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On Hankel Operators with Conjugate Analytic Symbols

Abstract. This paper is a slightly extended version of a lecture given at the Workshop on Complex Analysis. In Sections 2 and 3 the author's results from [11] and [12] are presented, whereas Section 4 contains some new, unpublished theorems.
E.g. for $f, g \in$ BMOA the compactness of the operator $H_{\bar f}^* H_{\bar g} - H_{\bar g}^* H_{\bar f}$ is equivalent to the condition : $\lim_{|\lambda|\to 1^-} (1-|\lambda|^2)^2 \{f'(\lambda)\overline{g'(\lambda)} - \overline{f'(\lambda)} g'(\lambda)\} = 0$.

1. Introduction

Let $\mathbb{D}$ denote the open unit disc in the complex plane $\mathbb{C}$ and let $d\sigma$ denote the area measure on $\mathbb{D}$ normalized so that the area of $\mathbb{D}$ is 1. For $1 \le p < \infty$ the Bergman space L_a^p is the subspace of $L^p(\mathbb{D}, d\sigma)$ consisting of analytic functions on $\mathbb{D}$. When $p = 2$, we obtain the Hilbert space with inner product given by

$$< f, g > = \int_{\mathbb{D}} f(z)\overline{g(z)}\, d\sigma(z) .$$

The space of bounded analytic functions on $\mathbb{D}$ will be denoted by H^∞ . The Bloch space $\mathcal{B}$ consists of analytic functions f on $\mathbb{D}$ such that

$$(1) \qquad \|f\|_{\mathcal{B}} = \sup\{|f'(z)|(1-|z|^2) \ : \ z \in \mathbb{D}\} < \infty .$$

The little Bloch space $\mathcal{B}_o$ is the subspace of $\mathcal{B}$ consisting of all functions f for which

$$\lim_{|z|\to 1^-}(1-|z|^2)|f'(z)|=0\,. \tag{2}$$

Some alternate descriptions of the spaces $\mathcal{B}$ and $\mathcal{B}_0$ are given e.g. in [3], [13].

Moreover, let H^p , $0<p<\infty$, denote the Hardy space of analytic functions f on $\mathbb{D}$ with the norm

$$\|f\|_{H^p}=\sup_{0\le r<1}\left(\int_{-\pi}^{\pi}|f(re^{i\theta}|^p d\theta/2\pi\right)^{1/p}\,. \tag{3}$$

For $1\le p\le\infty$ the boundary values of functions of H^p may be considered as a subspace of $L^p(\partial\mathbb{D},d\theta/2\pi)$.

Let BMOA and VMOA denote the spaces of functions f in H^2 whose boundary values have bounded mean oscillation and vanishing mean oscillation on $\partial\mathbb{D}$, respectively. There are several alternate characterizations of BMOA and VMOA spaces, ([7]) e.g. for f analytic on $\mathbb{D}$

$$f\in \mathrm{BMOA}\iff \sup_{a\in\mathbb{D}}\left\|f\left(\frac{z+a}{1+\bar{a}z}\right)-f(a)\right\|_{H^p}<\infty\,. \tag{4}$$

The proper inclusions

$$H^\infty\subset \mathrm{BMOA}\subset\mathcal{B} \tag{5}$$

are well known [9]. It has been also noticed ([3],[4]) that in many cases the roles BMOA and VMOA functions play in the H^p theory are analogous to those the Bloch space and the little Bloch space play in the theory of Bergman spaces. The results concerning Hankel operators on the Bergman space L^p_a we are going to discuss strengthen this analogy.

Let P denote the orthogonal projection of L^2 onto L^p_a . There is the following formula for P

$$P(f)(z)=\int_{\mathbb{D}}\frac{f(w)}{(1-\bar{w}z)^2}d\sigma(w)\,,\quad f\in L^2\,. \tag{6}$$

The above formula makes sense for $f \in L^p$, $1 \le p < \infty$, and we will use (6) to define $P : L^p \to L^p_a$. It is well known that the operator P is bounded for $1 < p < \infty$ and unbounded for $p = 1$ ([4], [17]). The Hankel operator $H_{\bar f} : L^p_a \to L^p$ is defined by

$$H_{\bar f}(g) = \bar f g - P(\bar f g)\,. \tag{7}$$

In 1986 S.Axler [3] proved the following theorems:

Theorem A. *Let $f \in L^2_a$. Then $H_{\bar f} : L^2_a \to (L^2_a)^{\perp}$ is bounded if and only if f is a Bloch function.*

Theorem B. *Let $f \in L^2_a$. Then $H_{\bar f} : L^2_a \to (L^2_a)^{\perp}$ is compact if and only if f is in the little Bloch space $\mathcal{B}_o$.*

We prove that the above theorems remain true for $1 < p < \infty$.

Now to see the analogy between H^p and L^p_a spaces let us consider the Szegö projection $S : L^p(\partial\mathbb{D}, d\theta/2\pi) \to H^p$ given by by the formula

$$Sf(z) = \int_{-\pi}^{\pi} \frac{f(e^{i\theta})}{1 - e^{-i\theta} z} d\theta/2\pi \quad , \quad z \in \mathbb{D} \tag{8}$$

and the Hankel operator defined by $H_{\bar f}(g) = \bar f g - S(\bar f g)$. It is known that for $1 < p < \infty$, $f \in H^2$, $H_{\bar f} : H^p \to L^p(\partial\mathbb{D}, d\theta/2\pi)$ is bounded iff $f \in$ BMOA and compact iff $f \in$ VMOA.

In Section 4 of this paper we consider only the case $p = 2$. For $f \in L^{\infty}$ the Toeplitz operator $T_f : L^2_a \to L^2_a$ is defined by $T_f(g) = P(fg)$. T_f is the multiplication operator on L^p_a for $f \in H^{\infty}$. We deal with some operators connected with commutators of the Toeplitz operators and involving Hankel operators. The main result of Section 4 is

Theorem 4. *Let $f, g \in BMOA$. If the operator $H^*_{\bar f} H_{\bar g} : L^2_a \to L^2_a$ is compact then*

$$\lim_{|z| \to 1^-} (1 - |z|^2)^2 |f'(z)||g'(z)| = 0\,. \tag{9}$$

2. Bounded Hankel operators on L^p_a , $1 \le p < \infty$

We shall start with the following

Theorem 1. *For* $f \in L^2_a$ *the Hankel operator* $H_{\bar f} : L^p_a \to L^p$, $1 < p < \infty$, *is bounded if and only if* $f \in \mathcal{B}$.

To prove the sufficiency we need the following generalization of Lemma 5 of [3]

Lemma 1. *If* $1 < p < \infty$ *and* $f \in \mathcal{B}$ *then there exists a constant* $c > 0$ *such that*

(10)
$$\int_{\mathbb{D}} |f(z) - f(w)||1 - \bar w z|^{-2}(1 - |w|^2)^{-\frac{1}{p}} d\sigma(w) \le c\|f\|_{\mathcal{B}}(1 - |z|^2)^{-\frac{1}{p}} .$$

Throughout this paper the letter c will denote a constant not necessarily the same at each occurence.

Proof of Theorem 1. Suppose that $f \in L^2_a$ and

$$\|H_{\bar f}(g)\|_p \le c\|g\|_p \quad , \quad \text{for } g \in L^p_a .$$

Define

$$k_w(z) = (1 - \bar w z)^{-2} \quad , w, z \in \mathbb{D} .$$

Then $k_w \in H^\infty \subset L^p_a$ and $H_{\bar f}(k_w) = (\bar f - \bar f(w))k_w$.
Now for $h \in L^{p'}$, $1/p + 1/p' = 1$, the integral

$$\int_{\mathbb{D}} H_{\bar f}(g(z))h(z)d\sigma(z)$$

defines a bounded linear operator on L^p_a and

$$(11) \qquad \left|\int_{\mathbb{D}} H_{\bar f}(g(z))h(z)d\sigma(z)\right| \le c\|H_{\bar f}(g)\|_p\|h\|_{p'} \le c\|g\|_p\|h\|_{p'} .$$

Replacing $g(z)$ by $k_w(z)$ and $h(z)$ by $z|1-\bar{w}z|^{-2}$ and using the well-known estimate (cf. [17, p.53])

$$(12)\qquad \int_{\mathbb{D}} \frac{d\sigma(z)}{|1-\bar{w}z|^{2+\alpha}} \le c\,\frac{1}{(1-|w|^2)^{\alpha}}\ , \qquad \alpha>0\ ,$$

we obtain

$$(13)\qquad \left|\int_{\mathbb{D}} H_{\bar{f}}(k_w(z))\frac{z}{|1-\bar{w}z|^2}d\sigma(z)\right| \le c\,\frac{1}{(1-|w|^2)^2}\ ,$$

On the other hand, in view of the identity,

$$\int_{\mathbb{D}} g(z)z\bar{f}(z)d\sigma(z) = \int_{\mathbb{D}} g(z)\bar{f}'(z)(1-|z|^2)d\sigma(z)\ , \quad f\in L^1_a\ , g\in H^\infty$$

we have

$$\left|\int_{\mathbb{D}} H_{\bar{f}}(k_w(z))\frac{z}{|1-\bar{w}z|^2}d\sigma(z)\right| = \left|\int_{\mathbb{D}} \frac{(\bar{f}(z)-\bar{f}(w))z}{(1-\bar{z}w)(1-z\bar{w})^3}d\sigma(z)\right|$$

$$= \left|\int_{\mathbb{D}} \frac{f'(z)(1-|z|^2)}{(1-z\bar{w})(1-\bar{z}w)^3}d\sigma(z) + \bar{w}\int_{\mathbb{D}} \frac{(f(z)-f(w))(1-|z|^2)}{(1-z\bar{w})^2(1-\bar{z}w)^3}d\sigma(z)\right| .$$

For all analytic $f\in L^1(\mathbb{D},(1-|z|^2)^\alpha d\sigma)$, $\alpha\ge 0$, the following formula holds

$$(14)\qquad f(a) = (\alpha+1)\int_{\mathbb{D}} \frac{(1-|z|^2)^\alpha}{(1-a\bar{z})^{\alpha+2}} f(z)d\sigma(z)\ , \quad a\in\mathbb{D}\ .$$

Applying (14) with $\alpha=1$ to the functions $f(z)(1-\bar{w}z)^{-1}$ and $(f(z)-f(w))(1-\bar{w}z)^{-2}$ we get the equality

$$\left|\int_{\mathbb{D}} H_{\bar{f}}(k_w(z))\frac{z}{|1-\bar{w}z|^2}d\sigma(z)\right| = \frac{|f'(w)|}{1-|w|^2}\ .$$

Combining this with (13) we see that $|f'(w)|(1-|w|^2)$, $w\in\mathbb{D}$, is bounded.

In view of Lemma 1 the proof of sufficiency can be performed similarly as in Theorem A ([3] p.328).

The bounded Hankel operators on the Bergman one space were characterized by K.R.M. Attele in 1992 [2]. He proved

Theorem C. *For $f \in L^2_a$ the Hankel operator $H_{\bar{f}} : L^1_a \to L^1$ is bounded if and only if*

$$\sup\{|f'(z)|(1-|z|^2)\log\frac{1}{1-|z|^2} \quad : \ z \in \mathbb{D}\} < \infty .$$

3. Compact Hankel operators

For Banach spaces X, Y let $B(X,Y)$ denote the collection of all bounded linear operators of X into Y. Let U be the open unit ball in X. An operator $T \in B(X,Y)$ is said to be compact if the closure of $T(U)$ is compact in Y.
In the case the space X is reflexive we have an alternate discription of compact operators.

Lemma 2 [14,p.107]. *If X is reflexive and $T \in B(X,Y)$ then T is compact if and only if $\|Tx_n - Tx\| \to 0$ whenever $x_n \to x$ weakly in X.*

An operator $T \in B(X^*,Y)$ is called $*$-compact if $\|Tx_n\| \to 0$ for every sequence $\{x_n\}$ which converges to 0 in the weak-star topology on X^*.

For $1 < p < \infty$, $1/p + 1/p' = 1$, the dual of L^p_a can be identified with $L^{p'}_a$ ($(L^p_a)^* \cong L^{p'}_a$). Moreover, under the pairing

$$< f, g >= \lim_{t\to 1^-} \int_{t\mathbb{D}} f\bar{g}\,d\sigma$$

we have

$$(L^1_a)^* \cong \mathcal{B} \quad , \quad (\mathcal{B}_o)^* \cong L^1_a .$$

We have the following theorems, cf. [12].

Theorem 2. *For $f \in L^2_a$ the Hankel operator $H_{\bar{f}} : L^p_a \to L^p$, $1 < p < \infty$, is compact if and only if $f \in \mathcal{B}_o$.*

Theorem 3. *For* $f \in L^2_a$ *the Hankel operator* $H_{\bar f} : L^1_a \to L^1$ *is* $*$*-compact if and only if*

$$(15)\qquad (1-|z|^2)|f'(z)|\log\frac{1}{1-|z|^2} \to 0 \quad as \quad |z| \to 1^- .$$

The next lemma is one of the main tools used in the proof of Theorem 2.

Lemma 3. *Let* $(z,w) \to \phi(z,w)$ *be a bounded measurable complex-valued function on* $\mathbb{D}\times\mathbb{D}$, $w \to \varphi(w)$ *be a continuous function on* $\overline{\mathbb{D}}$ *vanishing on* $\partial\mathbb{D}$. *Then, for* $1 < p < \infty$, *the operator* $S : L^p_a \to L^p_a$ *defined by the formula*

$$(16)\qquad S(h)(z) = \int_{\mathbb{D}} \frac{\phi(z,w)\varphi(w)}{(1-z\bar w)^2}\, h(w)d\sigma(w)\,, \quad z \in \mathbb{D}\,,$$

is compact.

4. Certain operators involving Hankel operators

In this section we shall deal only with the case $p = 2$. For $f, g \in H^\infty$ there are the following relations between commutators of the Toeplitz operators and the Hankel operators ([3], [4])

$$(17)\qquad T^*_f T_f - T_f T^*_f = H^*_{\bar f} H_{\bar f}\,,$$

$$(18)\qquad T^*_f T_g - T_g T^*_f = H^*_{\bar g} H_{\bar f}\,.$$

Hence T_f is essentially commuting for $f \in H^\infty$ if and only if $H_{\bar f}$ is compact. On the other hand, for $f \in L^2_a$, $H_{\bar f}$ is compact if and only if $f \in \mathcal{B}_o$ (so f need not be bounded).

The characterization of the functions $f, g \in H^\infty$ for which the operator $H^*_{\bar g} H_{\bar f}$ is compact was conjectured by S.Axler ([3]) and proved by D.Zheng[16] and independently by S. Axler and P. Gorkin ([6]). It states that the operator $H^*_{\bar g} H_{\bar f}$ is compact if and only if for each Gleason part G of the maximal ideal space of H^∞, except for the disc $\mathbb{D}$, either f or g is constant on G.

In his paper [16] D.Zheng proved that, if $f, g \in \mathcal{B}$ and

$$(19) \qquad \lim_{|z|\to 1} (1-|z|^2)^2 |f'(z)||g'(z)| = 0 \ ,$$

then $H_{\bar{g}}^* H_{\bar{f}}$ is compact. He also proved, using Hoffman's results on Gleason parts, that if $f, g \in H^\infty$ and $H_{\bar{g}}^* H_{\bar{f}}$ is compact then (19) holds.
Now it seems natural to ask if (19) is also a necessary condition for compactness of $H_{\bar{g}}^* H_{\bar{f}}$ for $f, g \in \mathcal{B}$. Theorem 4 states that it is if $f, g \in$ BMOA.

Let $\mathrm{Aut}(\mathbb{D})$ denote the set of analytic, one–to–one maps of $\mathbb{D}$ onto $\mathbb{D}$. In the proof of Theorem 4 we will use inclusion (5) and the fact that an analytic function f on $\mathbb{D}$ is a Bloch function if and only if it is is finitely normal i.e. the functions

$$(20) \qquad f_*(z) = f(h(z)) - f(h(0)) \ , \quad h \in \mathrm{Aut}(\mathbb{D})$$

form a normal family where the constant ∞ is not allowed as a limit, cf. [13].

For a continuous function u on $\mathbb{D}$ let $\mathcal{R}(u)$ denote the radialization of the function u, that is

$$(21) \qquad \mathcal{R}u(z) = \int_0^{2\pi} u(ze^{i\theta})\, d\theta/2\pi \ .$$

We will need the following lemma due to S. Axler and Ž.Čučković [5].

Lemma 4. *Suppose that* $u \in C(\mathbb{D}) \cap L^1(, d\sigma)$ *. Then* u *is harmonic on* $\mathbb{D}$ *if and only if*

$$(22) \qquad \int_{\mathbb{D}} u \circ h\, d\sigma = u(h(0))$$

and

$$\mathcal{R}(u \circ h) \in C(\bar{\mathbb{D}}) \quad \textit{for every} \quad h \in \mathrm{Aut}(\mathbb{D}) \ .$$

Proof of Theorem 4. Suppose the assertion of the theorem is false. Then we could find such a sequence $\{\lambda_n\}$ in $\mathbb{D}$ that $|\lambda_n| \to 1$ and

$$(23) \qquad \lim_{n\to\infty} (1-|\lambda_n|^2)^2 |f'(\lambda_n)||g'(\lambda_n)| = a > 0 \ .$$

For $\lambda \in \mathbb{D}$ let h_λ, φ_λ denote the functions defined by

$$(24) \qquad h_\lambda(z) = \frac{1-|\lambda|^2}{(1+\bar{\lambda}z)^2}\,, \quad z \in \mathbb{D}$$

and

$$(25) \qquad \varphi_\lambda(z) = \frac{\lambda + z}{1+\bar{\lambda}z}\,, \quad z \in \mathbb{D}\,.$$

The sequence $\{h_{\lambda_n}\}$ is weakly convergent to 0 in the space L^2_a. Hence, in view of Lemma 2,

$$(26) \qquad \begin{aligned} \lim_{n\to\infty} &< H_{\bar{f}}^* H_{\bar{g}} h_{\lambda_n}, h_{\lambda_n} >= \lim_{n\to\infty} < H_{\bar{g}} h_{\lambda_n}, H_{\bar{f}} h_{\lambda_n} > \\ &= \lim_{n\to\infty} \int_{\mathbb{D}} (f \circ \varphi_{\lambda_n} - f(\lambda_n))(\bar{g} \circ \varphi_{\lambda_n} - \bar{g}(\lambda_n))d\sigma = 0\,. \end{aligned}$$

Since f, g are finitely normal there exist a subsequence of $\{\lambda_n\}$, which we will also denote by $\{\lambda_n\}$, and analytic functions F, G on $\mathbb{D}$ such that

$$f \circ \varphi_{\lambda_n} - f(\lambda_n) \underset{n\to\infty}{\longrightarrow} F \quad \text{and} \quad g \circ \varphi_{\lambda_n} - g(\lambda_n) \underset{n\to\infty}{\longrightarrow} G$$

uniformly on compact subsets of $\mathbb{D}$.
We will show that the function $F\bar{G}$ is harmonic on $\mathbb{D}$.
By (26)

$$(27) \qquad \int_{\mathbb{D}} F(z)\bar{G}(z)d\sigma(z) = 0\,.$$

Now let $w \in \mathbb{D}$ be fixed. Then also

$$f \circ \varphi_{\lambda_n} \circ \varphi_w - f(\lambda_n) \underset{n\to\infty}{\longrightarrow} F \circ \varphi_w \text{ and } g \circ \varphi_{\lambda_n} \circ \varphi_w - g(\lambda_n) \underset{n\to\infty}{\longrightarrow} G \circ \varphi_w$$

uniformly on compact subsets of $\mathbb{D}$.
Let $\lambda'_n = \varphi_{\lambda_n}(w)$ and $\alpha_n = \arg(1 + \lambda_n \bar{w})$. Then

$$(28) \qquad \varphi_{\lambda'_n}(z) = \varphi_{\lambda_n} \circ \varphi_w(z e^{-2i\alpha_n})\,, \quad z \in \mathbb{D}\,.$$

Indeed we have

$$\begin{aligned}(29)\qquad \varphi_{\lambda_n}\circ\varphi_w(ze^{-2i\alpha_n}) &= \frac{\lambda_n+\frac{w+ze^{-2i\alpha_n}}{1+\bar{w}ze^{-2i\alpha_n}}}{1+\bar{\lambda}_n\frac{w+ze^{-2i\alpha_n}}{1+\bar{w}ze^{-2i\alpha_n}}}\\ &= \frac{1+\bar{w}\lambda_n}{1+w\bar{\lambda}_n}\,\frac{ze^{-2i\alpha_n}+\frac{\lambda_n+w}{1+\bar{w}\lambda_n}}{1+ze^{-2i\alpha_n}\frac{\bar{\lambda}_n+\bar{w}}{1+\bar{\lambda}_n w}} = \varphi_{\lambda'_n}(z).\end{aligned}$$

Since $|\lambda'_n|\to 1$ as $n\to\infty$, by Lemma 2 again

$$(30)\qquad \lim_{n\to\infty}\int_{\mathbb{D}}(f\circ\varphi_{\lambda'_n}-f(\lambda'_n))(\bar{g}\circ\varphi_{\lambda'_n}-\bar{g}(\lambda'_n))d\sigma = 0\,.$$

Moreover, by (28) and (29) the left hand side of (30) is equal to

$$\begin{aligned}&\lim_{n\to\infty}\int_{\mathbb{D}}(f\circ\varphi_{\lambda_n}\circ\varphi_w(ze^{-2i\alpha_n})-f(\lambda'_n))(\bar{g}\circ\varphi_{\lambda_n}\circ\varphi_w(ze^{-2i\alpha_n})\\ &\qquad -\bar{g}(\lambda'_n))d\sigma(z)\\ &= \lim_{n\to\infty}\int_{\mathbb{D}}(f\circ\varphi_{\lambda_n}\circ\varphi_w(z)-f(\lambda_n)+f(\lambda_n)-f(\lambda'_n))\\ &\qquad\qquad (\bar{g}\circ\varphi_{\lambda_n}\circ\varphi_w(z)-\bar{g}(\lambda_n)+\bar{g}(\lambda_n)-\bar{g}(\lambda'_n))d\sigma(z)\\ &= \int_{\mathbb{D}}(F\circ\varphi_w(z)-F\circ\varphi_w(0))(\bar{G}\circ\varphi_w(z)-\bar{G}\circ\varphi_w(0))d\sigma(z)\,.\end{aligned}$$

It follows that

$$(31)\qquad \int_{\mathbb{D}}F\circ\varphi_w(z)\bar{G}\circ\varphi_w(z)\sigma(z) = F\circ\varphi_w(0)\bar{G}\circ\varphi_w(0)\ ,\quad w\in\mathbb{D}\,.$$

It means that the function $F\bar{G}$ satisfies the area version of the invariant mean value property.

To prove that $F\bar{G}$ is harmonic it is enough, in view of Lemma 4, to show that for arbitrarily fixed $w\in\mathbb{D}$ the radialization $\mathcal{R}(F\circ\varphi_w\bar{G}\circ\varphi_w)$ extends continuously to $\bar{\mathbb{D}}$.

First we show that $F\circ\varphi_w$ and $G\circ\varphi_w$ are in H^2 .
It follows from the above considerations that

$$(32)\qquad F\circ\varphi_w(z)-F\circ\varphi_w(0) = \lim_{n\to\infty}(f\circ\varphi_{\lambda'_n}(ze^{2i\alpha_n})-f(\lambda'_n))\,.$$

Since $f \in$ BMOA, there is a common bound for $\|f\circ\varphi_{\lambda'_n} - f(\lambda'_n)\|_{H^2}$ ((4)). In this case, the function which is the limit of the sequence $\{f\circ\varphi_{\lambda'_n} - f(\lambda'_n)\}$ in the sense of uniform convergence on compacta is in H^2. To see this notice e.g. that for h analytic on $\mathbb{D}$ $\|h\|_{H^2}$ is equivalent to the quantity

$$\int_{\mathbb{D}} (1-|z|)|h'(z)|^2\, d\sigma(z)\ .$$

Thus $F\circ\varphi_w - F\circ\varphi_w(0) \in H^2$ which implies $F\circ\varphi_w \in H^2$. Clearly the same is true for $G\circ\varphi_w$. Now it follows that $\mathcal{R}(F\circ\varphi_w \bar{G}\circ\varphi_w) \in C(\bar{\mathbb{D}})$ (see [5, p. 9]).Hence $F\bar{G}$ is harmonic and $\partial^2(F\bar{G})/\partial\bar{z}\partial z = F'(z)\bar{G}'(z) = 0\, , z \in \mathbb{D}$. In particular $F'(0)\bar{G}'(0) = 0$. Thus

$$0 = F'(0)\bar{G}'(0) = \lim_{n\to\infty}(1-|\lambda_n|^2)^2 f'(\lambda_n)\bar{g}'(\lambda_n) \tag{33}$$

which contradicts (23).

Final remarks

Recently K. Stroethoeff [15] characterized the bounded harmonic functions f, g on $\mathbb{D}$ for which Toeplitz operators T_f and T_g are essentially commuting. He also obtained the following

Corollary . *Let $0 < r < 1$ be arbitarily fixed, $D(\lambda, r)$ denote the pseudohyperbolic disk $\{z \in \mathbb{D} : |\lambda - z|/|1-\bar{\lambda}z| < r\}$ and let f and g belong to H^∞ . Then the following statements are equivalent:*

(i) $H^*_{\bar{g}} H_{\bar{f}} - H^*_{\bar{f}} H_{\bar{g}}$ *is compact ;*

(ii) $\lim_{|\lambda|\to 1^-} \int_{r\mathbb{D}} |(\bar{f}\circ\varphi_\lambda - \bar{f}(\lambda))(g\circ\varphi_\lambda - g(\lambda)) - (\bar{g}\circ\varphi_\lambda - \bar{g}(\lambda))(f\circ\varphi_\lambda - f(\lambda))| = 0$;

(iii) $\lim_{|\lambda|\to 1^-} \int_{\mathbb{D}} |(\bar{f}\circ\varphi_\lambda - \bar{f}(\lambda))(g\circ\varphi_\lambda - g(\lambda)) - (\bar{g}\circ\varphi_\lambda - \bar{g}(\lambda))(f\circ\varphi_\lambda - f(\lambda))| = 0$;

(iv) $\lim_{|\lambda|\to 1^-} \int_{D(\lambda,r)} |\bar{f}'g' - f'\bar{g}'|\, d\sigma = 0$;

(v) $\lim_{|\lambda|\to 1^-} \frac{1}{|D(\lambda,r)|}\int_{D(\lambda,r)} |(\bar{f}\circ\varphi_\lambda - \bar{f}(\lambda))(g\circ\varphi_\lambda - g(\lambda)) - (\bar{g}\circ\varphi_\lambda - \bar{g}(\lambda))(f\circ\varphi_\lambda - f(\lambda))| = 0$;

(vi) $\lim_{|\lambda|\to 1^-}(1-|\lambda|^2)^2\{\bar{f}'(\lambda)g'(\lambda) - f'(\lambda)\bar{g}'(\lambda)\} = 0$.

In his paper K.Stroethoff asked if the above corollary holds for Bloch functions instead of H^∞ functions. Using the method of the proof of Theorem 4 and some results from [15] we easily show this corollary remains true for $f, g \in$ BMOA.

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