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Univalent Harmonic Mappings in the Plane

Abstract. Lately, G.Schober [S1] and P.Duren [D1] have written excellent surveys of univalent planar harmonic mappings. We give here an update of this theory.

1. Introduction

Let D be a domain of the extended complex plane $\overline{\mathbb{C}}$. A harmonic mapping is a complex-valued function $w = f(z) = u(z)+iv(z)$ which satisfies $f_{z\bar{z}} \equiv 0$ on D, i.e., u and v are real-valued harmonic functions on D. Observe that, in contrast to other authors, we do not require f to be univalent on D. For instance, any analytic or anti-analytic function is a harmonic mapping. Since u and v are real parts of locally analytic functions defined on D, it follows that f admits the representation

$$f(z) = h(z) + \overline{g(z)} \tag{1}$$

where h and g are locally analytic on D. For example, $f(z) = z - 1/\bar{z} + 2\ln|z|$ is a univalent harmonic mapping from the exterior of the

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unit disk U onto $\mathbb{C}\setminus\{0\}$ where $h(z) = z + \log z$ and $g(z) = \log z - 1/z$. If D is a simply connected domain of $\mathbb{C}$, then h and g are (globally) analytic functions on D. On the other hand,

$$h' = f_z = (\partial f/\partial x - i\partial f/\partial y)/2$$

and

$$\overline{g'} = f_{\overline{z}} = (\partial f/\partial x + i\partial f/\partial y)/2$$

are always (globally) analytic functions on D. In contrast to the linear space $H(D)$ of analytic functions, the product and the composition of two harmonic mappings are in general not harmonic. Furthermore, neither the reciprocal $1/f$ nor the inverse f^{-1} (whenever they exist) of a harmonic mapping f is in general harmonic. However, the composition of a harmonic mapping with a conformal premapping is a harmonic map. Moreover, an affine transformation applied to a harmonic map is also harmonic. E.Reich [R2], [R3] has given a complete description of the harmonic mappings f and g with the property that $g \circ f$ is also harmonic. In particular, as a special case, he obtains the following Choquet-Deny Theorem [C1]:

Theorem 1.1. *Suppose f is a sense-preserving harmonic homeomorphism and is neither analytic nor affine. Then f^{-1} is also harmonic if and only if*

$$f(z) = D + Az + B\log\frac{C - e^{-Az/B}}{\overline{C - e^{-Az/B}}}$$

where A, B, C and D are non-zero complex constants and $|C| > \sup_z |e^{-Az/B}|$.

Since the absolute value $|f|$ of a harmonic map f is subharmonic, it follows that f satisfies the maximum modulus principle. Furthermore, if f is not a constant, we conclude from (1) that the inverse image of a point is a union of points and analytic arcs. We say that a continuous map f is light, if the image of each continuum is a continuum. There are harmonic mappings which are not light. For instance, $z + \overline{z}$ and $z + \overline{z} - z^2 + \overline{z^2}$ map the imaginary axis onto the origin and $z - 1/\overline{z}$ maps the whole unit circle ∂U onto the origin. Let

p and q be two analytic polynomials of degree n and m. If $n \neq m$, then the harmonic polynomial $P = p + \overline{q}$ is light. A complete characterization of the local behaviour of a light harmonic mapping has been given by A. Lyzzaik in [L3]. It contains, as a special case, the following result due to J.Lewy [L2]

Theorem 1.2. *A harmonic mapping is locally univalent in a neighbourhood of a point z_0 if and only if its Jacobian $J_f(z)$ does not vanish at z_0.*

Remarks 1.1.

1.1.1 Theorem 1.2 fails to be true for harmonic mappings in higher dimensions. The following counter-example is due to J.C. Wood [W2]. Define

$$u(x,y,z) = x^3 - 3xz^2 + yz, \quad v(x,y,z) = y - 3xz,$$
$$w(x,y,z) = z.$$

Then the mapping is univalent on $\mathbb{R}^3$ but the Jacobian vanishes on the plane $x = 0$.

1.1.2 Starkov [S8] has studied in details the behaviour of locally univalent harmonic maps.

Open problem 1.1. Does Theorem 1.2 hold for complex-valued pluriharmonic mappings ?

Suppose that f is a univalent harmonic mapping defined on D. Then, either f is sense-preserving or sense-reversing. In the first case, the Jacobian

$$J_f(z) = |f_z|^2 - |f_{\overline{z}}|^2 = |h'|^2 - |g'|^2 \tag{2}$$

is strictly positive on D. If the second case holds, then $\overline{f}$ is sense-preserving. Suppose that f is a univalent harmonic sense-preserving mapping. The second dilatation function $a(z)$ of f is defined by

$$a(z) = g'(z)/h'(z) = \overline{f_{\overline{z}}(z)}/f_z(z) \tag{3}$$

which is an analytic function on D and satisfies $|a(z)| < 1$ for all $z \in D$.

More generally, we have

Theorem 1.3. *A non-constant complex-valued function f is a harmonic and sense-preserving mapping on D if and only if f is a solution of the elliptic partial differential equation*

$$\overline{f_{\bar{z}}(z)} = a(z) f_z(z), \quad a \in H(U), \quad |a| < 1 \tag{4}$$

on D.

Proof. The necessity follows directly from relation (3). Since $|a| < 1$ whenever $h' \neq 0$, the zeros of h' are removable singularities for a or else $a \equiv 0$. On the other hand, if $a \in H(U)$ and $|a| < 1$, the sufficiency follows from

$$\overline{f}_{\bar{z}z} = (\overline{f_{\bar{z}}})_z = (\overline{f_{\bar{z}}})_{\bar{z}} = (af_z)_{\bar{z}} = af_{z\bar{z}}$$

which implies that $f_{z\bar{z}} \equiv 0$ on U. □

Another interpretation of (4) is a generalized Cauchy–Riemann equation. Indeed, using real notation, equation (4) is equivalent to

$$\begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} -\mathrm{Im}\{p\} & \mathrm{Re}\{p\} \\ -\mathrm{Re}\{p\} & -\mathrm{Im}\{p\} \end{pmatrix} \cdot \begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

where p is the analytic function $(1+a)/(1-a)$. Observe that $\mathrm{Re}\{p\} > 0$ on U and that p belongs to the Toeplitz class $\mathcal{P}$ if $a(0) = 0$. One easily recognizes the Cauchy-Riemann equation if $a \equiv 0$.

It follows that univalent sense-preserving harmonic mappings are locally quasiconformal. Observe that we allow that $|a(z)|$ approaches 1 as z approaches the boundary ∂D of D. Therefore, univalent harmonic mappings do not have the same boundary behaviour as of quasiconformal mappings. The following two examples show the difference.

Example 1.1. The mappings $f(z) = z - 1/\bar{z} + C \ln |z|$, $|C| \leq 2$, are univalent sense–preserving and harmonic on the exterior Δ of the unit disk U and we have $f(\Delta) = \mathbb{C} \setminus \{0\}$. The whole unit circle is mapped onto the origin. For more details, see [HS3].

Example 1.2. The Carathéodory kernel theorem does not hold. The mappings

$$f_n(z) = \frac{1}{2}\log\frac{1+z}{1-z} + 2\mathrm{Re}\int \frac{(n-1)iz}{(n-(n-1)iz)(1-z^2)}dz$$

are univalent and harmonic on U and $f_n(U)$ is the horizontal strip $\Omega = \{w : |\mathrm{Im}\ w| < \pi/4\}$. Furthermore, the sequence converges locally uniformly to the univalent harmonic mapping

$$f(z) = \frac{1}{2}\log\frac{1+z}{1-z} + 2\mathrm{Re}\int \frac{iz}{(1-iz)(1-z^2)}dz.$$

The image $f(U)$ is the triangle with vertices $\pi/2+i\pi/4$, $-\pi/2+i\pi/4$ and $-i\pi/4$ which is not the kernel Ω. For more details, see [HS3].

In general, we have

Theorem 1.4. *The limit function f of a locally uniformly convergent sequence of univalent harmonic mappings f_n on D is either univalent on D, is a constant, or its image lies on a straight line.*

Proof. Suppose that f is not a constant. Since the mappings f_n are univalent on D, their Jacobians J_{f_n} do not vanish on D. We may assume that they are positive. Hence, the second dilatation functions a_n defined in (3) satisfy $|a_n| < 1$ on D. Moreover, it was shown in [HS3] that the sequence a_n converges locally uniformly to the dilatation function a of the limit function f. If $|a| < 1$ on D, then f is univalent (locally quasiconformal) on D. In the other case, we have $a(z) \equiv e^{i\beta}$ for some real β which implies by (3) that $g' \equiv e^{i\beta}h'$ i.e., $g \equiv e^{i\beta}h + \text{const}$. Therefore, we have $f(z) = 2e^{-i\beta/2}\mathrm{Re}\{h(z)e^{i\beta/2}\} + \text{const}.$. □

Let f be a sense-preserving harmonic mapping on D. Then f is locally of the form

$$f(z) = f(z_0) + A(z-z_0)^n + \overline{B(z-z_0)^n} + o(|z-z_0|^n);\ |B| < |A|,\ n \in \mathbf{N}. \tag{5}$$

If $f(z_0) = 0$, then we say that f has a zero of order n at z_0. It follows then that the argument principle holds on any cycle in D. It can be applied in order to get uniqueness results as for example in [BHH1] and [BHH2], or to prove univalence of certain mappings (see e.g., [BHH1] and [HS4]). J. Clunie and T. Sheil-Small [CS1] used the argument principle to show the following result:

Theorem 1.5. *Let f_n be a sequence of univalent harmonic mappings defined on D such that $f_n(z_0) = 0$ for some $z_0 \in D$ and suppose that they converge locally uniformly to f. Then $f(D)$ lies in the kernel of $\{f_n(D)\}$.*

In Section 2, we shall give a survey on univalent harmonic mappings defined on a simply connected domain D of $\mathbb{C}$. Section 3 deals with univalent harmonic mappings defined on multiply connected domains.

Recently several excellent survey articles have been written on harmonic mappings between Riemannian manifolds. For example, [EL1], [ES1], [J1], [J2], [S2] and [S3] are some of these.

2. Univalent harmonic mappings on a simply connected domain

2.1 Motivation

Univalent harmonic mappings are closely related to minimal surfaces. Let Ω be a domain in the (u, v) - plane and let S be a non-parametric surface over Ω. In other words, we suppose that the surface can be expressed by the function $s = s(u, v)$. Then the folowing characterisation holds:

Theorem 2.1. *A non-parametric surface S is a minimal surface if and only if there is a univalent harmonic mapping $f = u + iv$ from a domain D onto Ω such that $s_z^2 = -af_z^2 = -\overline{f_{\bar{z}}}f_z$ holds where a is defined in* (4).

It is interesting to note that the normal vector $\overrightarrow{n}$ of the surface S, called the Gauss map, depends only on the second dilatation function

a. Indeed, we have

(6) $$\vec{n} = (\mathrm{Im}\{\sqrt{a}\}, \mathrm{Re}\{\sqrt{a}\}, 1 - |a|)/(1 + |a|)\,.$$

Observe that $\vec{n}$ is vertical if and only if $a = 0$ and it is horizontal if and only if $|a| = 1$.

2.2 Univalent harmonic mappings defined on the plane

There are very few harmonic mappings which are univalent on $\mathbb{C}$. Indeed, J. Clunie and T. Sheil-Small have shown in [CS1]:

Theorem 2.2. *The only univalent harmonic mapping defined on the plane is the affine transformation*

(7) $$f(z) = Az + \overline{Bz} + C, \quad |A| \neq |B|.$$

The proof is based on the fact that the second dilatation function a of f is constant on $\mathbb{C}$ and we have $|a| \neq 1$. It follows then that $\Phi = f - \overline{af}$ is a univalent analytic function on $\mathbb{C}$. Hence, we have $\Phi = cz + d$ and Theorem 2.2 follows.

Theorem 2.2 says that there are no univalent mappings from the plane onto a proper subdomain of $\mathbb{C}$. Since, in general, the inverse of a univalent harmonic mapping is not harmonic, it is natural to ask if there are other univalent harmonic mappings whose image is $\mathbb{C}$. The answer is **no** [CS1] and we shall give a new proof for it after we show the following lemma .

Lemma 2.1. *Let $f = h + \overline{g}$ be a univalent harmonic and sense-preserving mapping from a domain D onto the domain Ω. Suppose that z_1 and z_2, $z_1 \neq z_2$, are two points in D such that the line segment $\gamma = \{w_t = tf(z_1) + (1-t)f(z_2) :\ 0 \leq t \leq 1\}$ belongs to Ω. Then we have*

(8) $$|h(z_2) - h(z_1)| > |g(z_2) - g(z_1)|.$$

Proof. The proof is essentially due to J. Clunie and T.Sheil-Small [CS1]. Let Ω_1 be a convex subdomain of Ω containing the line segment γ. Define $D_1 = f^{-1}(\Omega_1)$ and consider the mapping

$$\phi_\alpha(z) = e^{i\alpha}h(z) - e^{-i\alpha}g(z). \tag{9}$$

Then ϕ_α is a conformal mapping from D_1 onto a domain G_α which is convex in the horizontal direction. Therefore ϕ'_α does not vanish on D_1 and we have

$$h(z_2) - h(z_1) \neq e^{-2i\alpha}(g(z_2) - g(z_1))$$

for all $\alpha \in [0, 2\pi)$. Therefore we have $|h(z_2) - h(z_1)| \neq |g(z_2) - g(z_1)|$.

Let $z_t \in D$ be defined by $f(z_t) = w_t = tf(z_1) + (1-t)f(z_2)$; $0 \leq t \leq 1$. Then we get

$$\left|\frac{h(z_t) - h(z_1)}{z_t - z_1}\right| \neq \left|\frac{g(z_t) - g(z_1)}{z_t - z_1}\right|$$

for all $t \in (0,1)$. Passing to the limit $t \to 0$, we have $|h'(z_1)| > |g'(z_1)|$ since f is sense–preserving. Lemma 2.1 follows by a continuity argument. $\square$.

Theorem 2.3. *The only univalent harmonic mappings f satisfying $f(D) = \mathbb{C}$ are of the form* (7).

Proof. First, observe that the domain D is simply connected. By applying a conformal premapping from the unit disk U onto D, we may assume, without loss of generality, that $D = U$ and that $f(0) = h(0) = g(0) = 0$. Furthermore, we may assume that f is sense–preserving; indeed, if not, consider $\overline{f}$. Next, we have $\lim_{|z|\to 1} f(z) = \infty$.

By Lemma 2.1, we have $|f| \leq |h| + |g| < 2|h|$ which implies that $\lim_{r\to 1} |h(re^{it})| \equiv \infty$. But no such analytic function exists and Theorem 2.3 follows from Theorem 2.2. $\square$

Open problem 2.1. H.S. Shapiro posed in [S4] the following question: Is there a homeomorphism from the unit ball in $\mathbb{R}^3$ onto $\mathbb{R}^3$ whose coordinate functions are harmonic ?

Remarks 2.1.

2.1.1 J. Clunie and T. Sheil—Small [CS1] have shown that if f is a univalent sense–preserving harmonic mapping defined on the unit disk U, then each circle $\{w : |w - f(0)| = r|f_z(0)|\}$, $r \geq 2\pi\sqrt{6}/9$, contains at least one point of $\mathbb{C} \setminus f(U)$. The constant $2\pi\sqrt{6}/9 \approx 1.710$ is best possible.

2.1.2 We have given a new proof of the famous Bernstein's theorem which says that the only minimal surfaces over the whole plane are planes. Indeed, we have shown, that $D = \mathbb{C}$ and $a(z)$ is a constant. Bernstein's theorem follows now from the relation (6).

2.3 The classes S_H and S_H^0

Let D be a proper simply connected domain of $\mathbb{C}$ and f a univalent harmonic mapping from D to $\mathbb{C}$. Since the composition of a univalent harmonic mapping with a conformal premapping is a univalent harmonic map, we may assume that D is the unit disk U and that f is sense–preserving on U. Furthermore, since f_z does not vanish on U (Theorem1.2), we may normalize f by the transformation $(f(z) - f(0))/f_z(0)$. Then f admits the unique representation

$$f = h + \overline{g} = z + \sum_{k=2}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k}. \tag{10}$$

Observe that $b_1 = a(0)$.

Definition 2.1. The class S_H consists of all univalent harmonic and sense–preserving mappings $f = h + \overline{g}$ which are normalized by $g(0) = h(0) = 0$ and $f_z(0) = 1$.

Applying the affine postmapping $(w - \overline{a(0)w})/(1 - |a(0)|^2)$ to f we can transform f to a function whose dilatation function vanishes at the origin.

Definition 2.2. The class S_H^0 consists of all mappings $f = h + \overline{g} \in S_H$ such that $f_{\overline{z}}(0) = 0$.

Remark 2.2. The condition $f_{\overline{z}}(0) = 0$ is equivalent to the condition $a(0) = 0$, or to $g(z) = O(z^2)$ as $z \to 0$.

Since mappings in S_H^0 are K_r–quasiconformal on the disks $\{z : |z| \leq r\}$, $0 < r < 1$, where $K_r = (1+r)/(1-r)$, it follows that S_H^0 is compact with respect to the topology of locally uniform convergence. Furthermore, we have

$$\max_{f \in S_H} \max_{|z| \leq r} |f(z)| \leq 2 \max_{f \in S_H^0} \max_{|z| \leq r} |f(z)|$$

which shows that S_H is a normal family. Note that S_H is not compact. Indeed, the affine transformations $f_n(z) = z + n/(n+1)\overline{z}$ belong to the class S_H and the sequence converges locally uniformly to $f(z) = z + \overline{z}$ which is nowhere univalent. The following interesting distortion theorem is due to J. Clunie and T. Sheil–Small [CS1].

Theorem 2.4. *If $f \in S_H^0$, then $|f(z)| \geq |z/\{4(1-z)^2\}|$. In particular, we have $\{w : |w| < 1/16\} \subset f(U)$.*

Open problem 2.2. It is not known if the above estimate is sharp. There are some indications that perhaps the factor 1/4 can be replaced by 2/3. A possible candidate for the extremal function is the radial slit-mapping

$$f(z) = \frac{z - z^2/2 + z^3/6}{(1-z)^3} + \overline{\frac{z^2/2 + z^3/6}{(1-z)^3}}. \tag{11}$$

whose dilatation function is $a(z) = z$.

Remarks 2.3.

2.3.1 Let L be a linear continuous functional on the set $h(U)$ of all harmonic mappings defined on U. Then we have

$$L(f) = L(h + \overline{g}) = L_1(h) + \overline{L_2(g)}$$

where L_1 and L_2 belong to $H'(U)$, the topological dual space of $H(U)$.

2.3.2 Since S_H^0 is compact, each real continuous functional attains its maximum and its minimum on S_H^0. Hence, there are uniform bounds for the absolute value of the coefficients a_n and b_n in (10). Applying Schwarz's Lemma to the dilatation $a(z)$, one gets immediately the sharp inequality $|b_2| \leq 1/2$. The best known estimate for a_2 is so far $|a_2| \leq 49$. There is a conjecture that $|a_n| \leq (2n+1)(n+1)/6$ and $|b_n| \leq (2n-1)(n-1)/6$ and that equality is attained by the mapping given in (11). Another attractive conjecture is that $|a_n| - |b_n| \leq n$ holds for all $f \in S_H^0$. For further investigations see e.g. [CS1].

Y. Abu-Muhanna and A. Lyzzaik [AL1] have shown the following interesting result

Theorem 2.5. *Let $f = h + \overline{g}$ be a univalent harmonic mapping defined on the unit disk U. Then there is a universal $p > 0$ such that f belongs to the standard class h^p and that h and g belong to H^p.*

Open problem 2.3. Using the estimate $|a_2| \leq 49$ and following the arguments given by Y.Abu-Muhanna and A. Lyzzaik, we conclude that $f \in h^p$ for all $p \in (0, 10^{-4})$. Find the exact range for p.

2.4 Univalent harmonic mappings onto convex domains

2.4.1 The Rado–Kneser(–Choquet) theorem. In 1926, T. Radó [R1] asked to prove the following result:

Theorem 2.6. *Let f^* be a homeomorphism from the unit circle ∂U onto the boundary of a bounded convex domain Ω. Then the solution $f = u + iv$ of the Dirichlet problem $f_{z\overline{z}} \equiv 0$ on U and $f \equiv f^*$ on ∂U (the Poisson integral) is univalent on U.*

The same year, H. Kneser has shown in [K1] a much stronger result. Since his proof is not everywhere accessible, we reproduce it here.

Theorem 2.7. *Let f^* be a homeomorphism from the unit circle ∂U onto the boundary of a bounded Jordan domain Ω. Then the solution $f = u + iv$ of the Dirichlet problem $f_{z\overline{z}} \equiv 0$ on U and $f \equiv f^*$ on ∂U (the Poisson integral) is univalent on U if and only if $f(U) = \Omega$.*

Proof ([K1]). Observe that $\Omega \subset f(U)$ is a topological property which holds for all continuous extensions of f^*.

(a) Suppose that $f(U) \neq \Omega$. Then there is a $z_1 \in U$ and a $z_2 \in U$ such that $f(z_1) \in \Omega$ and $f(z_2) \notin \Omega$. If $|a| - 1$ changes the sign in U, then f is not univalent on U (see Theorem 1.2). Hence, suppose that $|a| \neq 1$ on U. Since f is not a constant, it follows that f is an open mapping. Let $\gamma = \{z(t) : 0 \leq t \leq 1\}$ be an arc in D from z_1 to z_2. Define $z_\tau = \inf\{t : z(t) \notin \Omega\}$. Then $f(z_\tau) = f(\zeta)$ for some $\zeta \in \partial\Omega$ which implies that there is a neighbourhood of $f(z_\tau)$ whose preimage consists of at least two components. This contradicts the univalence of f.

(b) Suppose now that $f(U) = \Omega$ and that the Jacobian $J_f(z))$ of f vanishes at a point $z_0 \in D$. Then the linear system $cu_x + dv_x = 0, cu_y + dv_y = 0$ admits a non-trivial solution (c, d). Define $\Psi = cu + dv$ and let Γ be the line segment in Ω passing through the point $w_0 = f(z_0)$. Denote the end-points of Γ by P and Q. Since f^* is a homeomorphism from the unit circle Ψ onto $\partial\Omega$, Ψ is not constant on D. Therefore, the preimage of $\gamma = f^{-1}(\Gamma)$ splits at z_0 in an even number of branches. Each branch may split again at other points; but they have to end at $f^{-1}(P)$ or $f^{-1}(Q)$. Hence, there is an open subset G of D such that Ψ restricted to ∂G is constant. This implies that Ψ is constant on G and hence on D which leads to a contradiction. So far, we have shown that f is locally univalent in D. If f is sense-preserving, then the argument principle shows that f is univalent. If f is sense-reversing, consider $\overline{f^*}$. □

Theorem 2.7 implies Theorem 2.6. Indeed, we have $f = \int f^* d\omega$, where the harmonic measure $d\omega$ is a probability measure. Therefore we have $\Omega \subset f(U) \subset \overline{\text{co}\Omega}$ If Ω is a bounded convex domain, then we conclude that $f(U) = \Omega$ and Theorem 2.6 follows.

Remarks 2.4.

2.4.1 In 1945, G. Choquet gave in [C1] another proof for Theorem 2.6, using the Poisson integral. One may also use the following arguments which were introduced by J. Clunie and T. Sheil-Small in [CS1]. Let $f = h + \overline{g}$ and define ϕ_α as in (9), i.e., $\phi_\alpha(z) = e^{i\alpha}h(z) - e^{-i\alpha}g(z)$. Then ϕ_α is a pointwise horizontal translation of $e^{i\alpha}f$. In other words, we have $\phi_\alpha(z) = e^{i\alpha}f(z) - 2\text{Re}\{e^{-i\alpha}g(z)\}$. The mappings ϕ_α are convex in the horizontal direction and hence, conformal for all real α. Therefore, ϕ'_α does not vanish on U which implies that the Jacobian J_f does not vanish on U.

2.4.2 Theorem 2.6 is false if Ω is not convex. This was already observed by G. Choquet [C1].

2.4.3 Theorem 2.6 does not hold if Ω is an unbounded convex domain.

2.4.4 An extension of Theorem 2.6 and Theorem 2.7 to multiply connected domains will be given in Section 3.

2.4.5 Theorem 2.6 and Theorem 2.7 do not hold in $\mathbb{R}^n, n \geq 3$. R. Laugesen gave an example of a homeomorphism $f^* = (f_1^*, f_2^*, f_3^*)$ from the unit sphere of $\mathbb{R}^3$ onto itself such that the Poisson integral $f = (f_1, f_2, f_3)$ maps the unit ball onto itself, but is not a univalent harmonic mapping.

2.4.6 A conformal mapping from U onto itself is uniquely determined by the correspondence of three boundary points. Theorem 2.6 shows that there are many univalent harmonic mappings from U onto U.

2.4.7 P.Duren and G.Schober, cf. [DS1] and [DS2], used Theorem 2.6 to develop a variation for univalent harmonic mappings from the unit disk U onto a fixed convex domain Ω. In particular, they gave for the case $\Omega = U$, sharp estimates for the coefficients and the distortion of the partial derivatives. A somewhat different approach is due to R.Wegmann [W1].

Definition 2.3. Let Ω be a simply connected Jordan domain of $\mathbb{C}$ and let Φ be a conformal mapping from U onto Ω.. A function f^* from ∂U into $\partial\Omega$ is called a **weak homeomorphism** from ∂U into $\partial\Omega$ if f^* is the pointwise limit of a sequence of homeomorphisms from ∂U onto $\partial\Omega$. In other words, f^* is a weak homeomorphism on ∂U if and only if $\psi(t) = \arg\Phi^{-1} \circ f^*(e^{it})$ (which exists a.e. on ∂U) is non-decreasing on $[0, 2\pi]$ and satisfies $\psi(2\pi) = \psi(0) + 2\pi$.

Note that a weak homeomorphism can be constant on an interval of ∂U and may have jumps; but it never can change the orientation. It follows immediately that Theorem 2.6. holds also for continuous weak homeomorphisms on ∂U. On the other hand, Theorem 2.7 holds true for f^*, a weak homeomorphism from ∂U into $\partial\Omega$, if its range consists of at least three different points.

Open problem 2.4. Does Theorem 2.7 hold for complex-valued pluriharmonic mappings ?

2.4.2 The class $\mathbf{K_H}$

Definition 2.4. A harmonic mapping f defined on the unit disk U belongs to the class K_H (K_H^0 resp.) if $f \in S_H$ ($f \in S_H^0$ resp.) and if $\Omega = f(U)$ is a convex domain.

Using the fact that the associated functions ϕ_α defined in (9) are univalent mappings onto domains convex in the horizontal direction if and only if f is univalent and $f(U)$ is a convex domain, J.Clunie and T.Sheil-Small[CS1] gave sharp estimates for the Fourier coefficients of f. They also have shown the remarkable result that $\{w : |w| < 1/2\} \subset f(U)$ whenever $f \in K_H^0$ which is already best possible for normalized conformal mappings onto convex domains.

2.4.3 Other special classes. We finish Section 2.4 with some remarks on harmonic mappings in S_H (S_H^0 resp. which are either close-to-convex or typically real. Recall that a domain Ω is close-to-convex if the complement of $\overline{\Omega}$ can be written as a union of non-crossing open half-lines. If $f = h + \overline{g} \in K_H$, then ϕ_α defined in (9) maps U onto a close-to-convex domain for all $\alpha \in \mathbb{R}$. It follows then (see [CS1]) that $h(z) - \zeta g(z)$ is a univalent close–to–convex mapping on U for all fixed $\zeta, |\zeta| \leq 1$. Conversely, J. Clunie and T. Sheil–Small [CS1] have shown the following interesting result

Theorem 2.8. *Let h and g be analytic in U and suppose that $|g'(0)| < |h'(0)|$. If $h(z) - \zeta g(z)$ is a univalent close-to-convex mapping defined on U for all fixed $\zeta, |\zeta| = 1$, then $f = h + \overline{g}$ is a univalent harmonic mapping from U onto a close-to-convex domain.*

Observe that the univalence of f follows directly from the univalence of the mappings $h(z) - \zeta g(z)$.

A harmonic mapping f on U is called typically real if $f(z)$ is real if and only if z is real. For example, a univalent harmonic mapping whose Fourier coeffients are real is typically real. Furthermore, $f = h + \overline{g}$ is typically real if and only if $\phi = h - g$ is typically real. Sharp coefficient estimates have been given in [CS1]. However, there are univalent (orientation-preserving) harmonic mappings $f = h + \overline{g}$ on U which are real on the real axis with $h'(0) > 0$ and $g'(0) > 0$ but fail to be typically real (see e.g. [BHH2]). Furthermore, there are univalent (orientation-preserving) harmonic non-typically real mappings $f = h + \overline{g}$ satisfying $f(0) = 0$ and $f_z(0) = 1$ and whose image $f(U)$ is symmetric with respect to the real axis. However, if the second dilatation function a has real coefficients, then, in both cases, f is typically real.

2.5 Mapping problems

Recall that harmonic and sense–preserving mappings defined on the unit disk U are solutions of the elliptic partial differential equation

$$\overline{f_{\overline{z}}(z)} = a(z) f_z(z); \; a \in H(U), \; |a| < 1$$

on U. It is natural to ask the question if for each given dilatation $a(z), a \in H(U), |a| < 1$, and for each given simply connected domain Ω there is a univalent solution of (4) which maps U onto Ω. Unfortunately the answer is no. Indeed, it has been shown in [HS2] that if a is a finite Blaschke product, there is no univalent harmonic mapping from U onto any bounded strictly convex domain. However, the following result has been given in [HS1] and [BHH1].

Theorem 2.9. *Let Ω be a given bounded domain of $\mathbb{C}$ such that its boundary $\partial\Omega$ is locally connected. Suppose that a satisfies $a \in H(U), |a| < 1$ on U. Choose w_0 in Ω. Then there exists a univalent solution of (4) having the following properties:*

(i) $f(0) = w_0$, $f_z(0) > 0$ *and* $f(U) \subset \Omega$.

(ii) *There is a countable set E on ∂U such that the unrestricted limits $f^*(e^{it}) = \lim_{z \to e^{it}} f(z)$ exist on $\partial U \setminus E$ and they are on $\partial\Omega$.*

(iii) *The functions $f^*_-(e^{it}) = \operatorname{ess\,lim}_{s \uparrow t} f^*(e^{is})$ and $f^*_+(e^{it}) = \operatorname{ess\,lim}_{s \downarrow t} f^*(e^{is})$ exist on ∂U*

(iv) *The cluster set of f at e^{it} is the line segment from $f^*_-(e^{it})$ to $f^*_+(e^{it})$.*

(v) *If, in addition, $|a| \le k < 1$ and Ω is a strictly starlike domain then f is uniquely determined. Uniqueness also holds for symmetric Ω if a has real coefficients* [BHH2].

Remarks 2.5.

2.5.1 If $|a| \leq k < 1$ then E is empty and f admits a continuous extension to $\overline{\Omega}$. Furthermore, we have $f(U) = \Omega$. If, in addition, Ω is a Jordan domain, then f extends to a homeomorphism from $\overline{U}$ onto $\overline{\Omega}$.

2.5.2 There is no analogue theorem for multiply connected domains.

Open problem 2.5. Prove or disprove the uniqueness of mappings satisfying Theorem 2.9. There are several kinds of uniqueness theorems for quasiconformal mappings. But none of them applies to our case. Suppose that the boundary $\partial\Omega$ is smooth enough. If one knows that two mappings f and F satisfy Theorem 2.9 and that $f_z(0) = F_z(0)$ then one can conclude that $f = F$ (see e.g. [GD2] and [B2]).

2.6 Boundary behaviour

If the second dilatation function a of a univalent harmonic mapping f satisfies $|a(z)| \leq k < 1$ for all $z \in U$, then f is a quasiconformal map and its boundary behaviour is the same as for conformal mappings. However, if a aproaches one as z tends to the boundary, then the boundary behaviour of f is quite different. It may happen that the boundary values are constant on an interval of ∂U or that there are jumps as the following example shows.

Example 2.1. The Poisson integral f of the boundary function

$$f^*(e^{it}) = \begin{cases} 1, & if\ |t| < \pi/3 \\ e^{2\pi i/3}, if & \pi/3 < t < \pi \\ e^{-2\pi i/3}, if & -\pi/3 > t > -\pi \end{cases}$$

is a univalent harmonic mapping from the unit disk onto the triangle with vertices 1, $e^{2\pi i/3}$ and $e^{-2\pi i/3}$.

Theorem 2.10. *Let*

(i) *Ω be a bounded domain of $\mathbb{C}$ such that its boundary $\partial\Omega$ is locally connected.*

(ii) *$a(z) \in H(U)$, $|a| < 1$ on U and $|a(e^{it})| = 1$ on an interval $J = \{e^{it}, \beta < t < \gamma\}$, $\beta < \gamma < \beta + 2\pi$.*

(iii) *$f(z)$ be a univalent solution of (4), such that $f(U) \subset \Omega$ and that $f^*(e^{it}) = \lim_{z \to e^{it}} f(z) \in \partial\Omega$ a.e*

Then we have

$$f^*(e^{it}) - \overline{a(e^{it}) f^*(e^{it})} + \overline{\int f^*(e^{it}) da(e^{it})} \equiv const \tag{12}$$

on J.

To prove Theorem 2.10, one shows that

$$d\mu_r(t) = df(re^{it}) - \overline{a(re^{it}) df(re^{it})} = ire^{it}(1 - |a(re^{it})|^2) h'(re^{it}) dt$$

converges weakly to the identical zero measure on J as r tends to one.

Corollary 2.1. *Let Ω, a, f and f^* be as in Theorem 2.8. Then, either f^* jumps at e^{it}, or is constant in a right or left neighborhood of e^{it}, or the curvature is strictly negative at $f^*(e^{it})$. In particular (see* [HS2] *and* [S6]*), if Ω is a bounded convex domain and if a is a finite Blaschke product containing N factors, then f^* is piecewise constant and $f(U)$ is a polygon with $N + 2$ edges.*

Y. Abu-Muhanna and A. Lyzzaik [AL1] gave a prime-end theory for univalent harmonic mappings. In particular, they have shown that no continuum of ∂U can be mapped onto a cusp. On the other hand, T. Sheil-Small [S6] considered harmonic mappings defined on U whose boundary function $f^*(e^{it})$ is a step function.

2.7 Univalent logharmonic mappings

Suppose we want to study minimal surfaces whose Gauss map (normal vector) is periodic. Then we are led to univalent harmonic mappings with periodic partial derivatives. We may restrict ourself to periods of $2\pi i$.

Let D be the left half-plane $\{z : \operatorname{Re} z < 0\}$ and consider the set $\mathcal{F}$ of all univalent harmonic and sense-preserving mappings $F = U + iV$

lefined on D such that

$$F(z+2\pi i) \equiv F(z) + 2\pi i \tag{13}$$

on D and

$$\mathrm{Re}\{F(-\infty)\} = \lim_{x\to-\infty} \mathrm{Re}\{F(x+iy)\} = -\infty. \tag{14}$$

It follows then that $F \in \mathcal{F}$ admits the representation

$$F(z) = z + 2\beta x + H(z) + \overline{G(z)} \tag{15}$$

where

(i) $\mathrm{Re}\beta > -1/2$,
(ii) H and G are analytic in D,
(iii) $G(-\infty) = \lim_{x\to-\infty} G(x+iy) = 0$,
(iv) $H(-\infty) = \lim_{x\to-\infty} H(x+iy)$ exists and is finite, and
(v) $H(z+2\pi i) \equiv H(z) + 2\pi i$ and $G(z+2\pi i) \equiv G(z) + 2\pi i$ on D.

Furthermore, the second dilatation function $A = G'/H'$ of F satisfies the properties:

$$\begin{aligned} &\text{(I)} \quad A \in H(D) \text{ and } |A| < 1 \text{ on } D,\\ &\text{(II)} \quad A(z+2\pi i) \equiv A(z) \text{ and}\\ &\text{(III)} \quad A(-\infty) = \lim_{x\to-\infty} A(x+iy) \text{ exists and is finite.}\end{aligned} \tag{16}$$

Observe that β defined in (15) depends only on $a(-\infty)$. Suppose now that the domain Ω has the property

$$\Omega = \{w = u + iv : \ -\infty < u < u_0(v)\ ,\ v \in \mathbb{R}\}\ , \tag{17}$$

where u_0 satisfies $u_0(v+2\pi) \equiv u_0(v)$, $v \in \mathbb{R}$.

The following mapping theorem corresponds to Theorem 2.9 and has been proved in [AH2].

Theorem 2.11. *Let Ω be given as in* (17) *and let A be as in* (16). *Then there exists a univalent solution F of* (4) *such that*
(i) *F is of the form* (15),

(ii) $H(-\infty)$ *exists and is real,*
(iii) $F(D) \subset \Omega$ *and*
(iv) $\lim_{z\to it} F(z)$ *exists and lies on* $\partial\Omega$ *for almost all* t.
(v) F *is uniquely determined if* Ω *is strictly convex in the horizontal direction, i.e., if each horizontal line intersects* $\partial\Omega$ *in exactly one point of* $\mathbb{C}$.

Again, if $|a| \le k < 1$ on D, then $f(D) = \Omega$. The proof uses the transformation $f(\zeta) = \exp(F(\log \zeta))$, $\zeta \in U$ or equivalently, $F(z) = \log f(e^z)$ Observe that f is univalent on U if and only if F is univalent on D and that f is a solution of the non-linear elliptic partial differential equation

$$(18) \qquad \overline{f_{\bar z}} = (a\overline{f}/f) f_z, \quad a \in H(U) \text{ and } |a| < 1$$

where $a(\zeta) = A(e^z)$. Any non-constant solution of (18) is called a logharmonic mapping. Such mappings have been studied in several papers, as for example [AB1], [AH1], [AH2] and [AH3]. In many cases, it is easier to work with logharmonic mappings than with harmonic maps of the form (15), even if the differential equation is nonlinear. For instance, it has been shown in [AH1] that f is a logharmonic automorphism on U satisfying $f(0) = 0$ and $f_z(0) > 0$ if and only if there is a normalized starlike conformal mapping ϕ and a $\beta > -1/2$ such that

$$(19) \qquad f(z) = |z|^{2\beta}\phi(z)/|\phi(z)|, \quad z \in U$$

with the branch $1^{2\beta} = 1$. Using the transformation $F(z) = \log f(e^z)$, we conclude that $F \in \mathcal{F}$ is an automorphism on the left half-plane D if and only if there is a $\beta > -1/2$ and a probability measure μ defined on the Borel σ-algebra over $[0, 2\pi)$ such that $F(z) = z + 2\beta x - 2i\int_0^{2\pi} \arg[1 - e^{it+z}]\,d\mu(t)$.

2.8 Constructive methods

There are several constructive methods for conformal mappings from a simply connected domain Ω containing the origin onto the unit disk U, or from U onto Ω. Some of them are based on extremal problems. For example, define $N(\Omega) = \{f \in H(\Omega) : f(0) = 0,\ f'(0) = 1\}$

and let Φ be the Riemann mapping from U onto Ω (Φ conformal, $\Phi(0) = 0$ and $\Phi'(0) > 0$). Then the unique solution $\hat{f}(z)$ of the extremal problem

$$\min_{f \in N(\Omega)} \int_{\Omega} |f'|^2 dxdy$$

is the conformal mapping $\Phi'(0)\Phi^{-1}(z)$ which maps Ω onto the disk of center 0 and radius $\Phi'(0)$. Another extremal problem is

$$\min_{f \in N(\Omega)} \sup_{z \in \Omega} |f(z)|$$

which has the same solution as in the previous optimization problem.

Other methods use the boundary correspondence together with the Cauchy-Riemann equations (e.g. Theodorsen method). While such methods may be modified for K-quasiconformal mappings, they are not applicable for univalent harmonic maps since collapsing may appear. Observe also that most known methods give approximations of the Riemann mapping Φ^{-1}. The mapping Φ can then be obtained by inverting Φ^{-1}. Such a procedure does not apply for univalent harmonic maps. Indeed, knowing the mapping f^{-1} we do not know how to retrieve f.

The following method was first introduced for conformal mappings by G.Opfer [O1] and [O2]. Let Ω be a strictly starlike domain (i.e. each radial line from the origin hits the boundary $\partial\Omega$ in exactly one finite point. Then $\partial\Omega$ admits the parametric representation $\omega(t) = R(t)e^{it}, 0 \leq t < 2\pi$. The Minkowski functional $\nu(w)$ is defined by

$$\nu(w) = \begin{cases} 0\,, & \text{if } w = 0 \\ |w|/R(t)\,, & \text{if } w = |w|e^{it} \neq 0. \end{cases}$$

If E is an arbitrary subset of $\mathbb{C}$, define $\mu(E) = \sup_{w \in E} \nu(w)$. Furthermore, for any complex-valued function f defined on a domain D, we put $\mu(f) = \mu(f(D))$. The following result has been shown in [BHH1].

Theorem 2.12. *Let $a \in H(U)$ and suppose that $|a| \leq k < 1$ on U. Denote by N_a the set of all solutions f of* (4) *which are of the form $f(z) = z + \overline{a(0)z} + o(|z|)$ as $z \to 0$. Denote by F the unique univalent solution of* (4) *which is normalized by $F(0) = 0$ and $F_z(0) > 0$ and* *which maps the unit disk U onto the strictly starlike domain Ω. Then there is a unique function $\hat{f} \in N_a$ which solves the extremal problem*

$$\min_{f \in N_a} \mu(f).$$

Furthermore, we have $\hat{f} = F/F_z(0)$.

To approximate f, we proceed in the following way (for more details see [BHH1]).

(i) Approximate $a(z)$ by a polynomial $a_1(z)$, $|a_1| < 1$.
(ii) Define

$$p_1(z) = z + \overline{\int_0^z a_1(s)ds}\ ,$$

$$p_n(z) = z^n + n\overline{\int_0^z s^{n-1}a_1(s)ds}\ , q_n(z) = i[z^n - n\overline{\int_0^z s^{n-1}a_1(s)ds}].$$

(iii) Put $V_N = \{p_1 + \sum_{n=2}^N \lambda_n p_n + \sum_{n=2}^N \mu_n q_n\}$, λ_n and μ_n real, and let f_N be a solution of $\min_{f\in V_N} \mu(f)$. Then f_N converges locally uniformly to the mapping $\hat{f} = F/F_z(0)$.
(iv) Define $\zeta_k = e^{2\pi i k/M}$, $1 \leq k \leq M$. Then the solution of the mathematical program

(20) $$\min C$$

$$\nu(p_1(\zeta_k) + \sum_{n=2}^N \lambda_n p_n(\zeta_k) + \sum_{n=2}^N \mu_n q_n(\zeta_k)) \leq C,$$

$$\lambda_n \in \mathbb{R},\ \mu_n \in \mathbb{R},\ 2 \leq n \leq N \ \text{ and } \ 1 \leq k \leq M$$

approximates the univalent harmonic mapping $\hat{f} = F/F_z(0)$.
(v) If, in addition, Ω is a bounded convex domain, then (20) becomes a standard linear program.

Constructive methods for univalent harmonic mappings defined on the exterior of the unit disk have been studied in [HN1].

3. Univalent harmonic mappings on multiply connected domains

3.1 Univalent harmonic mappings of the exterior of the unit disk

Let K be a compact of $\mathbb{C}$ such that K and its complement $\mathbb{C} \setminus K$ are connected. We are interested in sense–preserving univalent harmonic mappings f defined on the domain $D = \mathbb{C} \setminus K$ which keep infinity fixed. Applying the conformal premapping Φ from the exterior Δ of the unit disk U onto D normalized by $\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$ to f, we may assume without loss of generality that $D = \Delta$, $f(\infty) = \infty$, $f_z(\infty) = 1$ and that f is sense–preserving. So far, f can be written in the form

$$f(z) = z + \overline{Bz} + 2C \ln|z| + h(z) + \overline{g(z)} \tag{21}$$

where $h(z) = \sum_{n=0}^{\infty} a_n z^{-n}$ and $g(z) = \sum_{n=1}^{\infty} b_n z^{-n}$ are analytic functions on $\Delta \cup \{\infty\}$ and $|B| = |a(\infty)| < 1$. Furthermore, applying a translation, we may assume that $a_0 = 0$.

Definition 3.1. The class Σ_H consists of all univalent harmonic and sense–preserving mappings f defined on Δ which are of the form (21) and for which $a_0 = 0$.

Applying the affine postmapping $\Psi(w) = [w - \overline{a(\infty)w}]/[1 - |a(\infty)|^2]$ to f we can transform f to a function whose dilatation function vanishes at infinity.

Definition 3.2. The class Σ_H^0 consists of all mappings f in Σ_H such that $f_{\overline{z}}(\infty) = 0$.

In contrast to conformal mappings, there is no elementary isomorphism between S_H and Σ_H. Another difference is the fact that there are univalent harmonic mappings from Δ onto the whole plane minus a point. The following Theorem characterizes such mappings.

Theorem 3.1 [HS4]. *A harmonic function F is a univalent harmonic and sense-preserving mapping from Δ onto $\mathbb{C} \setminus \{p\}$ if and* *only if F is of the form*

$$F(z) = A\left[z + cd\,\overline{z} + 2(c+d)\ln|z| - cd/z - 1/\overline{z}\right] + p\,, \tag{22}$$

where $A \in \mathbb{C} \setminus \{0\}$, $|c| < 1$ *and* $|d| \leq 1$.

The corresponding second dilatation function a is of the form

$$a(z) = \frac{\overline{A}}{A}\left(\frac{cz+1}{z+\overline{c}}\right)\left(\frac{dz+1}{z+\overline{d}}\right). \tag{23}$$

It is interesting to note [HS4] that there is no mapping in Σ_H such that $\mathbb{C} \setminus f(\Delta)$ is a continuum and such that f is a solution of

$$\overline{f_{\overline{z}}(z)} = \frac{cz+1}{z+\overline{c}}\,\frac{dz+1}{z+\overline{d}}\,f_z(z), \quad |c| < 1 \text{ and } |d| \leq 1. \tag{24}$$

In particular, no such mapping exists from Δ onto $\Delta_r = \{z : r < |z|\}$ for any $r > 0$. However, for all other dilatation functions $a(z)$ we can find a solution of (4) which belongs to Σ_H and whose image is Δ_r for some $r > 0$ [HS4].

Finally, let us mention that some extremal problems concerning mappings in Σ_H or Σ_H^0 have been solved in [HS3].

3.2 Univalent harmonic ring mappings

Fix $r \in (0,1)$ and let $A(r,1)$ be the annulus $\{z : r < |z| < 1\}$. In this section we consider univalent harmonic mappings from $A(r,1)$ onto $A(R,1)$ for some $R \in [0,1)$. If f is conformal, then $R = r$ and f is a rotation $f(z) = e^{i\gamma}z$. However, there are univalent harmonic (and sense–preserving) mappings from $A(r,1)$ onto $A(0,1)$. For instance,

$$f(z) = \frac{z - r^2/\overline{z}}{1 - r^2}$$

is such a mapping. But there are many other ones, as we shall see in Theorem 3.4'. On the other hand, $R(r)$ cannot be arbitrarily close to one. J.C.C. Nitsche [N1] has given the following elegant proof of this fact.

Theorem 3.2. *For each $r \in (0,1)$ there is an $R_0(r) \in (0,1)$ such that, if $f = u + iv$ is a univalent harmonic mapping from $A(r,1)$ onto $A(R,1)$, then $R \leq R_0(r)$.*

Proof [N1]. Define $\gamma = \{z : |z| = (1+r)/2\}$. Then, by Harnack's inequality, there is a constant $K(\gamma) > 1$ such that $h(z_2) \leq Kh(z_1)$ for all positive harmonic functions h on $A(r,1)$ and all z_1 and $z_2 \in \gamma$. Define $h = 1 - u$. Then h is a positive harmonic function on $A(r,1)$. Next, there is a $z_1 \in \gamma$ such that $h(z_1) < 1 - R$ and there is a $z_2 \in \gamma$ such that $1 + R < h(z_2)$. Hence, $1 + R < h(z_2) \leq Kh(z_1) < K(1 - R)$ which implies that $R < (K-1)/(K+1) < 1$. □

Remarks 3.1.

3.1.1. The proof of Nitsche does not use the univalence of f but rather the fact, that $f(\gamma)$ contains a point in the region $\{w : R < \operatorname{Re} w < 1\}$ and a point in $\{w : -1 < \operatorname{Re} w < -R\}$.

3.1.2 The same proof can also be applied to other image domains as for example $\Omega = U \setminus [-R, R]$ or $\Omega = \{w : R < |w| \text{ and } |\operatorname{Re} w| < 1\}$.

Open problem 3.1. Find the value for $R_0(r)$. Since $f(z) = (z + r^2/\overline{z})/(1 + r^2)$ is univalent on $A(r,1)$, it follows that $2r/(1+r)^2 \leq R_0(r) < 1$. On the other hand, it is not likely that the lowest value of K which one can find in the above proof gives $R_0(r)$ (see item (ii) of the above remark).

3.3 Extensions of Kneser's Theorem

In this section we extend Kneser's result, Theorem 2.6 and Theorem 2.7 for multiply connected domains of $\mathbb{C}$.

Let D be a Jordan domain of finite connectivity N in $\mathbb{C}$ whose boundary is $\partial D = \cup_{k=0}^{N} C_k$ where C_0 is the outer boundary of D. Applying an appropriate conformal premapping, we may assume without loss of generality, that each component C_k is an analytic Jordan curve. Let Ω be a domain of $\mathbb{C}$ of connectivity N such that the outer boundary S_0 is a Jordan curve and such that each inner boundary component S_k, $1 \leq k \leq N$, is either a Jordan curve or a Jordan arc or a singleton. Denote by Φ_0 (Ψ_0 resp.) the conformal mapping from the unit disk U onto the bounded component of $\mathbb{C} \setminus C_0$ ($\mathbb{C} \setminus S_0$ resp.)

and let Φ_k (Ψ_k resp.) be the conformal mapping from the exterior Δ of U onto the unbounded component of $\mathbb{C} \setminus C_0$ ($\mathbb{C} \setminus S_0$, resp.)

Definition 3.2. Let D and Ω be as mentioned above. A function f^* from ∂D into $\partial\Omega$ is called **sense–preserving continuous weak homeomorphism** from ∂D onto $\partial\Omega$ if f^* is continuous on ∂D and $f^*(C_k) = S_k$, $0 \leq k \leq N$, and if S_k is not a singleton then

(i) $d \arg \Psi_k^{-1} \circ f^* \circ \Phi_k \geq 0$ and

(ii) $(2\pi)^{-1} \int_{\partial D} d \arg \Psi_k^{-1} \circ f^* \circ \Phi_k = 1$.

Modifying the proof of Kneser which we have given for Theorem 2.7, we get

Theorem 3.3. *Let D and Ω be as in* Definition 3.2 *and let f^* be a sense-preserving continuous weak homeomorphism from ∂D onto $\partial\Omega$. Then the solution f of the Dirichlet problem, $f = \int_{\partial D} f^* d\omega$ is univalent in D if and only if $f(D) = \Omega$.*

The next result is an extension of Theorem 2.6 to multiply connected domains..

Theorem 3.4. *Let D be a Jordan domain of finite connectivity N and suppose that $\partial D = \cup_{k=0}^{N} C_k$ where C_0 is the outer boundary of D. Let Ω be a bounded convex domain of $\mathbb{C}$ and suppose that f^* is a weak homeomorphism from C_0 onto $\partial\Omega$ (see* Definition 2.3*). Let f be a harmonic mapping defined on D which satisfies*

(i) *$f = h + \overline{g}$, $h \in H(D)$ and $g \in H(D)$.*

(ii) *$\lim_{z \to \zeta} f(z) = f^*(\zeta)$ for all $\zeta \in C_0$ and*

(iii) *the image of each inner boundary component C_k of D is a singleton $\{p_k\}$*

Then f is univalent on D.

Remarks 3.2.

3.2.1 There is at least one harmonic mapping which satisfies the conditions of Theorem 3.4.

3.2.2 Theorem 3.4 does not hold neither for unbounded convex domains nor for non–convex domains and there is no analoguous result for harmonic mappings in higher dimensions.

3.2.3 In general, one cannot prescribe the image points $f(C_k) = \{p_k\}$. However, if $N = 1$ and if D is the annulus $A(r,1) = \{w : r < |w| < 1\}$, then $p_1 = (2\pi)^{-1} \int_0^{2\pi} f^*(e^{it})dt$

3.2.4 There are univalent harmonic mappings which satisfy Theorem 3.4 without having the property (1). For instance, suppose that $D = A(r,1)$ and that $f^*(e^{it}) = e^{it}$. Then $f(z) = (z - r^2/\overline{z})/(1 - r^2) + 2C \ln|z|$ is univalent if and only if $|C| \leq r/(1-r^2)$.

3.2.5 It is a natural question to ask whether Theorem 3.4 holds if we replace condition (3) for f by the following weaker condition: (3') The image of each inner boundary component C_k of D is a horizontal line segment. The answer is negative. Indeed, consider $D = A(\sqrt{11/26}, 1)$ and $f(z) = 4z - \overline{z}/3 - 1/(6z) - 2/\overline{z}$. Then $f^* = f|_{C_0}$ is an sense–preserving homeomorphism from C_0 onto $\partial\Omega$ and the inner boundary of D is mapped onto the horizontal slit $[-16/\sqrt{286}, 16/\sqrt{286}]$ but f is not univalent on D.

Theorem 3.4 together with Remark 3.2.3 gives the particular case:

Theorem 3.4′. *Let $\Psi(t)$ be a non–decreasing function on $[0,2\pi)$ such that*

(i) $\int_{[0,2\pi)} d\Psi(t) = 2\pi$,

(ii) $\int_0^{2\pi} e^{i\Psi(t)}dt = 0$ *and*

(iii) *the image $\Psi([0,2\pi))$ contains at least three different points.*

Then the solution of the Dirichlet problem

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left\{1 + \sum_{n=1}^{\infty} \frac{(rR)^n}{R^{2n} - r^{2n}} \left[\left(\frac{ze^{-it}}{r}\right)^n - \left(\frac{r}{ze^{-it}}\right)^n\right]\right\} e^{i\Psi(t)}dt$$

is univalent on D.

Suppose that f is a harmonic mapping in a neighbourhood of the unit circle ∂U which maps ∂U onto a single point. In [HS5], necessary and sufficient conditions have been given for f to be univalent and orientation-preserving in an exterior neighbourhood of ∂U.

3.4 Canonical harmonic punctured plane mappings

It is well known that for any domain D of $\overline{\mathbb{C}}$ containing the point at infinity there is a conformal mapping $j_\beta(z)$ such that the image $j_\beta(D)$ is a parallel slit domain with inclination β with respect to the real axis and which satisfies $j_\beta(z) = z + o(1)$ as $z \to \infty$. If ∂D has countably many components then j_β is uniquely determined and we have

$$j_\beta(z) = e^{i\beta}[j_0(z)\cos\beta - i.j_{\pi/2}(z)\sin\beta]. \tag{25}$$

If ∂D has uncountably many components then j_β may not be unique; but there is one representative for each β such that (25) holds (see e.g. [A1]).

It is a natural question to ask whether there is for each domain D containing infinity a univalent harmonic mapping f such that $f(z) = z + o(1)$ in a neighbourhood of infinity and such that each component of $\partial f(D)$ is a singleton. The next theorem gives an affirmative answer.

Theorem 3.5. *Let D be a domain of $\overline{\mathbb{C}}$ containing the point at infinity. Then there exists a univalent harmonic mapping $\hat{F}$ from D onto $\overline{\mathbb{C}} \setminus \cup_{j\in J}\{p_j\}$ which is normalized at infinity by $f(z) = z + o(1)$. Furthermore, if ∂D has countably many components, then F is unique.*

Remarks 3.3.

3.3.1 The mapping $\hat{F} = \hat{H} + \overline{\hat{G}}$ defined in Theorem 3.5. is called the canonical harmonic punctured plane mapping.

3.3.2 Denote by $\hat{A}$ the second dilatation function of $\hat{F}$. Then there is no other solution f of (4) with respect to $\hat{A}$ which is univalent on D and satisfies $f(z) = z + o(1)$ as $z \to \infty$. Furthermore, if ∂D has N components then $\hat{A}$ assumes in D every value in the unit disk U exactly at $2N$ points and it does not assume any other values at all. In other words, $\hat{A}$ maps D onto a $2N-$ sheeted disk.

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