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Dariusz PARTYKA (Lublin)

On the Maximal Dilatation of the Douady-Earle Extension†

Abstract. This paper provides a new bound of the functional $|\varphi(0)|$ in the class $\mathbb{Q}^0(K;\Delta)$ of all K-quasiconformal self-mappings φ of the unit disc Δ normalized by a vanishing integral of their boundary values. Let Φ_K, $K\geq 1$, denote the Hersch-Pfluger distortion function. Using some properties of the function $[0,1]\ni r\mapsto\Phi_K^2(\sqrt{r})-r$ a bound of $|\varphi(0)|$, as well as an improved estimate of the maximal dilatation of the Douady-Earle extension of a quasisymmetric automorphism of the unit circle are derived.

0. Introduction. Notations. Statement of results

Let $\hat{\mathbb{C}} = \mathbb{C}\cup\{\infty\}$ denote the extended complex plane. A sense-preserving homeomorphism φ of a domain $\Omega\subset\hat{\mathbb{C}}$ onto a domain $\Omega'\subset\hat{\mathbb{C}}$ is said to be K-quasiconformal (abbreviated: K-qc.), $1\leq K<\infty$, if for every quadrilateral $Q=Q(z_1,z_2,z_3,z_4)$ whose closure is contained in Ω, $\operatorname{Mod}(\varphi(Q))\leq K\operatorname{Mod}(Q)$ (the geometric definition). Here $\operatorname{Mod}(Q)$ stands for the module of Q, cf. [LV]. We will write $\mathbb{Q}(K;\Omega,\Omega')$ for the class of all such mappings and $\mathbb{Q}(\Omega,\Omega') := \bigcup_{1\leq K<\infty}\mathbb{Q}(K;\Omega,\Omega')$. The value $K[\varphi]=\inf\{K\geq 1:$

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$\varphi \in \mathbb{Q}(K;\Omega,\Omega')\}$ is called the maximal dilatation of $\varphi \in \mathbb{Q}(\Omega,\Omega')$. In order to shorten the notation we write $\mathbb{Q}(K;\Omega)$ and $\mathbb{Q}(\Omega)$ for $\Omega = \Omega'$. If $\zeta \in \Omega$ is arbitrarily fixed then the notation $\varphi \in \mathbb{Q}_\zeta(K;\Omega)$ ($\mathbb{Q}_\zeta(\Omega)$) means that $\varphi \in \mathbb{Q}(K;\Omega)$ ($\mathbb{Q}(\Omega)$) and $\varphi(\zeta) = \zeta$. Assume $\Omega \subset \hat{\mathbb{C}}$ is a simply connected domain bounded by a Jordan curve $\Gamma = \partial\Omega \subset \hat{\mathbb{C}}$. If F is a complex-valued function on Ω then we put $\hat{\partial} F(z) = \lim_{u\to z} F(u)$ if the limit exists as u approaches z in Ω and $\hat{\partial} F(z) = 0$ otherwise. It is well known that every $\varphi \in \mathbb{Q}(\Omega)$ has a continuous extension to Γ being a sense-preserving homeomorphic self-mapping of Γ , cf. [LV]. Set $\hat{\partial}\mathbb{Q}(K;\Omega) = \{\hat{\partial}\varphi : \varphi \in \mathbb{Q}(K;\Omega)\}$ and $\hat{\partial}\mathbb{Q}(\Omega) = \{\hat{\partial}\varphi : \varphi \in \mathbb{Q}(\Omega)\}$. Let us denote by $\mathbf{\Delta}$, $\mathbf{T}$ and $\mathbb{C}_+$ the unit disk $\{z : |z| < 1\}$, the unit circle $\{z : |z| = 1\}$ and the upper half plane $\{z : \operatorname{Im} z > 0\}$, respectively.

In the famous paper [BA] Beurling and Ahlfors characterized the class $\hat{\partial}\mathbb{Q}(\mathbb{C}_+)$ by means of so-called quasisymmetric (abbreviated: qs.) homeomorphisms of the real axis $\mathbb{R}$, cf. also [LV]. Moreover, if $\varphi \in \mathbb{Q}(K;\mathbb{C}_+)$ then $\hat{\partial}\varphi$ is $\lambda(K)$-qs., cf. [LV] for the proof and the definition of the λ-distortion function. Conversely, if f is an M-qs. homeomorphism of $\mathbb{R}$, $k \geq 1$, then the extension formula of the Beurling–Ahlfors type generates $F \in \mathbb{Q}(\mathbb{C}_+)$ and the best bound known so far

$$K[F] \leq \max\{2M - 1\,,\, M^{3/2}\} \tag{0.1}$$

was found by Lehtinen in [Le].

Let $\operatorname{Hom}(\mathbf{T})$, ($\operatorname{Hom}^+(\mathbf{T})$) stand for the class of all (sense-preserving) homeomorphic self-mappings of $\mathbf{T}$. A counterpart of an M-qs. homeomorphism of $\mathbb{R}$ is an M-qs. automorphism γ of $\mathbf{T}$, i.e. $\gamma \in \operatorname{Hom}^+(\mathbf{T})$ satisfies the inequality $M^{-1} \leq |\gamma(I_1)|_1/|\gamma(I_2)|_1 \leq M$ for each pair of adjacent closed arcs $I_1, I_2 \subset \mathbf{T}$ of equal arc-length measure $0 < |I_1|_1 = |I_2|_1 \leq \pi$. Krzyż introduced this notion in [K1] and proved that $\gamma \in \operatorname{Hom}^+(\mathbf{T})$ is M-qs. iff there exists $\varphi \in \mathbb{Q}_0(K;\mathbf{\Delta})$ such that $\hat{\partial}\varphi = \gamma$ and the correspondence between M and K is the same as in the case of $\Omega = \mathbb{C}_+$, after a small modification of his proof. A more sophisticated but conformally invariant characterization of $\hat{\partial}\mathbb{Q}(\Omega)$ for arbitrary Ω by means of quasihomographies, or 1-dimensional qc. mappings of Γ due to many formal similarities to the class of plane qc. mappings, was studied by Zając in [Z]. Also cf. [K3].

We use the symbol $\mathcal{P}[f]$ to denote the Poisson integral of a complex–valued $|\cdot|_1$–integrable function f on $\mathbf{T}$, i.e.

$$(0.2)\qquad \mathcal{P}[f](z) = \frac{1}{2\pi}\int_{\mathbf{T}} f(u)\,\mathrm{Re}\,\frac{u+z}{u-z}|du|\,, \quad z \in \mathbf{\Delta}\,.$$

It follows from the noteworthy Kneser–Choquet theorem for convex domains, cf. [Kn], [C], that $\mathcal{P}[\gamma]$ is a sense–preserving diffeomorphic self–mapping of $\mathbf{\Delta}$ and obviously $\hat{\partial}\mathcal{P}[\gamma] = \gamma$ for each $\gamma \in \mathrm{Hom}^+(\mathbf{T})$. Consequently, for every $z \in \mathbf{\Delta}$ there exists the unique $w = F_\gamma(z) \in \mathbf{\Delta}$ satisfying the equality

$$(0.3)\qquad \mathcal{P}[h_z \circ \gamma](w) = 0$$

where

$$(0.4)\qquad h_a(u) = \frac{u-a}{1-\overline{a}u}\,, \quad a \in \mathbf{\Delta}\,, z \in \hat{\mathbb{C}}\,.$$

This shows that F_γ is a sense–preserving real–analytic diffeomorphic self–mapping of $\mathbf{\Delta}$, $\hat{\partial}F_\gamma = \check{\gamma}$ and

$$(0.5)\qquad F_{\hat{\partial}\mu\circ\gamma\circ\hat{\partial}\nu} = \check{\nu} \circ F_\gamma \circ \check{\mu}\,, \quad \mu\,, \nu \in \mathbb{Q}(1;\mathbf{\Delta})\,,$$

provided $\gamma \in \mathrm{Hom}^+(\mathbf{T})$, cf. [LP, Theorem 1.1]. Following [BS] we use the symbol $\check{f}$ to denote the inverse mapping of f if it exists, while $f^{-1} = 1/f$. The inverse mapping $E_\gamma := \check{F}_\gamma$ is a continuous extension of $\gamma \in \mathrm{Hom}^+(\mathbf{T})$ to $\mathbf{\Delta}$ conformally invariant, i.e.

$$(0.6)\qquad E_{\hat{\partial}\mu\circ\gamma\circ\hat{\partial}\nu} = \mu \circ E_\gamma \circ \nu\,, \quad \mu\,, \nu \in \mathbb{Q}(1;\mathbf{\Delta})\,,$$

by (0.5). As a matter of fact $E_\gamma := \check{F}_\gamma$ coincides with the mapping $E(\gamma)$ found by Douady and Earle in [DE, Theorem 1], and so we call E_γ the Douady–Earle extension of γ. It was the first conformally invariant analytic extension of $\gamma \in \mathrm{Hom}^+(\mathbf{T})$ to $\mathbf{\Delta}$. In the already mentioned eminent paper [DE] Douady and Earle showed that $E_\gamma \in \mathbb{Q}(\mathbf{\Delta})$ iff $\gamma \in \hat{\partial}\mathbb{Q}(\mathbf{\Delta})$. In fact, they proved that $K^* := \sup\{K[E_\gamma] : \gamma \in \hat{\partial}\mathbb{Q}(K;\mathbf{\Delta})\} < 4 \cdot 10^8 e^{35K}$, cf. [DE, Proposition 7], and given $\varepsilon > 0$ there exists $\delta > 0$ such that $K^* \leq K^{3+\varepsilon}$ if $K \leq 1+\delta$, cf.

[DE; Corollary 2]. This means that $K^* \to 1$ as $K \to 1^+$ and so their explicit estimate, starting from $4 \cdot 10^8 e^{35}$ for $K = 1$, is very inaccurate in the range of small K close to 1. Thus, analogously to (0.1), a natural problem appeared, to find an explicit estimate $L(K)$ of K^* for all $K \geq 1$ which is asymptotically sharp, i.e. $L(K) \to 1$ as $K \to 1^+$. The first bound L of this kind was found for small K, $1 \leq K \leq 1.01$, in [P1, Theorem] and then it was improved for all $K \geq 1$ in [P2, Theorem 3.1]. In this paper we proceed with the study of this topic. We extensively borrow from the techniques developed in [P1] and [P2]. However, an essential progress in this direction could be achieved due to two circumstances. The first one is the following equality, cf. [P5, Theorem 1.1, Corollary 1.2],

$$\max_{0 \leq r \leq 1} |\Phi_K^2(\sqrt{r}) - r| = M(K)\,, \quad K > 0\,, \tag{0.7}$$

where Φ_K is the Hersch–Pfluger distortion function, cf. [HP], [LV], and

$$\begin{aligned} M(K) &= 2\Phi^2_{\sqrt{K}}(1/\sqrt{2}) - 1 = \frac{\lambda(\sqrt{K}) - 1}{\lambda(\sqrt{K}) + 1}\,, \\ M(1/K) &= M(K)\,, \quad K \geq 1\,. \end{aligned} \tag{0.8}$$

The second one is the inequality (1.8). Combining these ideas we derive in Section 1 Theorem 1.4 which is the main proving tool of Lemmas 2.1 and 2.2 for K close to 1 in Section 2. The proof of (2.3) in Lemma 2.1 is an adaptation of the first part of the proof of Theorem 3.1 and the proof of Theorem 1.2 in [P2]. Roughly speaking, we modify those proofs by using the quasiconformal invariance of the harmonic measure instead of the quasisymmetric characterization of the class $\partial\mathrm{Q}_0(K;\Delta)$. Lemma 2.2 is an improvement of [P1, Lemma] for small $K \geq 1$. Lemmas 2.1 and 2.2 imply Theorem 2.3 which is our main result. It provides a new explicit and asymptotically sharp estimate $L(K)$ of K^* for all $K \geq 1$ which essentially improves those in [P1, Theorem] and [P2, Theorem 3.1*)]. Combining this

*) There is a minor error in the proof of this theorem. The theorem remains true after replacing the coefficient $1/(2\pi\sqrt{6})$ by $9\sqrt{3}/(32\pi)$ in the formula defining $F(K)$.

result with (0.1) yields a new bound of K^* which depends on the quasisymmetry constant M only. The problem of estimating K^* for quasihomographies was studied by Sakan and Zając in [SZ]. They also applied (0.7) to get asymptotically sharp estimate of K^*. Section 3 provides comments dealing with two previous sections.

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1. Some estimates involving the function $M(K)$

It was shown in [P5, Theorem 3.1] that

$$\max_{0\le r\le 1} |\hat{\partial}F(r) - r| \le M(K) \tag{1.1}$$

for every $F \in \mathbb{Q}(K;\mathbb{C}_+)$ satisfying $\hat{\partial}F(z) = z$, $z = 0,1,\infty$, and the equality is attained for some extremal mapping F_K such that $\hat{\partial}F_K(r_K) = 1 - r_K$ where $1 - 2r_K = M(K)$. Let $f(t) = \hat{\partial}F(t)$ for $0 \le t \le 1$. If $f(t) \ge t$ then we put $g(t) = f(t)$. Otherwise, we put $g(t) = a_t + b_t - f(a_t + b_t - t)$ where $(a_t, b_t) \subset [0,1]$ is the bigest open interval such that $f(r) < r$ for every $a_t < r < b_t$. g is an increasing function on $[0,1]$ because f does so. Furthermore, $0 < g(t) - t = a_t + b_t - t - f(a_t + b_t - t) < b_t - t \le 1 - t$ if $f(t) < t$. Therefore $0 < g(t) - t \le \min\{1-t, M(K)\}$ for every $0 \le t \le 1$ by (1.1) and the inequality $f(t) \le 1$. Since $\int_{a_t}^{b_t} |f(r) - r|dr = \int_{a_t}^{b_t}(g(r) - r)dr$ if $f(t) < t$, we obtain

$$\begin{aligned}\int_0^1 |\hat{\partial}F(r) - r|dr &= \int_0^1 (g(r) - r)dr \\ &\le \int_0^1 \min\{1 - r, M(K)\}dr = M(K) - \tfrac{1}{2}M^2(K)\end{aligned} \tag{1.2}$$

provided $F \in \mathbb{Q}(K;\mathbb{C}_+)$. In what follows we derive counterparts of the estimates (1.1) and (1.2) for the unit disk. We will use the symbol $\operatorname{Arg} z$ to denote the argument of $z \in \mathbb{C} \setminus \{0\}$, i.e. the unique t, $-\pi < t \le \pi$, satisfying $z = |z|e^{it}$.

Theorem 1.1. *If* $K \ge 1$, $\zeta \in \mathbf{T}$ *and* $\varphi \in \mathbb{Q}_0(K;\Delta)$ *satisfies*

$\hat{\partial}\varphi(\zeta) = \zeta$, $\hat{\partial}\varphi(-\zeta) = -\zeta$ *then*

$$\max_{z\in\mathbf{T}} |\operatorname{Arg}(\hat{\partial}\varphi(z)/z)| \le \pi M(K) , \tag{1.3}$$

as well as

$$\max_{z\in\mathbf{T}} |\hat{\partial}\varphi(z) - z| \le 2\sin(\pi M(K)/2) . \tag{1.4}$$

Proof. Let $K \ge 1$ and $\varphi \in \mathbb{Q}_0(K;\Delta)$ satisfies $\hat{\partial}\varphi(\zeta) = \zeta$, $\hat{\partial}\varphi(-\zeta) = -\zeta$ for some $\zeta \in \mathbf{T}$. Without loss of generality we can assume that $\zeta = 1$. It can be always achieved after a suitable rotation. Following Krzyż, cf. [K1], we assign to φ a K–qc. self–mapping F of $\mathbb{C}_+$ such that

$$\varphi(e^{\pi i z}) = e^{\pi i F(z)} , \quad z \in \mathbb{C}_+ , \tag{1.5}$$

and $\hat{\partial}F$ keeps the points 0, 1, 2 fixed. It follows from (1.1) that $|\hat{\partial}F(t) - t| \le M(K)$ for every $t \in \mathbb{R}$. By this and (1.5) we have $|\operatorname{Arg}(\hat{\partial}\varphi(e^{it})e^{-it})| = |\pi\hat{\partial}F(t/\pi) - t| \le \pi M(K)$ for all $t \in \mathbb{R}$, which proves (1.3); (1.4) is an obvious consequence of (1.3). □

Remark. Unfortunately, the estimates (1.3) and (1.4) are not sharp for $K > 1$. It is caused by the fact that the extremal function F_K is not periodic with the period 2. Therefore the strict inequality holds in (1.3) and (1.4) for $K > 1$. However, the obtained results seem to be fairly accurate at least for K close to 1.

Theorem 1.2. *If* $K \ge 1$, $\zeta \in \mathbf{T}$ *and* $\varphi \in \mathbb{Q}_0(K;\Delta)$ *satisfies* $\hat{\partial}\varphi(\zeta) = \zeta$, $\hat{\partial}\varphi(-\zeta) = -\zeta$ *then*

$$\frac{1}{2\pi}\int_{\mathbf{T}} |\operatorname{Arg}(\hat{\partial}\varphi(z)/z)||dz| \le \pi(M(K) - \tfrac{1}{2}M^2(K)) . \tag{1.6}$$

Moreover,

$$\frac{1}{2\pi}\int_{\mathbf{T}} |\hat{\partial}\varphi(z) - z||dz| \le 2\sin(\tfrac{\pi}{4}(2M(K) - M^2(K))) . \tag{1.7}$$

Proof. Let φ and F be as in the proof of the previous theorem. Applying (1.2) and (1.5) we get

$$\frac{1}{2\pi}\int_{\mathbf{T}} |\operatorname{Arg}(\hat{\partial}\varphi(z)/z)||dz| \leq \frac{1}{2\pi}\int_0^{2\pi} |\pi\hat{\partial}F(t/\pi) - t|dt$$
$$\leq \pi(M(K) - \tfrac{1}{2}M^2(K)) ,$$

which proves (1.6). Similarly, by using Jensen's inequality for concave functions, we have

$$\frac{1}{2\pi}\int_{\mathbf{T}} |\hat{\partial}\varphi(z) - z||dz| = \frac{1}{2\pi}\int_0^{2\pi} 2\sin\tfrac{1}{2}|\pi\hat{\partial}F(\tfrac{t}{\pi}) - t|dt \leq$$
$$2\sin(\frac{1}{2\pi}\int_0^{2\pi} \tfrac{1}{2}|\pi\hat{\partial}F(\tfrac{t}{\pi}) - t|dt) \leq 2\sin(\tfrac{\pi}{4}(2M(K) - M^2(K))) ,$$

which proves (1.7). □

We proceed with extending the above theorem to any $\varphi \in \mathbb{Q}(K;\mathbf{\Delta})$. We first prove the basic statement in this paper.

Lemma 1.3. *For every* $a \in \mathbf{\Delta}$ *and* $\gamma \in \operatorname{Hom}^+(\mathbf{T})$

$$\frac{1}{2\pi}\int_{\mathbf{T}} |\operatorname{Arg}((h_a \circ \gamma)(z)/h_a(z))||dz| \leq \max_{z\in\mathbf{T}} |\operatorname{Arg}(\gamma(z)/z)| . \tag{1.8}$$

Proof. Fix $a \in \mathbf{\Delta}$ and $\gamma \in \operatorname{Hom}^+(\mathbf{T})$. Let

$$m = \max_{z\in\mathbf{T}} |\operatorname{Arg}(\gamma(z)/z)| . \tag{1.9}$$

Clearly, if $m = \pi$ then (1.8) holds. Assume $m < \pi$. For any $z, w \in \mathbf{T}$ we denote by $I(z,w)$ the closed arc directed counterclockwise from z to w. Consider the function $f : \mathbf{T} \to \mathbb{R}$ defined by $f(z) = |I(h_a(z), h_a \circ \gamma(z))|_1$ as $\operatorname{Arg}(\gamma(z)/z) \geq 0$ and $f(z) = |I(h_a \circ \gamma(z), h_a(z))|_1$ otherwise. We assign to f two functions f_+ and f_- defined on $\mathbf{T}$ as follows: $f_+(z) = f(z)$ for $\operatorname{Arg}(\gamma(z)/z) > 0$ and $f_+(z) = 0$ otherwise, $f_-(z) = f(ze^{im})$ for $\operatorname{Arg}(\gamma(ze^{im})/ze^{im}) < 0$

and $f_-(z) = 0$ otherwise. Evidently, $f(z) = f_+(z) + f_-(ze^{-im})$ and consequently

$$\begin{aligned}\int_{\mathbf{T}} f(z)|dz| &= \int_{\mathbf{T}} f_+(z)|dz| + \int_{\mathbf{T}} f_-(ze^{-im})|dz| \\ &= \int_{\mathbf{T}} (f_+(z) + f_-(z))|dz| \,. \end{aligned} \tag{1.10}$$

Since γ and h_a are sense–preserving, we conclude from (1.9) that $f_+(z) + f_-(z) \le |I(h_a(z), h_a(ze^{im}))|_1$. Hence by (1.10) and Fubini's and Cauchy's integral theorems

$$\begin{aligned}\frac{1}{2\pi}\int_{\mathbf{T}} f(z)|dz| &\le \frac{1}{2\pi}\int_{\mathbf{T}} |I(h_a(z), h_a(ze^{im}))|_1|dz| \\ &= \frac{1}{2\pi}\int_{\mathbf{T}} |h_a(I(z, ze^{im}))|_1|dz| = \frac{1}{2\pi}\int_{\mathbf{T}}\int_0^m \frac{1-|a|^2}{|1-\overline{a}ze^{it}|^2}dt|dz| \\ &= \frac{1}{2\pi}\int_0^m\int_{\mathbf{T}} \frac{1-|a|^2}{|1-\overline{a}ze^{it}|^2}|dz|dt = \frac{1}{2\pi}\int_0^m 2\pi dt = m\,.\end{aligned}$$

This and the obvious inequality $|\operatorname{Arg}((h_a \circ \gamma)(z)/h_a(z))| \le f(z)$, $z \in \mathbf{T}$, imply (1.8). □

Theorem 1.4. *If $K \ge 1$, $\varphi \in \mathbb{Q}(K;\mathbf{\Delta})$ and $a = \varphi(0)$ then $\gamma := \hat{\partial}\varphi$ satisfies*

$$\min_{\theta\in\mathbb{R}} \tfrac{1}{2\pi}\int_{\mathbf{T}} |\operatorname{Arg}(h_{-a}(e^{i\theta}z)/\gamma(z))||dz| \le \pi M(K)\,, \tag{1.11}$$

as well as

$$\min_{\theta\in\mathbb{R}} \tfrac{1}{2\pi}\int_{\mathbf{T}} |h_{-a}(e^{i\theta}z) - \gamma(z)||dz| \le 2\sin(\tfrac{\pi}{2}M(K))\,. \tag{1.12}$$

Proof. Fix $K \ge 1$ and $\varphi \in \mathbb{Q}(K;\mathbf{\Delta})$. By the Darboux property there exist two points $\zeta_1, \zeta_2 \in \mathbf{T}$ such that $\hat{\partial}(h_a \circ \varphi)(\zeta_1) = \zeta_2$ and $\hat{\partial}(h_a \circ \varphi)(-\zeta_1) = -\zeta_2$. Then, setting $e^{i\theta} = \zeta_2/\zeta_1$ and $\psi(z) := h_a \circ$

$\varphi(e^{-i\theta}z)$, $z \in \Delta$, we see that $\psi \in \mathbb{Q}_0(K;\Delta)$ and $\hat{\partial}\psi$ keeps the points ζ_2, $-\zeta_2$ fixed. Applying Lemma 1.3 and Theorem 1.1 we have

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\mathbf{T}} |\operatorname{Arg}(h_{-a}(e^{i\theta}z)/\gamma(z))||dz| \\
&= \frac{1}{2\pi}\int_{\mathbf{T}} |\operatorname{Arg}(h_{-a}(z)/h_{-a}\circ h_a\circ\gamma(e^{-i\theta}z))||dz| \\
&= \frac{1}{2\pi}\int_{\mathbf{T}} |\operatorname{Arg}(h_{-a}(z)/h_{-a}\circ\hat{\partial}\psi(z))||dz| \\
&\le \max_{z\in\mathbf{T}} |\operatorname{Arg}(\hat{\partial}\psi(z)/z)| \le \pi M(K)\,,
\end{aligned}$$

which proves (1.11). Hence by Jensen's inequality for concave functions

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\mathbf{T}} |h_{-a}(e^{i\theta}z)-\gamma(z)||dz| \\
&= \frac{1}{2\pi}\int_{\mathbf{T}} 2\sin\tfrac{1}{2}|\operatorname{Arg}(h_{-a}(e^{i\theta}z)/\gamma(z))||dz| \\
&\le 2\sin\Big(\frac{1}{4\pi}\int_{\mathbf{T}} |\operatorname{Arg}(h_{-a}(e^{i\theta}z)/\gamma(z))||dz|\Big) \le 2\sin\big(\tfrac{\pi}{2}M(K)\big)\,,
\end{aligned}$$

and this yields (1.12). □

2. An estimate of $K[E_\gamma]$ for $\gamma \in \hat{\partial}\mathbb{Q}(K;\Delta)$

Suppose $\Omega \subset \mathbb{C}$ is a domain and φ is a sense-preserving diffeomorphism of Ω onto $\Omega' = \varphi(\Omega)$. Then the Jacobian $|\partial\varphi(\zeta)|^2 - |\bar{\partial}\varphi(\zeta)|^2$ is positive at every $\zeta \in \Omega$. Define $k[\varphi](\zeta) = |\bar{\partial}\varphi(\zeta)/\partial\varphi(\zeta)|$ for $\zeta \in \Omega$ and $k[\varphi] = \sup_{\zeta\in\Omega} k[\varphi](\zeta)$. It is well known that $\varphi \in \mathbb{Q}(\Omega,\Omega')$ if $k[\varphi] < 1$, and

$$K[\varphi] = (1+k[\varphi])(1-k[\varphi])^{-1}\,, \tag{2.1}$$

cf. [LV]. We denote by $\mathbb{Q}^0(K;\Delta)$ the class of all $\varphi \in \mathbb{Q}(K;\Delta)$ normalized by

$$\mathcal{P}[\hat{\partial}\varphi](0) = 0\,. \tag{2.2}$$

It follows from (0.3) that $\varphi \in \mathbb{Q}^0(K;\Delta)$ iff $\varphi \in \mathbb{Q}(K;\Delta)$ and $F_{\partial\varphi}(0) = 0$.

Lemma 2.1. *If* $K \geq 1$, $\varphi \in \mathbb{Q}^0(K;\Delta)$, $\gamma := \partial\varphi$ *and* $r_l = \cos(\pi/2^{l+1})$, $r_l' = \sin(\pi/2^{l+1})$, $l = 1,2,3$, *then*

(2.3)
$$k^2[F_\gamma](0) \leq 1 - \frac{2^7\sqrt{2}}{\pi}\left(\frac{1-|\varphi(0)|}{1+|\varphi(0)|}\right)^5 \times$$
$$\Phi_K^2(r_1)\Phi_{1/K}^2(r_1')\Phi_K^2(r_2)\Phi_{1/K}^2(r_2')\Phi_K(r_3)\Phi_{1/K}(r_3')(2\Phi_K^2(r_3)-1)$$

and

$$\begin{aligned} k^2[F_\gamma](0) &\leq \sin 2P(K,|\varphi(0)|) \\ &+ (1-\sin 2P(K,|\varphi(0)|))(2\sin(\pi M(K)/2) \\ &+ |\varphi(0)|^2 + \sin P(K,|\varphi(0)|)) \end{aligned} \tag{2.4}$$

where

$$P(K,r) = \tfrac{\pi}{4} - 2(1-r)(1+r)^{-1}\arccos\Phi_K(\cos\tfrac{\pi}{8})\ . \tag{2.5}$$

Moreover,

$$\begin{aligned} &k[F_\gamma](0) \leq \sin 2P(K,|\varphi(0)|) \\ &+ \sin P(K,|\varphi(0)|)\cos^2 2P(K,|\varphi(0)|)(1-2\sin(\pi M(K)/2) \\ &- |\varphi(0)|^2 - \sin P(K,|\varphi(0)|)\sin 2P(K,|\varphi(0)|))^{-1} \end{aligned} \tag{2.6}$$

as the denominator is positive and $P(K,|\varphi(0)|) \leq \pi/8$.

Proof. Fix $K \geq 1$ and $\varphi \in \mathbb{Q}^0(K;\Delta)$. Let $a = \varphi(0) \in \Delta$. Then $\psi := h_a \circ \varphi \in \mathbb{Q}_0(K;\Delta)$. Assume I is an arbitrary subarc of $\mathbf{T}$. It follows from the quasi-invariance of the harmonic measure ω that

$$\begin{aligned} \tfrac{1}{K}\mu(\cos(\tfrac{\pi}{2}\omega(0,\Delta)[I])) &\leq \mu(\cos(\tfrac{\pi}{2}\omega(\psi(0),\Delta)[\partial\psi(I)])) \\ &\leq K\mu(\cos(\tfrac{\pi}{2}\omega(0,\Delta)[I]))\ , \end{aligned} \tag{2.7}$$

cf. [H]. Here μ stands for the module of the Grötzsch extremal domain $\Delta \setminus [0,r]$, cf. [LV]. Since $\psi(0) = 0$ and $2\pi\omega(0,\Delta)[I] = |I|_1$ for any arc $I \subset \mathbf{T}$, we get by (2.7) and the definition of Φ_K

$$\Phi_{1/K}\left(\cos\tfrac{|I|_1}{4}\right) \leq \cos\tfrac{|h_a\circ\gamma(I)|_1}{4} \leq \Phi_K\left(\cos\tfrac{|I|_1}{4}\right)\ . \tag{2.8}$$

Set $\alpha_{K,l} = 4\arccos\Phi_K(r_l)$, $l = 1,2,3$. An easy computation applying the identity

$$(2.9)\qquad \Phi_K^2(r) + \Phi_{1/K}^2(\sqrt{1-r^2}) = 1\,, \quad 0 \le r \le 1\,,$$

cf. [AVV, Theorem 3.3], shows that

$$(2.10)\qquad \sin(\alpha_{K,l}/2) = 2\Phi_K(r_l)\Phi_{1/K}(r_l')\,, \quad l = 1,2\,,$$

and

$$(2.11)\qquad \sin(\alpha_{K,3}) = 4\Phi_K(r_3)\Phi_{1/K}(r_3')(2\Phi_K^2(r_3) - 1)\,.$$

It follows from (2.8) and the inequality $|h_{-a}(I)|_1 = \int_I |h'_{-a}(z)||dz| \ge (1-|a|)(1+|a|)^{-1}|I|_1$ that

(2.12)
$$\begin{aligned}|\gamma(I)|_1 = |h_{-a}\circ h_a\circ\gamma(I)|_1 &\ge (1-|a|)(1+|a|)^{-1}|h_a\circ\gamma(I)|_1\\ &\ge (1-|a|)(1+|a|)^{-1}\alpha_{K,l}\end{aligned}$$

for $|I|_1 = \pi/2^{l-1}$, $l = 1,2,3$. The inequalities (2.12), $l = 1,2$, correspond to (1.2) and (1.3) in [P2] after replacing $2\pi/(1+k)$ and $2\pi/(1+k)^2$ by $\alpha_{K,1}$ and $\alpha_{K,2}$, respectively. Define for every $\eta \in \mathrm{Hom}(\mathbf{T})$ and any integers $n, m \in \mathbb{Z}$

$$(2.13)\qquad \eta_m^n := \frac{1}{2\pi}\int_{\mathbf{T}} z^m(\eta(z))^n|dz|\,.$$

A calculation similar to that in the proof of Theorem 1.2 in [P2] shows that

$$(2.14)\qquad \begin{aligned}&|\gamma_0^2| \le \cos\left(\alpha_{K,2}\tfrac{1-|a|}{1+|a|}\right) = \sin 2P(K,|a|)\,;\\ &|\gamma_1^1| \le \cos\left(\tfrac{\pi}{4} + \tfrac{\alpha_{K,2}}{2}\tfrac{1-|a|}{1+|a|}\right) = \sin P(K,|a|)\,;\\ &1 \ge |\gamma_{-1}^1|^2 - |\gamma_1^1|^2 \ge \tfrac{2\sqrt{2}}{\pi}\sin^2\left(\tfrac{\alpha_{K,1}}{2}\tfrac{1-|a|}{1+|a|}\right)\sin\left(\alpha_{K,3}\tfrac{1-|a|}{1+|a|}\right)\,.\end{aligned}$$

Since $\varphi \in \mathrm{Q}^0(K;\mathbf{\Delta})$, $F_\gamma(0) = 0$. Differentiating at the point $z = 0$ both sides of the equality $\mathcal{P}[h_z\circ\gamma](F_\gamma(z)) = 0$, $z \in \mathbf{\Delta}$, we see that

$$\gamma_{-1}^1\partial F_\gamma(0) + \gamma_1^1\overline{\bar\partial F_\gamma(0)} = 1\quad,\quad \gamma_{-1}^1\bar\partial F_\gamma(0) + \gamma_1^1\overline{\partial F_\gamma(0)} = -\gamma_0^2\,,$$

hence

$$(2.15)\qquad \partial F_\gamma(0) = \frac{\overline{\gamma^1_{-1}} + \overline{\gamma^2_0}\gamma^1_1}{|\gamma^1_{-1}|^2 - |\gamma^1_1|^2} \quad , \quad \bar\partial F_\gamma(0) = \frac{-\gamma^1_{-1}\overline{\gamma^2_0} - \gamma^1_1}{|\gamma^1_{-1}|^2 - |\gamma^1_1|^2} \, ,$$

and finally

$$(2.16)\qquad 1 - \left|\frac{\bar\partial F_\gamma(0)}{\partial F_\gamma(0)}\right|^2 = \frac{(1-|\gamma^2_0|^2)(|\gamma^1_{-1}|^2 - |\gamma^1_1|^2)}{|\overline{\gamma^1_{-1}} + \overline{\gamma^2_0}\gamma^1_1|^2} \, .$$

From this, (2.16) and (2.14)
(2.17)

$$\begin{aligned} 1 - \left|\frac{\bar\partial F_\gamma(0)}{\partial F_\gamma(0)}\right|^2 &= \frac{(1-|\gamma^2_0|^2)(|\gamma^1_{-1}|^2 - |\gamma^1_1|^2)}{|\overline{\gamma^1_{-1}} + \overline{\gamma^2_0}\gamma^1_1|^2} \geq \frac{1-|\gamma^2_0|}{1+|\gamma^2_0|}(|\gamma^1_{-1}|^2 - |\gamma^1_1|^2) \\ &\geq \frac{2\sqrt{2}}{\pi}\tan^2\left(\frac{\alpha_{K,2}}{2}\frac{1-|a|}{1+|a|}\right)\sin^2\left(\frac{\alpha_{K,1}}{2}\frac{1-|a|}{1+|a|}\right)\sin\left(\alpha_{K,3}\frac{1-|a|}{1+|a|}\right) \\ &\geq \frac{2\sqrt{2}}{\pi}\left(\frac{1-|a|}{1+|a|}\right)^5 \sin^2\frac{\alpha_{K,1}}{2}\sin^2\frac{\alpha_{K,2}}{2}\sin\alpha_{K,3} \, . \end{aligned}$$

Hence by (2.10) and (2.11) the bound (2.3) follows. We derive now (2.4). By (2.13) we get for every $\theta \in \mathbb{R}$

$$\begin{aligned} ||\gamma^1_{-1}| - 1| &\leq |\gamma^1_{-1} - e^{i\theta}| = \tfrac{1}{2\pi}\left|\int_{\mathbf{T}} (\gamma(z)\bar z - e^{i\theta})|dz|\right| \\ &\leq \tfrac{1}{2\pi}\int_{\mathbf{T}} |\gamma(z) - h_{-a}(e^{i\theta}z)||dz| + \tfrac{1}{2\pi}\left|\int_{\mathbf{T}} (h_{-a}(e^{i\theta}z)\bar z - e^{i\theta})|dz|\right| \\ &= \tfrac{1}{2\pi}\int_{\mathbf{T}} |\gamma(z) - h_{-a}(e^{i\theta}z)||dz| + |a|^2 \, . \end{aligned}$$

Furthermore, by Theorem 1.4

$$(2.18)\qquad \begin{aligned} 1 - |\gamma^1_{-1}| &\leq \min_{\theta\in\mathbb{R}} \tfrac{1}{2\pi}\int_{\mathbf{T}} |\gamma(z) - h_{-a}(e^{i\theta}z)||dz| + |a|^2 \\ &\leq 2\sin(\tfrac{\pi}{2}M(K)) + |a|^2 \, . \end{aligned}$$

It follows from (2.16) that

$$1 - \left|\frac{\bar\partial F_\gamma(0)}{\partial F_\gamma(0)}\right|^2 \geq \frac{(1-|\gamma^2_0|^2)(|\gamma^1_{-1}|^2 - |\gamma^1_1|^2)}{(1+|\gamma^2_0|)(|\gamma^1_{-1}|+|\gamma^1_1|)} = (1-|\gamma^2_0|)(|\gamma^1_{-1}| - |\gamma^1_1|) \, .$$

Combining this, (2.14) and (2.18) gives (2.4). The last bound (2.6) is a direct conclusion from (2.14), (2.15), (2.18) and the following estimate

$$k[F_\gamma](0) = \frac{|-\overline{\gamma^1_{-1}}\gamma^2_0 - \gamma^1_1|}{|\gamma^1_{-1} + \overline{\gamma^2_0}\gamma^1_1|} \leq |\gamma^2_0| + |\gamma^1_1|(1-|\gamma^2_0|^2)(|\gamma^1_{-1}| - |\gamma^2_0||\gamma^1_1|)^{-1} . \square$$

The estimate of $k[F_\gamma](0)$ in the above lemma depends on K and $|a|$. The next lemma provides a bound of $|a|$ which depends on K only. Consider in the class $\mathbb{Q}^0(K;\mathbf{\Delta})$ the basic distortion functional

$$\rho(K) = \sup\{|\varphi(0)| : \varphi \in \mathbb{Q}^0(K;\mathbf{\Delta})\} . \tag{2.19}$$

Lemma 2.2. *For every $K \geq 1$*

$$\rho(K) \leq p(K) := \min\{2\sin(\tfrac{\pi}{2}M(K)), \quad 1 - 2(\sqrt{3}\Phi_K(\sqrt{3}/2)\Phi^{-1}_{1/K}(1/2)+1)^{-1}\} . \tag{2.20}$$

Proof. Fix $K \geq 1$, $\varphi \in \mathbb{Q}^0(K;\mathbf{\Delta})$ and set $\gamma = \hat{\partial}\varphi$, $a = \varphi(0)$. It follows from $\mathcal{P}[\gamma](0) = 0$ that for every arc $I \subset \mathbf{T}$ of length $|I|_1 = 2\pi/3$, $|\gamma(I)|_1 \leq 4\pi/3$, cf. [LP] for details. Applying now (2.8) gives $|\varphi(0)| \leq 1/2 + \sqrt{3}/2\cot(\pi/3 + \arccos\Phi_K(\sqrt{3}/2))$, cf. [P1, (4)]. Hence by (2.9) we derive

$$\rho(K) \leq 1 - 2(\sqrt{3}\Phi_K(\sqrt{3}/2)\Phi^{-1}_{1/K}(1/2)+1)^{-1} . \tag{2.21}$$

This estimate is not sharp because the right hand side tends to 1/2 as $K \to 1^+$. To improve it for small K close to 1 we will use Theorem 1.4. Since $\mathcal{P}[\gamma](0) = 0$, we have for every $\theta \in \mathbb{R}$

$$|\varphi(0)| = |a| = \tfrac{1}{2\pi}\left|\int_{\mathbf{T}} h_{-a}(e^{i\theta}z)|dz| - \int_{\mathbf{T}}\gamma(z)|dz|\right| \leq \tfrac{1}{2\pi}\int_{\mathbf{T}} |h_{-a}(e^{i\theta}z) - \gamma(z))||dz| .$$

Now, applying Theorem 1.4 gives $|\varphi(0)| \le 2\sin(\pi M(K)/2)$ and so $\rho(K) \le 2\sin(\pi M(K)/2)$ for $K \ge 1$. Combining this and (2.21) leads to (2.20). □

Consider the second distortion functional $\rho_*(K)$ in the class $\mathbb{Q}^0(K;\boldsymbol{\Delta})$ given by

$$\rho_*(K) = \sup\left\{ \tfrac{1+|\varphi(0)|}{1-|\varphi(0)|} : \varphi \in \mathbb{Q}^0(K;\boldsymbol{\Delta})\right\} . \tag{2.22}$$

In view of the above lemma $\rho(K) < 1$ and so

$$\rho_*(K) = \tfrac{1+\rho(K)}{1-\rho(K)} \le p_*(K) := \tfrac{1+p(K)}{1-p(K)} < \infty\,, \quad K \ge 1\,. \tag{2.23}$$

Now we are ready to prove the main result in this section which improves Theorem in [P1] and Theorem 3.1 in [P2] in the whole range of $K \ge 1$.

Theorem 2.3. *If $K \ge 1$ and $\gamma \in \partial\mathbb{Q}_{\mathbf{T}}(K)$ then F_γ and E_γ are $(1+k)(1-k)^{-1}$-qc. mappings and $k = k[E_\gamma] = k[F_\gamma]$ satisfies*

(2.24)
$$k^2 \le 1 - \tfrac{2^7\sqrt{2}}{\pi} p_*^{-5}(K)\times$$
$$\Phi_K^2(r_1)\Phi_{1/K}^2(r_1')\Phi_K^2(r_2)\Phi_{1/K}^2(r_2')\Phi_K(r_3)\Phi_{1/K}(r_3')(2\Phi_K^2(r_3)-1) \le$$
$$1 - \tfrac{2^7\sqrt{6}}{3^3\pi}\Phi_K^{-5}\left(\tfrac{\sqrt{3}}{2}\right)\Phi_{1/K}^5\left(\tfrac{1}{2}\right)\times$$
$$\Phi_K^2(r_1)\Phi_{1/K}^2(r_1')\Phi_K^2(r_2)\Phi_{1/K}^2(r_2')\Phi_K(r_3)\Phi_{1/K}(r_3')(2\Phi_K^2(r_3)-1)$$

and

$$\begin{aligned} k^2 &\le \sin 2P(K) + (1-\sin 2P(K))(2\sin(\tfrac{\pi}{2}M(K)) \\ &\quad + 4\sin^2(\tfrac{\pi}{2}M(K)) + \sin P(K)) \end{aligned} \tag{2.25}$$

where

$$P(K) = P(K, p(K)) = \tfrac{\pi}{4} - 2p_*^{-1}(K)\arccos\Phi_K(\cos\tfrac{\pi}{8})\,, \tag{2.26}$$

and r_l, r_l', $l = 1,2,3$, were defined in Lemma 2.1. Moreover,

$$\begin{aligned} k &\le \sin 2P(K) + \sin P(K)\cos^2 2P(K)(1 - 2\sin(\tfrac{\pi}{2}M(K)) \\ &\quad - 4\sin^2(\tfrac{\pi}{2}M(K)) - \sin P(K)\sin 2P(K))^{-1} \end{aligned} \tag{2.27}$$

whenever $P(K) \le \pi/8$.

Proof. Fix $K \ge 1$ and $\gamma \in \hat{\partial}\mathbb{Q}(K;\mathbf{\Delta})$. If $z \in \mathbf{\Delta}$ then $\gamma_z := h_z \circ \gamma \circ h_{-F_\gamma(z)} \in \hat{\partial}\mathbb{Q}(K;\mathbf{\Delta})$ and by (0.5) $h_{-F_\gamma(z)} \circ F_{\gamma_z}(0) = F_\gamma(h_{-z}(0)) = F_\gamma(z)$. Hence $F_{\gamma_z}(0) = h_{F_\gamma(z)}(F_\gamma(z)) = 0$ and applying (0.5) once again we have $k[F_\gamma](z) = k[h_{F_\gamma(z)} \circ F_\gamma \circ h_{-z}](0) = k[F_{\gamma_z}](0)$ for every $z \in \mathbf{\Delta}$. Consequently,

(2.28)
$$k := k[F_\gamma] = \sup_{z\in\mathbf{\Delta}} k[F_{\gamma_z}](0) \le \sup\{k[F_{\partial\varphi}](0) : \varphi \in \mathbb{Q}^0(K;\mathbf{\Delta})\} .$$

It follows from (2.5), (2.23) and (2.26) that for each $\varphi \in \mathbb{Q}^0(K;\mathbf{\Delta})$, $P(K, |\varphi(0)|) \le P(K)$. From this, (2.28) and the formulas (2.3), (2.4) and (2.6) we easily derive corresponding bounds (2.24), (2.25) and (2.27)*) in our theorem. Moreover, by (2.20) we get $p_*(K) \le \sqrt{3}\Phi_K(\sqrt{3}/2)\Phi_{1/K}^{-1}(1/2)$, which completes the proof of (2.24). By definition, $E_\gamma = \check{F}_\gamma$. Therefore $k = k[E_\gamma] = k[F_\gamma]$ and $E_\gamma \in \mathbb{Q}((1+k)(1-k)^{-1};\mathbf{\Delta})$. □

Corollary 2.4. *If γ is an M-qs. automorphism of* $\mathbf{T}$, $1 \le M < \infty$, *then F_γ and E_γ are $(1+k)(1-k)^{-1}$-qc. mappings and $k = k[E_\gamma] = k[F_\gamma]$ satisfies the inequalities* (2.24), (2.25) and (2.27) after K has been replaced by $\min\{M^{3/2}, 2M-1\}$.

Proof. Modifying the proof of Krzyż's Theorem from [K1] by applying Lehtinen's result (0.1) we deduce that M-qs. automorphism of $\mathbf{T}$ has a K-qc. extension to $\mathbf{\Delta}$ with $K \le \min\{M^{3/2}, 2M-1\}$. In this way the corollary follows immediately from Theorem 2.3. □

3. Complementary remarks

Remark 1. Let $\phi_K(x) = \min\{4^{1-1/K}x^{1/K}, 1\}$ and $h(x) = (1-x)(1+x)^{-1}$ for all $0 \le x \le 1$, $K > 0$. Consider the following functions

$$\Phi_0[K,t](x) = \Phi_t \circ \phi_K \circ \Phi_{1/t}(x) ,$$
$$\Phi_1[K,t](x) = h \circ \Phi_0[1/K,t] \circ h(x) , \quad K > 0 ,$$
$$\Phi[K,t](x) = \begin{cases} \min\{\Phi_0[K,t](x) , \Phi_1[K,t](x)\} & , K \ge 1 \\ \max\{\Phi_0[K,t](x) , \Phi_1[K,t](x)\} & , 0 < K \le 1 \end{cases}$$

*) see Remark 1 in Section 3 for the completion of the proof of (2.27).

for $0 \le x \le 1$, $t > 0$. Since, as shown in [LV], $\Phi_2(r) = 2\sqrt{r}(1+r)^{-1}$, $0 \le r \le 1$, and $\Phi_{2^{n+1}} = \Phi_{2^n} \circ \Phi_2$, $\Phi_{2^{-n}} = \overset{-1}{\Phi}_{2^n}$, $n \in \mathbb{N}$, all functions $\Phi[K, 2^n]$, $K > 0$, $n \in \mathbb{Z}$, are elementary. Moreover, it follows from [P3, Theorem 1.3, Corollary 1.4] that $\Phi[K, 2^n]$ approaches monotonically Φ_K as $n \to \infty$. Moreover, it follows from [P3, Theorem 1.5, Corollary 1.6] that

$$0 \le \Phi_0[K, 2^n](x) - \Phi_K(x) \le x^{2^{n+1}/K} \Phi_0[K, 2^n](x)$$

for $K \ge 1$, $n = 2, 3, 4, \ldots$,

$$0 \le \Phi_1[K, 2^n](x) - \Phi_K(x) \le 2\big((1 - h(x)^{2^{n+1}})^{-K2^{-n}} - 1\big) h^K(x)$$

for $K \ge 1$, $n = 1, 2, 3, \ldots$,

$$0 \le \Phi_K(x) - \Phi_0[K, 2^n](x) \le \big((1 - x^{2^{n+1}})^{-1/K2^n} - 1\big) \Phi_0[K, 2^n](x)$$

for $0 < K \le 1$, $n = 1, 2, 3, \ldots$, and

$$\begin{aligned} 0 &\le \Phi_K(x) - \Phi_1[K, 2^n](x) \\ &\le 2\big((1 - h(x)^{K2^{n+1}})^{-1} - 1\big) \min\{4^{1-K} h^K(x), 1\} \end{aligned}$$

for $0 < K \le 1$, $n = 2, 3, 4, \ldots$.

All bounds in Lemma 2.2 and Theorem 2.3 depend on Φ_K. Applying the approximating sequence $\Phi[K, 2^n]$, $n = 0, 1, \ldots$, of Φ_K, we can estimate the right-hand side of (2.24), (2.25) and (2.27) by elementary functions with arbitrarily preassigned accuracy due to the above inequalities, cf. [P4, Theorem 3.1]. For example, we can determine the constants K_1 and K_2 such that the bound (2.27) is better than that given in (2.25) for $1 \le K < K_1$ and the bound (2.25) is better than that in (2.24) for $1 \le K < K_2$. Relevant computer calculations give $0 < K_1 - 1.053180 < 10^{-6}$ and $0 < K_2 - 1.113057 < 10^{-6}$. Moreover, $P(1.1) > \pi/8$ and $2\sin(\frac{\pi}{2}M(1.1)) + 4\sin^2(\frac{\pi}{2}M(1.1)) + \sin P(1.1) \sin 2P(1.1) < 1$, which completes the proof of (2.27).

Remark 2. Theorems 1.1 and 1.2 are counterparts of Corollaries 2.4 and 2.7 in [K2], respectively, for M–qs. functions h on $\mathbb{R}$ such that σ, $\sigma(t) = h(t) - t$, is 2π–periodic on $\mathbb{R}$ normalized by

$\int_0^{2\pi} \sigma(t)dt = 0$. They enable us to adopt some Krzyż's result from [K2] for functions of the form $\mathbb{R} \ni t \mapsto h(t) - t \in \mathbb{R}$ where $h : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying $\gamma(e^{it}) = e^{ih(t)}$, $-\pi < h(0) \leq \pi$ and $\gamma \in \partial \mathrm{Q}_0(K; \Delta)$, but we will not develop this point here.

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Instytut Matematyki UMCS
Plac M. Curie Skłodowskiej 1
20–031 Lublin, Poland
e–mail: partyka@golem.umcs.lublin.pl