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Peter PFLUG (Vechta)
Gerald SCHMIEDER (Oldenburg)

Remarks on the Ilieff-Sendov Problem

Abstract. Let p be a polynomial whose zeros are contained in the closed unit disk $\overline{E}$. According to the Ilieff-Sendov conjecture there exist a, $z^* \in \overline{E}$ such that

$$(*) \qquad p(a)=0, \ p'(z^*)=0, \ |a-z^*| \leq 1 .$$

The authors define certain polynomials $\beta_k = \beta_k(a, z_2, \dots, z_n)$, $k=0,\dots,n-1$, and establish inequalities involving the values of β_k at $a \in [0,1]$ and $z_2,\dots,z_n \in \overline{E}$ which imply the existence of z^* such that $(*)$ holds.

The problem known for more than thirty years as the Ilieff-Sendov conjecture is as follows:

Let $p \in \mathbb{C}[z]$ be a polynomial whose zeros belong to the closed unit disc $\bar{E}$ with $E := \{\lambda \in \mathbb{C} : |\lambda| < 1\}$. Let $p(a) = 0$. Then it is asked whether there exists $z^* \in a + \bar{E}$ with $p'(z^*) = 0$.

There is already a long list of papers attacking this question (cf. [1]-[27]). But so far it is only known that the conjecture is true if the degree of p is less or equal than 6 (cf. [11]). Proofs use very ad hoc methods. So the aim of our short note is to reduce the above problem to a more geometric question.

Fix $a \in [0, 1]$ and let $z_2, \dots, z_n \in \bar{E}$ ($n \geq 3$). We define

$$\sigma_k := \sigma_k(a, z_2, \dots, z_n)$$
$$:= \sum_{2 \leq j_1 < \dots < j_k \leq n} (a - z_{j_1}) \dots (a - z_{j_k}), \quad k = 1, \dots, n-1,$$

and

$$\beta_k := \beta_k(a, z_2, \ldots, z_n)$$
$$:= \frac{k+1}{\binom{n-1}{k}} \sigma_{n-1-k}(a, z_2, \ldots, z_n), \quad k = 0, \ldots, n-1,$$

with

$$\sigma_0 := 1.$$

We will prove the following:

Theorem . *Let $p(z) = (z-a)\prod_{j=2}^{n}(z-z_j)$ with $a, z_2, \ldots, z_n$ as above, $n \geq 3$. Assume that at least one of the following conditions is fulfilled:*
1) there exists some $\zeta \in \bar{E}$ such that for some $\nu \in \{3, \ldots, n\}$ the inequality $|\zeta\beta_1 + \beta_0| \leq |\zeta\beta_{\nu-1} + \beta_{\nu-2}|$ holds
or
2) $|\beta_0| \leq |\beta_{n-1}| (= n)$.
Then there exists $z^* \in a + \bar{E}$ with $p'(z^*) = 0$.

Proof. Put $q(z) := p(z+a)$ and

$$q'(z) = \sum_{\nu=0}^{n-1} b_\nu z^\nu = \sum_{\nu=0}^{n-1} \binom{n-1}{\nu} \beta_\nu z^\nu$$

For $1 \leq n - \nu \leq n-3$ we consider the polynomial

$$L(z) = 1 + (-1)^{n-\nu}\binom{n-1}{\nu}\ell_{n-\nu}\, z^{n-\nu}$$
$$+ (-1)^{n-2}\binom{n-1}{2}\ell_{n-2}\, z^{n-2} .$$

Note that $L(z)$ has all its zeros in $\bar{E}$ if and only if $Q(z) = z^{n-2}\, L(1/z)$ does not vanish in E. And this is obviously true in the case that

$$(*) \qquad 1 + \binom{n-1}{\nu}|\ell_{n-\nu}| \leq \binom{n-1}{2}|\ell_{n-2}| .$$

Assume that $(*)$ holds and take some $\zeta \in \bar{E}$.

Then all the zeros of the test-polynomial

$$L_\zeta(z) = -L(z)(z-\zeta) =$$
$$\zeta + (-1)^1 \binom{n-1}{1} \frac{1}{n-1} z + (-1)^{n-\nu} \binom{n-1}{n-\nu} \frac{n-\nu}{\nu} \zeta \, \ell_{n-\nu} \, z^{n-\nu}$$
$$+ (-1)^{n-\nu+1} \binom{n-1}{n-\nu+1} \frac{(n-\nu+1)(n-\nu)}{\nu(\nu-1)} \, \ell_{n-\nu} \, z^{n-\nu+1}$$
$$+ (-1)^{n-2} \binom{n-1}{n-2} \frac{n-2}{2} \zeta \, \ell_{n-2} \, z^{n-2}$$
$$+ (-1)^{n-1} \binom{n-1}{n-1} \binom{n-1}{2} \ell_{n-2} \, z^{n-1}$$

are as well contained in $\bar{E}$.

The (somewhat mysterious) theorem of Grace (cf. [26] Satz 1′) can be stated as follows:

Let $\lambda_0, \ldots, \lambda_N$, $a_0, \ldots, a_N$ be complex numbers with $\lambda_N \neq 0$ which fulfill the following "apolarity condition"

$$\lambda_0 a_N + \lambda_1 a_{N-1} + \cdots + \lambda_N a_0 = 0 \, .$$

Then every closed disk D in $\mathbb{C}$ containing all the roots of the polynomial

$$\Lambda(z) = \lambda_0 - \binom{N}{1} \lambda_1 z^1 + \binom{N}{2} \lambda_2 z^2 - + \cdots + (-1)^N \lambda_N z^N$$

contains at least one zero of the polynomial

$$A(z) = a_0 + a_1 z + \cdots + a_N z^N \, .$$

Hence in order to apply the theorem of Grace we have to make sure that there are $\ell_{n-\nu}$ and ℓ_{n-2} with $(*)$ and $\ell_{n-2} \neq 0$ such that the apolarity condition holds (a separate discussion of the cases $\nu = n-1$ and $\nu = 3$ (two exponents in L_ζ coincide) is not necessary):

$$\zeta\beta_{n-1} + \binom{n-1}{n-2}\frac{1}{n-1}\beta_{n-2} + \binom{n-1}{\nu-1}\frac{n-\nu}{\nu}\,\zeta\,\ell_{n-\nu}\beta_{\nu-1}$$
$$+\binom{n-1}{\nu-2}\frac{(n-\nu+1)(n-\nu)}{\nu(\nu-1)}\,\ell_{n-\nu}\beta_{\nu-2} + \binom{n-1}{1}\frac{n-2}{2}\,\zeta\,\ell_{n-2}\beta_1$$
$$+\binom{n-1}{2}\ell_{n-2}\beta_0 = 0\,.$$

This is equivalent to

$$\ell_{n-2}\binom{n-1}{2}(\zeta\beta_1+\beta_0)$$
$$= -\left(\zeta\beta_{n-1} + \beta_{n-2} + \binom{n-1}{\nu}\ell_{n-\nu}[\zeta\beta_{\nu-1}+\beta_{\nu-2}]\right)\,.$$

From $(*)$ we obtain

$$\left(1+\binom{n-1}{\nu}|\ell_{n-\nu}|\right)|\zeta\beta_1+\beta_0|$$
$$\le \left|\zeta\beta_{n-1}+\beta_{n-2}+\binom{n-1}{\nu}\ell_{n-\nu}[\zeta\beta_{\nu-1}+\beta_{\nu-2}]\right|\,.$$

Division by $|\ell_{n-\nu}|$ and considering the limit $\ell_{n-\nu}\to\infty$ gives the assumption 1) in the theorem. Therefore, whenever 1) is fulfilled then the above apolarity condition for $L_\zeta(z)$ and $q'(z)$ is fulfilled for a suitable choice of $\ell_{n-\nu}$, ℓ_{n-2}. From $(*)$ we obtain the desired zero z^* by the theorem of Grace.

The polynomial $\ell(z) = -\beta_0/\beta_{n-1} + (-1)^{n-1}z^{n-1}$ is always apolar to $q'(z)$ and therefore the assumption 2) in the theorem comes out to be sufficient for the claimed zero z^*. □

Remarks.

1. Note that the condition 1) is equivalent to the following statement which is free of the parameter ζ:

(1a) $$|\beta_0| \le |\beta_1|$$

or, for some $\nu \in \{3, \dots, n\}$,

$$(1b) \qquad |\beta_0|^2 + |\beta_1|^2 - |\beta_{\nu-1}|^2 - |\beta_{\nu-2}|^2 \le 2|\beta_{\nu-1}\overline{\beta_{\nu-2}} - \beta_1\overline{\beta_0}|$$

This follows from the fact that a Moebius transformation is an inner function of the unit disk iff the image of ∂E is part of $\bar{E}$ and there is no pole inside E.

Hence, in order to solve the Ilieff-Sendov conjecture it suffices to show that there are no points $a, z_2, \dots, z_n \in \bar{E}$ (as above) satisfying the following inequalities:

$$|\sigma_{n-1}(a, z_2, \dots, z_n)| > n,$$
$$|\sigma_{n-1}(a, z_2, \dots, z_n)| > \frac{2}{n-1}\,|\sigma_{n-2}(a, z_2, \dots, z_n)|$$

and

$$|\sigma_{n-1}(a, z_2, \dots, z_n)|^2 + \frac{4}{(n-1)^2}|\sigma_{n-2}(a, z_2, \dots, z_n)|^2 - \left[\frac{\nu!(n-\nu)!}{(n-1)!}|\sigma_{n-\nu}(a, z_2, \dots, z_n)|\right]^2 - \left[\frac{(\nu-1)!(n-\nu+1)!}{(n-1)!}|\sigma_{n-\nu+1}(a, z_2, \dots, z_n)|\right]^2$$
$$> 2\left|\frac{\nu!(\nu-1)!(n-\nu)!(n-\nu+1)!}{[(n-1)!]^2}\sigma_{n-\nu}\cdot\bar{\sigma}_{n-\nu+1} - \frac{2}{(n-1)}\sigma_{n-2}\bar{\sigma}_{n-1}\right|$$

if $\nu = 3, 4, \dots, n$.

2. Setting $\alpha = 0$ our Theorem shows that the Ilieff-Sendov conjecture is true for every polynomial $p(z) = (z-a)\prod_{\nu=2}^{n}(z - z_\nu)$ with $q(z) = p(z+a)$ and $q'(z) = \sum_{\nu=0}^{n-1}\binom{n-1}{\nu}\beta_\nu z^\nu$ for which the coefficients fulfill

$$|\beta_0| \le |\beta_k| \quad \text{for some} \quad k \in \{1, \dots n-1\},$$

i.e. $\binom{n-1}{k} \le (k+1)\left|\sum_{2\le j_1<\dots<j_k\le n}(a - z_{j_1})^{-1}\dots(a - z_{j_k})^{-1}\right|$ for some k.

3. From the above inequality for $k = n - 1$ we see that it suffices to have

$$\left|\prod_{j=2}^{n}(a - z_j)\right| \le n\,.$$

Therefore the desired zero z^* of q' can be found if

$$|a - z_j| \le \sqrt[n-1]{n}$$

for all $j = 2, \dots, n$. Roughly speaking: the conjecture is true for a if the other zeros are pretty close to a. In particular, if $a < -1 + \sqrt[n-1]{n}$, then the conjecture is always true for a.

4. Observe that the inequality $|\beta_0| > |\beta_1|$ covers the often discussed case that $\sum_{j=2}^{n} 1/r_j^2 \le (n-1)/(1+a)$, where $r_j := |z_j - a|$.

Moreover, $|\beta_0| > |\beta_1|$ implies that

$$\frac{n-1}{2} > \left|\sum_{j=2}^{n} \frac{1}{z_j - a}\right| \ge \sum_{j=2}^{n} \operatorname{Re}\frac{-1}{z_j - a} \ge \frac{n-1}{2a} - \frac{1-a}{2a}\sum_{j=2}^{n}\frac{|z_j| + a}{r_j^2},$$

which gives $r_j < \sqrt{|z_j| + a}$ for at least one j.

Hence we have the following Corollary:

Corollary . *If $p(z) = (z - a)\prod_{j=2}^{n}(z - z_j)$ is as above and if $|z_j - a| \ge \sqrt{|z_j| + a}$ for all j, then the Ilieff-Sendov conjecture is true for a, i. e. there exists $z^* \in a + \bar{E}$ with $p'(z^*) = 0$.*

We emphasize that this remark solves the Ilieff-Sendov conjecture for all n if $|z_j - a| \ge \sqrt{1 + a}$ for $j = 2, \dots, n$, i. e. if one zero of p (here a) is strongly separated from the remaining $n - 1$ zeros. Roughly speaking this means: the conjecture is true for a if the other zeros are pretty far away from a.

Using the Corollary it is easy to see that already the assumption $r_2 \le \left(n/\sqrt{1+a}\right)^{1/(n-2)}$ (cf. Remark 3) implies that the Ilieff-Sendov conjecture is true for a.

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Universität Osnabrück
Standort Vechta
Fachbereich Mathematik
Postfach 15 53
49364 Vechta
Germany

Universität Oldenburg
Fachbereich 6 Mathematik
Postfach 2503
26111 Oldenburg
Germany