^2-holomorphic function equal to f on $A \setminus T$ and g on B can be computed according to the following infinite diagram [SKW 13]. (The rows in the diagram correspond to piecewise L^2-holomorphic functions and converge in L^2 to the Bergman projection. The letter r stands for the restriction)

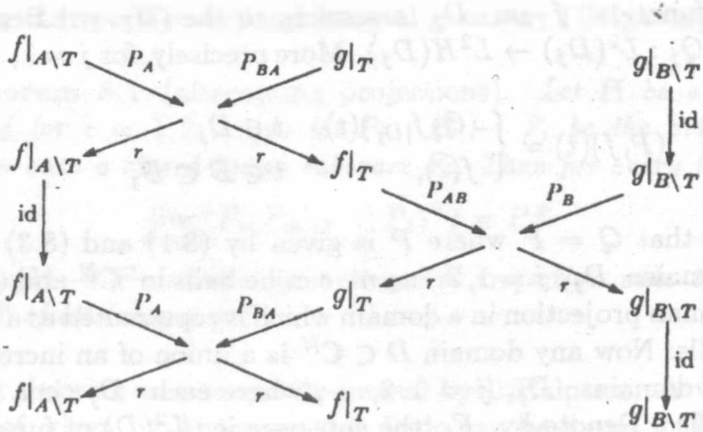

Fig. 1

In some (relatively simple) cases the procedure in fig.1 can be carried out by explicit analytic calculations leaving some interesting byproducts [RS]*, [SKW 13,14,15]. The rate of convergence of the alternating procedure is related to the L^2–angle $\gamma(A, B)$ between domains A and B , which by definition is equal to the angle in $L^2(D)$ between subspaces F_1 (functions holomorphic on A) and F_2 (functions holomorphic on B). More precisely, if $F := F_1 \cap F_2$ then $\gamma(A, B) \in [0, \pi/2]$ and

$$(8.4) \qquad \cos\gamma(A, B) = \sup\left[\frac{|<f_1, f_2>|}{\|f_1\|\,\|f_2\|} : f_i \in F_i \setminus \{0\},\ f_i \perp F\right].$$

It follows from the properties of canonical isometry that the L^2–angle is invariant under biholomorphic mapping of D: If A, B are mapped onto A', B' respectively, then $\gamma(A', B') = \gamma(A, B)$. The equality $\gamma(A, B) = \pi/2$ means that both spaces $L^2H(A)$, $L^2H(B)$ are trivial. This is impossible for $N = 1$ but it can occur for $N > 1$ [SKW 14]. So far computed L^2–angles between plane domains are presented in the following table

A	B	Restrictions	$\cos^2\gamma$	Reference
$\mathrm{Re}z < r$	$\mathrm{Re}z > 0$	$r > 0$	1	[RS 1]
$0 < \mathrm{Re}z < 1$	$\mathrm{Re}z > r$	$0 < r < 1$	r	[SKW 13]
$-\pi < \mathrm{Re}z < s\pi$	$r\pi < \mathrm{Re}z < \pi$	$-1 < r < s < 1$	$\frac{(1-s)(1+r)}{1+s)(1-r)}$	[SKW 14]
$\|z\| < 1$	$\|z\| > r$	$0 < r < 1$	r^2	[RS 1]
$a < \|z\| < c$	$b < \|z\| < d$	$0 < a < b < c < d$	$\frac{\ln(b/a)\ln(d/c)}{\ln(d/b)\ln(c/a)}$	[JM 1]
$-\delta < \mathrm{Arg}(-z) < \delta$	$-\alpha < \mathrm{Arg}\, z < \beta$	$0 < \delta < \min(\alpha, \beta)$ $\alpha + \beta < \pi$	1	[RZ 2]
$z \notin [0, \infty)$	$z \notin (-\infty, 0]$	none	1	[SKW 1]

As shown in [JM 2]* the L^2–angle between the domains A, B in $\mathbb{C}^N$ behaves nicely under cartesian product with a domain C in $\mathbb{C}^M$ such that $L^2H(D) \neq \{0\}$. Namely

$$(8.5) \qquad \gamma(A \times C, B \times C) = \gamma(A, B)\,.$$

The study of endogeneous operators, alternating projections and L^2–angles is greatly facilitated by the use of Genchev transforms (see next section). With the use of multidimensional Genchev transform some L^2-angles between multidimensional tubes were computed in [HYB 2]. A relation between the Bergman function of a tube over ball $B \subset \mathbb{R}^N$ and the Bessel function was found in [HYB 1].

9. Genchev transforms. Multipliers for edogeneous operators

Informally speaking the Genchev transform is an L^2-variant of the Laplace transform due to Dzhrbashyan [DM]* and Genchev [GNV 1,2]. Some simplifications can be found in [SKW 13]. Fundamental relations between Genchev transforms , endogeneous operators and L^2-angles were described in [SKW 13,14,15]. Therefore there is a clear link between Fourier analysis and holomorphic geometry. Perhaps this is one more explanation for importance of complex analysis in the scientific study of natural phenomena.

In this section we shall assume $N = 1$. We shall need

Lemma 9.1. *Let $D = \{\text{Re } z \in J\}$ be the tube over an open (possibly unbounded) interval $J \subset \mathbb{R}$. For every $f \in L^2H(D)$ and every $x \in J$ the function $g(y) := f(x + iy)$ belongs to $L^2(R)$. (The latter is certainly true for almost every $x \in J$ in view of Fubini theorem.)*

The proof is easy [SKW 13]. Fix $r > 0$ such that $(x-r, x+r) \subset J$. For every $y \in \mathbb{R}$ the function $h(z) := f(z+iy)$ is L^2-holomorphic in the square $Q = (x - r, x + r) \times (-r, r)$. Write $z = u + iv$. The value of h at the centre of Q is $h(x) = f(x + iy)$. Since Q contains concentric disc with centre r, we have the inequality

$$|f(x + iy)|^2 \leq (\pi r^2)^{-1} \|h\|_Q^2 .$$

The desired conclusion follows after integrating both sides over $y \in \mathbb{R}$ with respect to the Lebesgue measure. Since this measure is translation invariant, the Fubini theorem yields

$$\int_{y\in\mathbb{R}} \|h\|_Q^2\,dy = \int_{u\in(x-r,x+r)}\int_{v\in(-r,r)}\int_{y\in\mathbb{R}} |f(u+iv+iy)|^2\,dy\,dv\,du$$
$$= \int_{u\in(x-r,x+r)}\int_{v\in(-r,r)}\int_{y\in\mathbb{R}} |f(u+iy)|^2\,dy\,dv\,du$$
$$= 2r\|f\|_D^2 < \infty\,.$$

This completes the proof. The (inverse) Fourier transform of g given by the formula

$$(\mathbb{F}^{-1}g)(t) = \int_{y\in\mathbb{R}} e^{2\pi ity} f(x+iy)\,dy$$

depends on $x \in J$ in a very explicit way. In fact the function

$$G_f(t) := e^{2\pi tx}(\mathbb{F}^{-1}g)(t)$$
$$= i^{-1}\lim_{E\to\infty}\int_{x-iE}^{x+iE} e^{2\pi tz} f(z)\,dz$$

does not depend on x. It is called the Genchev transform of f. Note that G_f is completely determined by the values of f on one line $\mathrm{Re}\, z = x$. Therefore two functions (respectively in tubes D_1 and D_2) which agree on one vertical line have the same Genchev transforms. In view of Plancherel theorem

$$\|f\|_D^2 = \int_{\mathbb{R}} |g_f(t)|^2 w_J(t)\,dt$$
$$w_J := \int_J e^{-4\pi tx}\,dx\,.$$

More precisely, we have the fundamental

Theorem 9.2 (Dzhrbashyan,Genchev). *Let $D = \{\mathrm{Re}\, z \in J\}$ be a one-dimensional tube. The formula $f \mapsto G_f$ defines a unitary mapping of $L^2H(D)$ onto $L^2(\mathbb{R}, w_D)$.*

In terms of Genchev transforms some continuous linear operators between Bergman spaces in tubes are determined by multipliers. Therefore Genchev transforms are useful and important in holomorphic geometry .

Definition 9.3. A mesurable function $\mu : \mathbb{R} \to \mathbb{C}$ is called a multiplier for $P : L^2H(D_1) \to L^2H(D_2)$ if $G_{Pf} = \mu G_f$ a.e. for every $f \in L^2H(D_1)$.

For example let us consider the standard strip $D = \{-\pi < \operatorname{Re} z < \pi\}$ and a smaller strip $T = \{r\pi < \operatorname{Re} z < s\pi\}$ where $-1 \leq r < s \leq 1$. Then it is known that the endogeneous operator $P_{TD} : L^2H(T) \to L^2H(D)$ has a multiplier

$$\mu(t) = \frac{q^r - q^s}{q^{-1} - q}\ , \quad q = \exp(-4\pi^2 t),\ t \neq 0\ . \tag{9.1}$$

For the proof see [RZ 2].

10. Interpolation in Bergman Spaces

As time passes by the interest within functional analysis changes from the very abstract to more and more concrete objects. The study of abstract linear topological spaces is often replaced by the study of concrete spaces of functions harmonic or holomorphic in some "model" domain like the unit disc or its multidimensional generalization. See [DZH 1,2]. At the next stage of development more complicated "model" domains are admitted. When considered spaces are "biholomorphically invariant" (bounded holomorphic functions, square integrable holomorphic functions) the merger with complex analysis (or even with holomorphic geometry) becomes inevitable. Interpolation problems arise naturally in the theory of function spaces. Let us consider the interpolation problem for $H^\infty(D)$, the space of bounded holomorphic functions. A sequence $p_n \in D$ is called a bounded interpolation sequence if for every bounded sequence $c_n \in \mathbb{C}$ there exists $f \in H^\infty(D)$ such that $f(p_n) = c_n$ for $n = 1, 2, \ldots$. The classical solution for the unit disc D is given in the following

Theorem 10.1 (Carleson). *For $p_n \in \Delta$ to be the bounded interpolation sequence it is necessary and sufficient that for some $\delta > 0$*

$$\prod_{n \neq k} \rho_\Delta(p_n, p_k) \geq \delta\ , \quad k = 1, 2, \ldots\ .$$

The interpolation problem in Bergman spaces is perhaps less popular if not more difficult. From our point of view [SHA 1] and [SS]* are good introduction to the subject. The following result is quite elementary

Theorem 10.2 (see [SHA, p.88]). *Consider the Bergman space $L^2H(D)$, a subset $D_0 \subset D$ and a function $F: D_0 \to \mathbb{C}$. Let $M \subset L^2H(D)$ be a (possibly empty) subset consisting of all f which are extensions of F. The set M is not empty iff it contains an element f^* from the closed linear span of $\{\chi_D(t) : t \in D\}$. In this case f^* is the (unique) element of minimal norm in M.*

A sequence $\{p_n\}$ in D is called a universal interpolation sequence (u.i.s.) for $L^2H(D)$ if the interpolation problem $f(p_n) = c_n$ is solvable whenever $\sum |c_n|^2 K_D(p_n, p_n)^{-1} < \infty$. The motivation for this definition stems from the study of nearly orthogonal, or nearly complete systems of vectors in a Hilbert space. See [SS]*, [TAL]. An answer to the universal interpolation problem is given in the following

Theorem 10.3 (see [SHA, p.89]). *The necessary and sufficient condition for $\{p_n\}$ to be u.i.s. is that the eigenvalues of the section of the infinite Hermitian matrix (a_{ij}) be bounded away from zero, where*

$$a_{ij} = \frac{K_D(p_i, p_j)}{K_D(p_i, p_i)^{1/2} K_D(p_j, p_j)^{1/2}} . \tag{10.1}$$

The above condition is satisfied if in particular for some $c > 0$

$$\sum_{j \neq i} |a_{ij}| \leq 1 - c \, , \quad i = 1, 2, \ldots . \tag{10.2}$$

Note that $|a_{ij}| = 1 - \rho_D^2(p_i, p_j)$.

Some more general interpolations problems in a context of boun - ded symmetric domains were recently studied in [WOL 1].

11. Stability and mean square approximation

In holomorphic geometry one has to consider various quantities $J(D_1, \ldots, D_m)$ determined by a domain (e.g. the Bergman function

K_D) or by a system of domains (e.g. the L^2–angle $\gamma(D_1, D_2)$). Assume that (in some specified sense of convergence) $\lim_{s\to s_0}(D_1(s), \ldots, D_m(s)) = (D_1, \ldots, D_m)$ implies $\lim_{s\to s_0} J(D_1(s), \ldots, D_m(s)) = J(D_1, \ldots, D_m)$. Then we say that the system $D_1, \ldots, D_m$ is J–stable. Otherwise it is called unstable.

For example the classical theorem of I. Ramadanov [RAM], [SKW 7] says that the Bergman function K_D is stable when D is approached from inside by domains $D_n \subset D$. (The latter means that $F^{\text{compact}} \subset D$ implies $F \subset D_n$ for all sufficiently large n.

An analogous problem for a decreasing sequence of domains [IS]*,[SKW 7] was discussed for a bounded domain D which is equal to the interior of $\operatorname{cl} D$ and such that ∂D has a negligible $2N$–volume. We say that a sequence $D_n \supset D$ approaches D from outside if $\operatorname{cl} D \subset G^{\text{open}}$ implies $D_n \subset G$ for all sufficiently large n. Then the stability of D is equivalent to the following approximation property:

$$\overline{H(\operatorname{cl} D)} = L^2 H(D) \,. \tag{11.1}$$

Here $H(\operatorname{cl} D)$ denotes the linear subset of $L^2H(D)$ consisting of functions f which have a holomorphic extension to an open neighbourhood of $\operatorname{cl} D$ (which depends on f).

The theory of holomorphic mean square approximation in several complex variables is not yet well developed. In one variable the situation is much better (much of the progress originated in [SNN]). See for e.g. [BRN], [HAV], [HED 1], [CAR], [RUT]. An excellent survey of this area is contained in [HED 2].

We assume $N = 1$ in the following. Therefore in (11.1) we may replace $H(\operatorname{cl} D)$ by the linear set $R(\operatorname{cl} D)$ of all rational functions with poles off $\operatorname{cl} D$ (Runge approximation theorem).

The fundamental theorems of Havin, see [HED 2; Theorems 1.5, 1.18] yield immediately the following

Corollary 11.1. *Let $D \subset \mathbb{C}$ be as above. The set $R(\operatorname{cl} D)$ is dense in the Hilbert space $L^2H(D)$ iff the set of points on ∂D which do not belong to the fine closure of the exterior of D is polar. (Informally speaking this means that, with the exception of a very small set* *of boundary points, the exterior of D is "sufficiently massive" near each point of ∂D.)*

This leads to the following example , see [HED 2, example 1.17], [SKW 7, corollary V.10].

Example 11.2. Let D be the domain obtained from $\Delta \setminus [-1/2, 1/2]$ by removing infinitely many disjoint closed discs Δ_n in such a way that the union of Δ_n has $[-1/2, 1/2]$ as the set of adjoint points. Furthermore we can choose the sequence diam Δ_n converging to 0 so rapidly that the exterior of D is thin (informally :not "sufficiently massive") at each point of the segment $[-1/2, 1/2]$. We know from the potential theory [HLM] that a segment is not a polar set. Hence according to Corollary 11.1 the condition (11.1) is not satisfied in D. As a consequence, D is not K_D–stable in a sense of approximation from the outside.

Of course Example 11.2 is rather special. Corollary 11.1 assures K_D–stability for "nice" domains, in particular for each domain bounded by a finite number of Jordan curves.

For L^2–angles the study of stability is only in the beginning. Some examples of non–stability were discovered in [JM 1,2]*. The first positive and general result was obtained recently as a corollary of a result for abstract Hilbert spaces.

Theorem 11.3 (R. Goebel). *Let $\{L_n\}, \{M_n\}$ be two increasing sequences of closed subspaces in a Hilbert space $\mathbf{H}$. Let L be the closed subspaces spanned by the union of $\{L_n\}$. Similarily let M be the closed subspace spanned by the union of $\{M_n\}$. Assume further that $L \cap M = L_n \cap M_n = F$ is independent of n and that $L_1 \neq F$, $M_1 \neq F$. Then*

$$\lim \gamma(L_n, M_n) = \gamma(L, M)\,, \tag{11.2}$$

where $\gamma(L, M) \in [0, \pi/2]$ stands for the (usual) angle between L and M .

Corollary 11.4 (R. Goebel). *Assume that the domains A and B have L^2-angle $\gamma(A, B)$ and satisfy the assumptions of* Corollary 11.1. *Moreover A is approached from outside by a decreas-*

ing sequence of domains A_n *and* B *is approached from outside by a decreasing sequence of domains* B_n. *Assume further that* $A \cup B = A_n \cup B_n = C$ *is independent of* n. *Then*

$$\lim \gamma(A_n, B_n) = \gamma(A, B) . \tag{11.3}$$

We recall the idea of the proof [GBR]. The subspace in $L^2(C)$ consisting of functions holomorphic on A_n (resp. B_n) is denoted by L_n (resp. M_n). By Corollary 11.1 the closed subspace L (resp. M) spanned in $L^2(C)$ by the union of L_n (resp. M_n) consists of functions which are holomorphic on A (resp. on B). Hence $L \cap M = L_n \cap M_n = L^2H(D)$ is independent of n . Therefore Theorem (11.3) yields equality (10.2) which in the considered case can be rewritten as the desired equality (11.3).

12. Weighted Bergman space. Some Physical interpretations

An eminent example of a weighted Bergman space plays a significant role in quantum physics [JPR 1]*, [KS]*, [BK]*. It is called the Fock space. A heuristic introduction follows. Let us start with an abstract Hilbert space $\mathbf{H}$. Let $h_0, h_1, \dots$ be an orthonormal basis in $\mathbf{H}$. We shall introduce two (densely defined) operators in $\mathbf{H}$, annihilation operator A and creation operator A^*. These operators are defined on basic vectors as follows

$$Ah_k = \begin{cases} 0, & \text{if } k = 0 \\ k^{1/2} h_{k-1}, & \text{if } k > 0 \end{cases} \tag{12.1}$$

$$Ah_k^* := (k+1)^{1/2} h_{k+1} .$$

Each of these operators assignes to $f \in \mathbf{H}$ (by linearity) a formal infinite linear combination of basic vectors ; if its coefficients are square summable, we consider f in the domain of the operator. We are interested in eigenvectors of A (A^* has no eigenvectors). Assuming that $f = \sum f(k) h_k \neq 0$ belongs to the eigenvalue $z \in \mathbb{C}$, we conclude that

$$(k+1)^{1/2} f(k+1) = z f(k) , \quad k = 0, 1, \dots .$$

With no loss of generality we may additionally assume that $f(0) = 1$, hence

$$f(k) = \frac{z^k}{(k!)^{1/2}}\,, \quad k = 0, 1, \ldots \,. \tag{12.2}$$

On the other hand it is easy to verify that the vector (12.2) is an eigenvector of A with the eigenvalue z.

So far the space $\mathbf{H}$ and its orthonormal system has been arbitrary. A new idea is to choose for $\mathbf{H}$ a weighted Bergman space $L^2H(\mathbb{C}, \phi)$ in which monomials (12.2) define a complete orthonormal system . It is easy to verify that the Fock space $F(\mathbb{C}) =$ $L^2H(\mathbb{C}, \exp(-|z|^2)/\pi)$ has the above property. In this particular case annihilation and creation operators have a simple form, namely

$$Af = (\partial/\partial z)f \quad A^*f = zf \tag{12.3}$$

and one verifies immediately the Heisenberg relation

$$[A, A^*] = \mathrm{id}\,. \tag{12.4}$$

Eigenvectors for annihilation operator represent coherent states. (They minimize an error in the Heisenberg uncertainty principle). The Bergman function in the space $F(\mathbb{C})$ is given by

$$K_F(z,t) = \sum_{k=0}^{\infty} \frac{z^k}{(k!)^{1/2}} \frac{\bar{t}^k}{(k!)^{1/2}} = \exp(z\bar{t})\,. \tag{12.5}$$

A look at Fourier coefficient suffices to verify that $\chi(t) = K_F(\cdot, t)$ is an eigenfunction for A which belongs to the eigenvalue $\bar{t}$ (hence represents a coherent state).

In conclusion we remark that many constructions described in this article generalize easily to Bergman weighted spaces. For example (see [MAZ]) the Bergman metric tensor in the case of the Fock space is equal to the euclidean metric tensor.

REFERENCES

[ARN] Aronszajn, N., *Theory of reproducing kernels*, Trans. Amer. Math. Soc. 68 (1950), 337–404.

[BK]* Bargmann, V., P. Butera, L. Girardello and J. Klauder, *On the completeness of the coherent states*, Reports on math. physics 2 (1971), 221–228.

[BAR] Barker II, W., *Kernel functions on domains with hyperelliptic double*, Trans. Amer. Math. Soc. 231 (1977), 339–347.

[BLD] Bell, D., *Some properties of the Bergman kernel function*, Compositio Math. 21 (1969), 329–330.

[BLS] Bell, S., *Non vanishing of the Bergman kernel function at boundary points of certain domains in* $\mathbb{C}^N$, Math. Ann. 244 (1979), 69–74.

[BL]* Bell, S., and E. Ligocka, *A simplification and extension of Fefferman's theorem on biholomorphic mappings*, Invent. Math. 57 (1980), 283–289.

[BER 1] Berezin, F., *Covariant and contravariant symbols of operators*, Math. USSR Izvestia 6 (1972), 1117–1151.

[BER 2] Berezin, F., *Quantization*, Math.USSR Izvestia 8 (1974),1109-1163.

[BER 3] Berezin, F., *Quantization in complex symmetric spaces*, Math.USSR Izvestia 9 (1975),341-379.

[BE 1] Bergman, S., *Ueber die Entwicklung der harmonischen Funktionen der Ebene und des Raumes nach Orthogonalfunktionen*, Math. Ann. 86 (1922), 237-271.(Thesis,University of Berlin,1921)

[BE 2] Bergman, S., *Ueber die Kernfunktion eines Bereiches und ihr Verhalten am Rande*, J. Reine Angew. Math. 169 (1933), 1–42 & 172 (1934), 89–128.

[BE 3] Bergman, S., *The kernel function and conformal mapping*, Math. Surveys 5 , Amer Math. Soc. 1950 second ed. 1970.

[BS]* Bergman, S., and M. Schiffer, *Kernel functions and conformal mapping*, Compositio Math 8 (1951), 205–249.

[BIE] Bieberbach, L., *Zur Theorie und Praxis der konformen Abbildung*, Rend. Circ. Math. Palermo 38 (1914), 98–112

[BSR] Boas, R., *Fundamental sets of entire functions*, Annals of math. 147 (1946), 21–32.

[BSH] Boas, H., *Counterexample to the Lu Qi-keng conjecture*, Proc. Amer. Math. Soc. 97 (1986), 374–375.

[BCH] Bochner, S., *Monotone Funktionen, Stieltjesche Integrale und harmonische Analyse*, Math. Annalen 108

[BOR] Bohr, H., *Ueber streckentreue und konforme Abbildungen*, Math. Z. 1 (1918), 403–420.

[BRM 1] Bremermann, H., *Holomorphic continuation of the kernel and the Bergman metric. In: Lectures on functions of a complex variable*, Univ.of Michigan Press 1955 ,349-383.

[BRM 2] Bremermann, H., *Distributions,complex variables and Fourier Transforms*, Addison-Wesley, Reading 1965.

[BRM 3] Bremermann, H., *Ueber die Aequivalenz der pseudokonvexen Gebiete und der Holomorphiegebiete im Raum von n komplexen Veränder lichen*, Math. Ann. 128 (1954), 63–91.

[BRM 4] Bremermann, H., *Die Holomorphiehüllen der Tuben- und Halbtubengebiete*, Math. Ann. 127 (1954), 406–423.

[BRN] Brennan, J., *Invariant subspaces and rational approximation*, J.Funct. Anal. 7 (1971), 285–310.

[CAR] Carleson, L., *Selected problems on exceptional sets*, Van Nostrand, Princeton 1967.

[CHO 1] Chojnacki, W., *On some holomorphic dynamical systems*, Quart. J. Math. Oxford 39 (1988), 159–172.

[CHO 2] Chojnacki, W., *On some functionals on Bergman spaces*, Bull. Polish Acad. Sci. 37 (1989), 351–353.

[CHO 3] Chojnacki, W., *On the Bergman metric tensor*, Bull. Polish Acad. Sci. 39 (1991), 147–150.

[DEU] Deutsch, F., *Applications of von Neumann's alternating projections algorithm*, In: *Proceedings of the Conference on "MAthematical Methods in Operations Research"*, Sofia, Bulgaria, Oc. 24–29, 1983 ed. P. Kenderov, 44–51.

[DIE] Diederich, K., *Some recent developments in the theory of the Bergman kernel function: a survey*, Proc. Symp. Pure Mat. 30 (1977), 127–137.

[DUR] Duren, P., *The theory of H^p spaces*, Academic Press. New York and London 1970.

[DZH 1] Dzhrbashyan, M., *On canonical representation of functions meromorphic in the unit disc*, (in Russian). Proceedings of the Academy of Sciences of the Armenian SSR, 1945.

[DZH 2] Dzhrbashyan, M., *On a problem of representation of analytic functions*, Communications of the Inst. Math. Mech. Acad. Sci. Armenian SSR, 1948, 3–39.

[DM]* Dzhrbashyan, M., and V. Martirosyan, V. *Integral representations for functions holomorphic in a strip or in a half plane*, Anal. Math. 12 (1986), 191–212.

[EPS] Epstein, B., *Orthogonal families of analytic functions*, Macmillan, N.York 1965.

[FNR] Fechner, B., *Przykłady brzegu idealnego w klasie nieograniczonych obszarów Reinharda*, MS Thesis, Warsaw Univ. 1978.

[FTZ] Fitzgerald, C.H., *Analytic continuation of a funcion defined on a sequence of points*, Bull. London Math. Soc. 3 (1971), 286–290.

[GAM] Gamelin, T., *Uniform algebras*, Prentice Hall, Englewood Cliffs 1969.

[GNV 1] Genchev, T., *Paley–Wiener type theorems for functions holomorphic in a half plane*, Comptes rendus Acad. Bulg. Sci. 37 (1983), 141–144.

[GNV 2] Genchev, T., *Integral representations for functions holomorphic in tube domains*, Comptes rendus Acad. Bulg. Sci. 37 (1984), 717–720.

[GBR] Goebel, R., *Funkcja Bergmana, rzuty alternujące i stabilność L^2-kątów*, MS dissertation, UMCS, Lublin 1994.

[GOL] Golusin, G., *A solution of fundamental planar problems of mathematical physics in the case of Laplace equation in multiconnected domains bounded by circles (method functional equations)*,(in Russian), Matem. Sbornik 41 (1934), 246–275.

[GUI] Guillemin, V., *Toeplitz operators in n dimensions*, Integral equations and operator theory 7 (1984), 145–205.

[HA] Hahn, K., *On completeness of the Bergman metric and its subordinate metric*, Proc. Nat. Acad. Sci. USA (1976), 4294. See also Pac. J. Math. 68 (1977), 437–446.

[HAV] Havin, V.P., *Approximation in the mean by analytic functions*, Dokl. Akad. Nauk SSSR 178 (1968), 1025–1028 (Soviet Math. Dokl.9 (1968), 245–248).

[HED 1] Hedberg, L., *Bounded point evaluation and capacity*, J. Funct. Anal. 10 (1972), 269–280.

[HED 2] Hedberg, L., *Spectral synthesis and stability in Sobolev spaces*, In: *Euclidean harmonic analysis*, Springer L.N.# 779, Berlin 1980, 73–103.

[HLG] Helgason, S., *The Radon transform*, Birkhauser 1980.

[HLM] Helms, L.L., *Introduction to potential theory*, Wiley Interscience, N.York 1969.

[HI 1] Hille, E., *Introduction to the general theory of reproducing kernels*, Rocky Mount. J. 2 (1972), 321–368.

[HI 2] Hille, E., *Analytic function theory*, vol.I,II. Ginn and Co. N.York 1959.

[HS]* H e n g a r t n e r, W., and G. S c h o b e r, *Interpolation, continuation and quadratic inequalities*, Pac. J. Math. 100 (1982), 139–151.

[HEN] H e n k i n, G., *The method of integral representations in complex analysis*, (in Russian) Itogi nauki i tehniki,VINITI 7 (1985),23-124.

[HOF] H o f f m a n, K., *Banach spaces of analytic functions*, Prentice Hall, Englewood Cliffs 1962, (Russian transl. Moscov 1962).

[HU 1] H u a L o–k e n g, *Harmonic analysis of functions of several complex variables in the classical domains*, Science Press, Peking 1958;Engl.transl., Transl. Math. Monographs 6, Amer. Math.Soc. 1963.

[HU 2] H u a L o–k e n g, *On the Riemannian curvature in the space of several complex variables*,Riemann Tagung,see also On the Riemann curvature of the non-euclidean space of several complex variables,Acta math.Sinica 4(1954),141-168 and On non- continuable domain with constant curvature, ibid.

[HYB 1] H y b, W., *The Bergman function of the tube over an open ball*, Classical Analysis, Kazimierz Dolny"91, ed. T. Mazur, World Scientific, Singapore 1992, 17–28.

[HYB 2] H y b, W., *Bergman function, Genchev transform and L^*–angles for multidimensional tubes*, (in preparation).

[IS]* I w i ń s k i, T.B., and M. S k w a r c z y ń s k i, *The convergence of Bergman functions for a decreasing sequence of domains*, In: *Approximation theory*, Z.Ciesielski and J. Musielak, PWN Warszawa 1975, 117–120.

[JAK] J a k ó b c z a k, P., *The exceptional sets for functions from the Bergman space*, Portugaliae Math. 50 (1993), 115–128.

[JM 1]* J a k ó b c z a k, P., and T. M a z u r, *On discontinuity of L^2–angle*, J. Austral. Math. Soc. 47 (1989), 269–279.

[JM 2]* J a k ó b c z a k, P., and T. M a z u r, *Some properties of the L^2–angle between complex domains*, Radovi Matematicki 7 (1991), 109–121.

[JPR]* J a n s o n, S., J. P e e t r e, and J. R o c h b e r g, *Hankel forms and the Fock space*, Uppsala University Dpt.of Math. Reports, May 1986, 1–89.

[JP 1]* J a r n i c k i, M., and P. P f l u g, *Bergman completeness of complete circular domains*, Ann. Polon. Math. 50 (1989), 219–222.

[JP 2]* J a r n i c k i, M., and P. P f l u g, *Invariant distances and metrics in complex analysis*, W.de Gruyter, Berlin 1993.

[KAU] K a u p, G., *Ueber das Randverhalten von holomorphen Auto- morphismen beschraenkter Gebiete*, manuscripta math. 3 (1970), 257–270.

[KS 1]* K e r z m a n, N., and E. S t e i n, *The Cauchy Kernel,the Szegö kernel and the Riemann mapping function*, Math. Ann. 236 (1968), 85–93.

[KS 2]* Kerzman, N., and E. Stein, *Integral representations and boundary behavior in several complex variables*, (book to appear).

[KS]* Klauder, J., and E. Sundarshan, *Fundamentals of quantum optics*, Benjamin, N.York 1968.

[KOB 1] Kobayashi, S., *Geometry of bounded domains*, Trans. Amer. Math. Soc. 92 (1959), 267–290.

[KOB 2] Kobayashi, S., *On complete Bergman metrics*, Proc. Amer. Math. Soc. 13 (1962), 511–513.

[KR 1] Krantz, S., *Function theory of several complex variables*, Wiley and Sons, N.York 1982.

[KR 2] Krantz, S., *Recent progress and future directions in several complex variables*, In: *Complex Analysis*, Springer Lecture Notes 1268 , Berlin 1987.

[KP]* Krantz, S., and H. Parks, *Distance to C^k hypersurface*, Journal of dif.eq. 40 (1981), 116–120.

[KRZ] Krzyż, J., *On a theorem of Kubo concerning functions regular in an annulus*, Colloquium Math. 16 (1967), 43–47.

[LIG 1] Ligocka, E., *Domains of existence of real analytic functions and the inverse of the Bremermann theorem*, Bull. Acad. Sci. Pol. 26 (1972), 495–499.

[LIG 2] Ligocka, E., *How to prove Fefferman's theorem without use of differential geometry*, Ann. Polon. Math. 39 (1981), 117–130.

[LUK] Lu Qi-keng, *On Kaehler manifolds with constant curvature*, Acta Math. Sinica 16 (1966), 269–281.

[LI] Lichnerowicz, A., *Variétés complexes et tenseur de Bergman*, Ann. Inst. Fourier 15 (1965), 345–108.

[MAC] Mackey, G.W., *Unitary group representations in physics,probability and number theory*, Benjamin/Cummings 1978.

[MAZ] Mazur, T., *Some topics in analysis related to the Bergman function*, (in preparation).

[MS]* Mazur, T., and M. Skwarczyński, *Spectral properties of holomorphic automorphism with fixed point*, Glasgow Math. J. 28 (1986), 25–30.

[MPS]* Mazur, T., P. Pflug, and M. Skwarczyński, *Invariant distances related to the Bergman function*, Proc. Amer. Math. Soc. 94 (1985).

[MES] Meschkowski, H., *Hilbertsche Räume mit Kernfunktion*, Springer, Berlin 1962 .

[NAR] Naruki, I., *The holomorphic equivalence problem for a class of Reinhardt domains*, Publ.RIMS Kyoto Univ.4 (1968), 527–543.

[PW]* Paley, R., and N. Wiener, *Fourier Transforms in the Complex Domain*, Amer. Math. Soc., N.York 1934 (Russian transl.Moscov 1964).

[RAM] Ramadanov, I.P., *Sur une propriete de la fonction de Bergman*, Comt. Rend. Acad. Bulg. Sci. 20 (1967), 759–762.

[RS]* Ramadanov, I.P., and M.L. Skwarczyński, *An angle in $L^2(\mathbb{C})$ determined by two plane domains*, Bull. Polish Acad. Sci. 32 (1984), 653–659.

[ROS] Rosenthal, P.L., *On the zeros of the Bergman function in doubly-connected domains*, Proc. Amer. Math. Soc. 21 (1969), 33–35.

[RUT] Ruta, M., *O twierdzeniu aproksymacyjnym dla L^2 holomorficznych funkcji i punktach cienkości dla zbiorów otwartych*, Doctoral dissertation, University of Lódź 1992.

[RZ 1] Rządkowski, G., *On an angle between two rings*, Bull. Austal. Math. Soc. 43 (1991), 79–87.

[RZ 2] Rządkowski, G., *An L^2-angles between plane sectors*, Demonstratio Math., to appear.

[SAI] Saitoh, S., *Theory of reproducing kernels and its applications*, Pitman Research Notes # 189, Longman 1988.

[SHF 1] Schiffer, M., *Stefan Bergman*(1895-1977),In Memoriam, Annales Polonici Mathematici 39 (1981), 5–9.

[SHF 2] Schiffer, M., *Fredholm eigenvalues and Grunsky matrices*, Annales Polonici Mathematici 39 (1981), 149–164.

[STT 1] Schottky, F., *Ueber die konforme Abbildung mehrfach zusamenhän gender ebener Flächen*, Crelles J. 83 (1877), 300–351.

[STZ] Schwarz, L., *Sous-espaces hilbertiens d'espaces vectoriels topologiques et noyaux associés*, (noyaux reproduisants), J. Analyse Math. 13 (1964), 115–256.

[SHA] Shapiro, H., *Topics in approximation theory*, Springer Lecture Notes # 187, Berlin 1971.

[SS]* Shapiro, H., and A. Shields, *On some interpolation problems for analytic functions*, Amer. J. Math. 83 (1961), 513–532.

[SIC 1] Siciak, J., *Extremal plurisubhamonic functions and capacities in $\mathbb{C}^n$*, Sophia Kokyuroku in Mathematics 14 ,1982.

[SIC 2] Siciak, J., *On removable singularities of L^2-holomorphic functions of several complex variables*, Prace Matematyczno–Fizyczne WSI Radom 1982 , 73–82.

[SNN] Sinanjan, S.O., *Approximation by polynomials and analytic functions in the areal mean*, Mat. Sb. 69 (1966), 546–578; Amer. Math. Soc.

Transl. 74 (1968), 91–124.

[SIN 1] S i n g e r, I., *Bases in Banach spaces* I, Springer Verlag Berlin 1970.

[SIN 2] S i n g e r, I., *Bases in Banach spaces* II, Springer Verlag Berlin 1981.

[SKW 1] S k w a r c z y ń s k i, M., *Stefan Bergman* (1895-1977),(In Polish), Wiadomości Matematyczne 23 (1981), 189–204.

[SKW 2] S k w a r c z y ń s k i, M., *The Bergman function and semiconformal mappings*, (Thesis, University of Warsaw 1969).

[SKW 3] S k w a r c z y ń s k i, M., *The invariant distance in the theory of pseudo-conformal transformations and the Lu Qi-keng conjecture*, Proc. Amer. Math. Soc. 22 (1969), 305–310.

[SKW 4] S k w a r c z y ń s k i, M., *A continuation theorem for holomorphic mappings into a Hilbert space*, Ann. Polon. Math. 23 (1970), 281–284.

[SKW 5] S k w a r c z y ń s k i, M., *The Bergman function and semiconformal mappings*, Bulletin Acad. Polon. Sci. 22 (1974), 667–673.

[SKW 6] S k w a r c z y ń s k i, M., *A new notion of boundary in the theory of several complex variables*, Bull. Acad. Polon. Sci. 24 (1976), 327–330.

[SKW 7] S k w a r c z y ń s k i, M., *Biholomorphic invariants related to the Bergman function*, Dissertationes Math. 173, Warszawa 1980.

[SKW 8] S k w a r c z y ń s k i, M., *A remark on holomorphic isometries with respect to the induced Bergman metric. Analytic functions*, Kozubnik'ed. J. Lawrynowicz, Springer Lecture Notes # 798 ,

[SKW 9] S k w a r c z y ń s k i, M., *Evaluation functionals on spaces of square integrable holomorphic functions*, Prace Matem. –fizyczne WSI Radom 1982, 65–72.

[SKW 10] S k w a r c z y ń s k i, M., *The ideal boundary of a domain in* C^n, Ann. Polon. Math. 39 (1981), 213–226.

[SKW 11] S k w a r c z y ń s k i, M., *Alternating projections in complex analysis*, Complex analysis and applications'83, Sofia 1985.

[SKW 12] S k w a r c z y ń s k i, M., *A general description of the Bergman projection*, Ann. Polon. Math. 46 (1985), 311–315.

[SKW 13] S k w a r c z y ń s k i, M., *Alternating projections between a strip and a half plane*, Math. Proc. Camb. Phil. Soc. 102 (1987), 121–129.

[SKW 14] S k w a r c z y ń s k i, M., L^2*-angles between one-dimensional tubes*, Studia Math. 40 (1988), 213–233.

[SKW 15] S k w a r c z y ń s k i, M., *The punctured plane; alternating projections and* L^2*-angles*, Ann. Polon. Math. 52 (1991), 293–301.

[SM]* S o m m e r, F., and J. M e h r i n g, *Kernfunktion und Hllenbildung in der Funktionentheorie von mehreren Ver¨nderlichen*, Math. Ann. 131

(1956), 1–16.

[STN] S t o n e, M.H., *Linear transformations in Hilbert space and their application to analysis*, Amer. Math. Soc. Providence 1979.

[SUE] S u e t i n, P., *Polynomials orthogonal over domain and Bieberbach polynomials*, Trudy Inst. Steklova 100 , Moskwa 1974.

[SY]* S u i t a, N., and A. Y a m a d a, *On the Lu Qi-keng conjecture*, Proc. Amer. Math. Soc. 59 (1976), 222–224.

[SUN] S u n a d a, T., *Holomorphic equivalence problem for bounded Reinhardt domains*, Math. Ann. 235 (1978), 111–128.

[SUS] S u s z c z y ń s k i, A., MS thesis, Warsaw University 1976, Zeszyty Naukowe Filii UW w Białymstoku 39 (1984), 113–118.

[TAL] T a l d y k i n, A.T., *Sistemy i rjady elementov*, Nauka, Moskva 1971.

[WAL] W a l s h, J.L., *Interpolation and approximation by rational functions in the complex domain*, Amer. Math. Soc. 1960.

[WIE] W i e g e v r i n c k, J., *Domains with finite dimensional Bergman spaces*, Math. Z. 187 (1984), 559–562.

[WOL 1] W o l n i e w i c z, T., *Inclusion operators in Bergman spaces on bounded symmetric domains in* C^N, Warsaw Uniwersity 1982, preprint IMUW No 6/82.

[WOL 2] W o l n i e w i c z, T., Ph.D Thesis (in Polish), Warsaw University 1984 .

[ZAK 1] Z a r a n k i e w i c z, K., *Sur la représentation conforme d'un domaine doublement connexe sur un anneau circulaire*, C. R. Acad. Sci. Paris 198 (1934), 1347–1349.

[ZAK 2] Z a r a n k i e w i c z, K., *Ueber ein numerisches Verfahren zur konformen Abbildung zweifach zusamenhängender Gebiete*, Zeitschrift für angew. Math. und Mech. 14 (1934), 97–104.

[ZAR] Z a r e m b a, S., *Sur le calcul numerique des fonctions demandées dans le probléme de Dirichlet et le probléme hydrodynamique*, Bull. Internat. Acad. Sci. Crocovie 1 (1909), 125–195.

[ZIN 1] Z i n o v i e v, B.S., *On reproducing kernels for multicircular domains of holomorphy*, (in Russian), Sib. Matem. Z. 15 (1974), 35–48.

[ZIN 2] Z i n o v i e v, B.S., *Reproducing Bergman kernels for n-circular domains their "extension" and relation with Szegõ kernels*, (in Russian), Trudy Tomskovo Universiteta , 210, 23–33.

Szkoła Główna Gospodarstwa Wiejskiego
Katedra Zastosowań Matematyki
Nowoursynowska 166, 02–766 Warszawa, Poland