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# Harmonic Locally Quasiconformal Mappings

ABSTRACT. Analogously with the universal linearly-invariant families $\mathcal{U}_\alpha$ (see: [1]) of analytic functions, in this paper we introduce and investigate linearly-invariant families $H(\alpha, K)$ of functions locally $K$-quasiconformal and harmonic in the unit disc. Not all of properties of $\mathcal{U}_\alpha$ have their counterparts in $H(\alpha, K)$.

In this paper we consider functions complex-valued and harmonic in the unit disc $\Delta = \{z : |z| < 1\}$. In eighties univalent and locally univalent harmonic functions in $\Delta$ were extensively studied. Various classes were introduced by an analogy with regular functions and their geometric characterizations such as convexity, close-to-convexity, univalence, symmetry and so on. In this paper we investigate classes of harmonic functions whose definition is based on properties of local quasiconformality and linear invariance.

Ch. Pommerenke [1] defined a linearly-invariant family of functions of the order $\alpha$ ($\alpha \geq 1$) as a set $\mathcal{M}$ of functions $\phi(z) = z + d_2(\phi)z^2 + \ldots$ regular in $\Delta$ which satisfy the following conditions:

a) $\phi'(z) \neq 0$ in $\Delta$ (local univalence);
b) for every conformal automorphism $b(z) = e^{i\theta}\frac{z+a}{1+\bar{a}z}$ of the unit disc $\Delta$ and for every function $\phi \in \mathcal{M}$ the function

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$$\frac{\phi(b(z))-\phi(b(0))}{\phi'(b(0))b'(0)}=z+...\in\mathcal{M}$$

(invariance with respect to Möbius automorphisms of $\Delta$);
c) the order of the family $\mathcal{M}$ is equal to $\alpha$, i.e.

$$\operatorname{ord}\mathcal{M}=\sup_{\phi\in\mathcal{M}}|d_2(\phi)|=\sup_{\phi\in\mathcal{M}}\frac{|\phi''(0)|}{2}=\alpha.$$

The universal linearly-invariant family $\mathcal{U}_\alpha$ of order $\alpha$ is defined by Ch. Pommerenke as the union of all linearly-invariant families of order less than or equal to $\alpha$. It is clear that $\mathcal{U}_\alpha$, $\alpha\in[1,\infty]$, contains all normalized conformal mappings $\phi(z)$ of the disc $\Delta$.

Most classes of functions regular and univalent or locally univalent are linearly-invariant. Because of this they have several general properties which depend only on their order $\alpha$. On the other hand, introducing the universal linearly-invariant family $\mathcal{U}_\alpha$ allows us to investigate all locally univalent functions of a finite order.

In this paper we extend some ideas connected with $\mathcal{U}_\alpha$ to the class of harmonic functions. Such functions can be represented in the following form:

$$f(z)=h(z)+\overline{g(z)}, \tag{1}$$

where

$$h(z)=\sum_{n=0}^{\infty}a_n(f)z^n \quad\text{and}\quad g(z)=\sum_{n=1}^{\infty}\overline{a_{-n}(f)}z^n$$

are functions regular in $\Delta$. We consider functions of the form (1) preserving the orientation in $\Delta$, i.e. the Jacobian $J_f(z)$ satisfies

$$J_f(z)=|h'(z)|^2-|g'(z)|^2>0 \quad\text{in}\quad \Delta. \tag{2}$$

Thus the functions considered are locally homeomorphic and harmonic in $\Delta$.

In what follows formal derivatives $f_z$, $f_{\bar z}$ will be also denoted by $\partial f$ and $\overline{\partial} f$ in order to avoid ambiguity in symbols like $f_z(z)$ and so on.

**Definition 1.** If there exists a number K such that the function $f(z)$ of the form (1) satisfies

$$\frac{|\partial f|+|\overline{\partial} f|}{|\partial f|-|\overline{\partial} f|}=\frac{|h'|+|g'|}{|h'|-|g'|}\leq K=\frac{1+k}{1-k} \quad \text{in} \quad \Delta,$$

then $f(z)$ is said to be locally $K$-quasiconformal in $\Delta$.

**Definition 2.** Let us denote by $H(\alpha, K)$ the set of all functions $f(z)=h(z)+\overline{g(z)}$ locally $K$-quasiconformal and harmonic in $\Delta$ with the normalization $a_0(f)=0,\ a_1(f)+a_{-1}(f)=1$, and such that $h'(z)/h'(0)\in\mathcal{U}_\alpha$.

The classes $H(\alpha, K)$ expand if $\alpha$ and $K$ increase and they include all functions $f(z)$ with the above normalization sense-preserving and harmonic in $\Delta$. We consider the case, when $\alpha$ and $K$ are finite.

**Theorem 1.** *For all $\alpha\in[1,\infty)$, $K\in[1,\infty)$, the classes $H(\alpha, K)$ are compact with respect to the topology of almost uniform convergence in $\Delta$ (i.e. uniform convergence on compact subsets of $\Delta$).*

**Proof.** Let a sequence $f_n(z)=h_n(z)+\overline{g_n(z)}\in H(\alpha,K)$ . Then $a_1(f_n)+a_{-1}(f_n)=1$ . Since $|\overline{\partial} f_n(0)/\partial f_n(0)|\leq k$ , we have $|a_{-1}(f_n)/(1-a_{-1}(f_n))|\leq k$ . Thus $|a_{-1}(f_n)|\leq k/(1-k)$ and consequently

$$|a_1(f_n)|\leq 1+|a_{-1}(f_n)|\leq 1/(1-k).$$

By the definition of $H(\alpha,K)$ we have that $h_n(z)/a_1(f_n)\in\mathcal{U}_\alpha$. Thus (see [1]):

$$\frac{(1-|z|)^{\alpha-1}}{(1+|z|)^{\alpha+1}}\leq\left|\frac{h_n'(z)}{a_1(f_n)}\right|\leq\frac{(1+|z|)^{\alpha-1}}{(1-|z|)^{\alpha+1}}. \tag{3}$$

Thus $h_n'(z)$ are uniformly bounded on compact subsets of $\Delta$. Moreover, $|g_n'(z)|\leq k|h_n'(z)|$. Now, our theorem follows from the principle of compactness. □

Let us observe that the inequality $|a_{-1}(f)/a_1(f)|\leq k$ and the normalization $a_1(f)+a_{-1}(f)=1$ for $f\in H(\alpha,K)$ imply $1/(1+k)\leq |a_1(f)|$. Thus we have

$$\forall_{f\in H(\alpha,K)}\ \frac{1}{1+k}\leq|a_1(f)|\leq\frac{1}{1-k},\quad |a_{-1}(f)|\leq\frac{k}{1-k}. \tag{4}$$

The inequalities (4) are sharp which follows by examples of functions from the class $H(\alpha, K)$ given below.

The derivative of a complex-valued function $f(z)$ in the direction of vector $e^{i\theta}$ at the point $z$ will be denoted by

$$\partial_\theta f(z) := \lim_{\rho\to+0} \frac{f(z+\rho e^{i\theta}) - f(z)}{\rho}.$$

For harmonic functions of the form (1) we have:

$$\partial_\theta f(z) = h'(z)e^{i\theta} + \overline{g'(z)}e^{-i\theta} = \partial f(z)e^{i\theta} + \overline{\partial} f(z)e^{-i\theta}.$$

By an analogy to the definition of the linearly-invariant family of regular functions we give the following

**Definition 3.** A family $\mathcal{H}$ of functions harmonic in $\Delta$ is called linearly-invariant , if for all functions $f \in \mathcal{H}$:

(a) the conditions (1) and (2) hold,

(b) $a_0(f) = 0, \quad a_1(f) + a_{-1}(f) = 1,$ (5)

(c) for all $a \in \Delta$ and $\theta \in [0, 2\pi)$ the function

$$f_\theta(z,a) = \frac{f(b(0)) - f(b(0))}{\partial_\theta f(b(0))(1-|a|^2)} \in \mathcal{H}.$$

Let us observe that some classes of harmonic functions considered so far, are linearly-invariant. For example: the class $K_H$ - of univalent harmonic functions mapping $\Delta$ onto convex domains, the class $C_H$ - of close-to-convex harmonic functions, the class $S_H$ - of univalent harmonic functions. The above classes were introduced in [2] and later on were dealt with by some other authors. Linear-invariance of the class $S_H$ and some of its subclasses was used by T. Sheil-Small [3], but he considered the normalization $a_1(f) = 1$ instead of $a_1(f) + a_{-1}(f) = 1$. He observed that the behaviour of $f(z) = h(z) + \overline{g(z)} \in S_H$ depends of the order (in the sense of Ch. Pommerenke) of the function $h(z)/h'(0)$. The same holds in the case of the families $H(\alpha, K)$.

If $f \in H(\alpha, K)$ and $f_\theta(z,a) = h_\theta(z,a) + \overline{g_\theta(z,a)}$, where

$$h_\theta(z,a) = \frac{h(\frac{z+a}{1+\overline{a}z}e^{i\theta}) - h(ae^{i\theta})}{(1-|a|^2)\partial_\theta f(ae^{i\theta})},$$

then $h_\theta(z,a)/h'_\theta(0,a) \in \mathcal{U}_\alpha$ and

$$\left|\frac{\overline{\partial} f_\theta(z,a)}{\partial f_\theta(z,a)}\right| = \left|\frac{g'(\frac{z+a}{1+\bar{a}z}e^{i\theta})}{h'(\frac{z+a}{1+\bar{a}z}e^{i\theta})}\right| \le k \quad \text{in} \quad \Delta.$$

Thus $H(\alpha, K)$ are linearly-invariant families of harmonic functions. Observe that $H(\alpha,1) = \mathcal{U}_\alpha$.

**Theorem 2.** *For every $f(z) = h(z) + \overline{g(z)} \in H(\alpha, K)$ we have the following inequality*

$$\frac{1}{K}\frac{(1-|z|)^{\alpha-1}}{(1+|z|)^{\alpha+1}} \le |\partial_\theta f(z)| \le K\frac{(1+|z|)^{\alpha-1}}{(1-|z|)^{\alpha+1}}. \tag{6}$$

*The equalities in (6) are attained for $\theta = \pm\frac{\pi}{2}$. Moreover, if $z = re^{i\phi}$, then the equality on the right is attained for*

$$h(z) = \frac{e^{i\phi}}{2\alpha(1-k)}\left[\left(\frac{1+ze^{-i\phi}}{1-ze^{-i\phi}}\right)^\alpha - 1\right] \quad , \quad g(z) = -kh(z); \tag{7}$$

*and the equality on the left is attained for*

$$h(z) = \frac{e^{i\phi}}{2\alpha(1+k)}\left[\left(\frac{1-ze^{-i\phi}}{1+ze^{-i\phi}}\right)^\alpha - 1\right] \quad , \quad g(z) = kh(z).$$

**Proof.** If $f(z) = h(z) + \overline{g(z)} \in H(\alpha, K)$, then

$$\left|\frac{\overline{\partial} f(z)}{\partial f(z)}\right| = \left|\frac{g'(z)}{h'(z)}\right| \le k.$$

Thus there exists a function $\omega$, regular in $\Delta$ such that $|\omega(z)| \le 1$ and $g'(z) = k\omega(z)h'(z)$. Moreover, the equality

$$\partial_\theta f(z) = h'(z)e^{i\theta} + \overline{g'(z)e^{i\theta}} = h'(z)e^{i\theta} + \overline{k\omega(z)h'(z)e^{i\theta}}$$

implies

$$|\partial_\theta f(z)| = |h'(z)| \cdot \left|1 + k\frac{\overline{h'(z)}}{h'(z)}\overline{\omega(z)}e^{-2i\theta}\right|.$$

This and the inequality (3) imply

$$(1-k)|h'(0)|\,\frac{(1-|z|)^{\alpha-1}}{(1+|z|)^{\alpha+1}} \le |h'(z)|(1-k) \le |\partial_\theta f(z)|$$

$$\le |h'(z)|(1+k) \le (1+k)|h'(0)|\,\frac{(1+|z|)^{\alpha-1}}{(1-|z|)^{\alpha+1}}.$$

Now, using (4) we obtain

$$\frac{1-k}{1+k}\frac{(1-|z|)^{\alpha-1}}{(1+|z|)^{\alpha+1}} \le |\partial_\theta f(z)| \le \frac{1+k}{1-k}\frac{(1+|z|)^{\alpha-1}}{(1-|z|)^{\alpha+1}}.$$

□

For $K=1$, we obtain from (6) a known estimate $|\phi'(z)|$ for $\phi \in \mathcal{U}_\alpha$, cf. [1]. One can give a more precise estimate $|\partial_\theta f(z)|$ in $H(\alpha, K)$ according to $|h'(z)|$ and $\arg h'(z)$.

**Corollary 1.** *Let $f \in H(\alpha, K)$; $z_1, z_2 \in \Delta$. Then for any real $\theta$ and $\gamma$*

$$\left|\log|\partial_\gamma f(z_1)| - \log|\partial_\theta f(z_2)| + \log\frac{|1-z_1\overline{z_2}|^2 - |z_1-z_2|^2}{(1-|z_2|^2)^2}\right| \le$$

$$\alpha \log\frac{1+R}{1-R},$$

*where $R = |(z_1-z_2)/(1-z_1\bar{z}_2)|$. Moreover, for any $z_1, z_2 \in \Delta$ there exist real $\theta$ and $\gamma$, and a function $f \in H(\alpha, K)$, such that the equality holds.*

Indeed, for fixed $a, z \in \Delta$ choose $\psi \in \mathbf{R}$ such that $e^{i\psi}(1+\bar{a}z)^{-2} > 0$. Put $z_1 = e^{i\theta}(z+a)(1+\bar{a}z)^{-1}$, $z_2 = ae^{i\theta}$. For any $\theta, \gamma \in \mathbf{R}$ we have

$$\partial_{\psi-\theta+\gamma} f_\theta(z,a) = \frac{\partial f(e^{i\theta}\frac{z+a}{1+\bar{a}z})\frac{e^{i\theta}e^{i(\psi-\theta+\gamma)}}{(1+\bar{a}z)^2} + \overline{\partial} f(e^{i\theta}\frac{z+a}{1+\bar{a}z})\overline{(\frac{e^{i(\psi+\gamma)}}{(1+\bar{a}z)^2})}}{\partial_\theta f(ae^{i\theta})}$$

$$+\frac{\partial_\gamma f(z_1)}{\partial_\theta f(z_2)|1+\bar{a}z|^2} = \frac{\partial_\gamma f(z_1)}{\partial_\theta f(z_2)}\frac{|1-z_1\overline{z_2}|^2}{(1-|z_2|^2)^2}.$$

Thus we get

$$\frac{1}{K}\frac{(1-R)^{\alpha-1}}{(1+R)^{\alpha+1}} \le \frac{|\partial_\gamma f(z_1)|}{|\partial_\theta f(z_2)|}\frac{|1-z_1\overline{z_2}|^2}{(1-|z_2|^2)^2} \le K\frac{(1+R)^{\alpha-1}}{(1-R)^{\alpha+1}}.$$

This implies our inequality. The equality statement follows from Theorem 2.

If $\theta = \phi$ we obtain by Theorem 2 the following

**Corollary 2.** *If $f \in H(\alpha, K)$, $re^{i\phi} \in \Delta$, then for the derivative of $f(z) = f(re^{i\phi})$ with respect to $r$ the following sharp estimates hold*

$$\frac{1}{K}\frac{(1-r)^{\alpha-1}}{(1+r)^{\alpha+1}} \le |f'_r(re^{i\phi})| \le K\frac{(1+r)^{\alpha-1}}{(1-r)^{\alpha+1}},$$

*with equalities for $\phi = \pm\frac{\pi}{2}$ and functions for the left and right side as in Theorem 2, respectively.*

Let us denote by $F = F_f = f(\Delta)$ a two-dimensional manifold being the univalent image of the disc $\Delta$ under a locally homeomorphic mapping $f \in H(\alpha, K)$. Let $w_1, w_2 \in F$, $\Gamma$ being a rectifiable curve joining $w_1$ and $w_2$ in $F$. Let diam $\Gamma$ be the diameter of the projection of $\Gamma$ onto the complex plane and $l(\Gamma)$ the length of the projection of $\Gamma$ onto the complex plane. Denote

$$d(w_1, w_2) = d_F(w_1, w_2) = \inf \operatorname{diam}\Gamma,$$
$$l(w_1, w_2) = l_F(w_1, w_2) = \inf l(\Gamma),$$

where the infimum is considered for all curves $\Gamma \subset F$ joining $w_1$ and $w_2$. It is clear that $|w_1 - w_2| \le d(w_1, w_2) \le l(w_1, w_2)$.

**Theorem 3.** *Let $f \in H(\alpha, K)$, $r \in (0,1)$. Then the manifold with the boundary $F(r) = \{f(z) : |z| \le r\}$ includes the disc of center 0 and radius $[1 - (\frac{1-r}{1+r})^\alpha]/2\alpha K$.*

**Proof.** Let $\rho$ be the radius of the largest disc of center 0 and contained in $F(r)$. Then for some $z_0$, $|z_0| = r$, we have $|f(z_0)| = \rho$. Moreover, the segment $[0, f(z_0)] \subset F(r)$. Let $\Gamma$ be a curve joining 0 and $z_0$ in the disc $\{z : |z| \le r\}$ which is the preimage of $[0, f(z_0)]$ for the mapping $f$; $\Gamma(t)$, $t \in [0,1]$ is a smooth parametrization of $\Gamma$,

$\Gamma(0)=0$, $\Gamma(1)=z_0$. Then using the left inequality in (6) we get for $\theta=\arg\Gamma'(t)$

$$\rho=|f(z_0)|=\left|\int_0^1(f[\Gamma(t)])'_t dt\right|=\int_0^1|(f[\Gamma(t)])'_t|dt$$
$$=\int_0^1|\partial_\theta f(\Gamma(t))||\Gamma'(t)|dt\geq\frac{1}{K}\int_0^1\frac{(1-|\Gamma(t)|)^{\alpha-1}}{(1+|\Gamma(t)|)^{\alpha+1}}|d\Gamma(t)|$$
$$=\frac{1}{K}\int_\Gamma\frac{(1-|z|)^{\alpha-1}}{(1+|z|)^{\alpha+1}}|dz|\geq\frac{1}{K}\int_0^r\frac{(1-|z|)^{\alpha-1}}{(1+|z|)^{\alpha+1}}d|z|$$
$$=\frac{1}{2\alpha K}[1-(\frac{1-r}{1+r})^\alpha].$$

Now, let us observe that for the function

$$(8)\quad f(z)=h(z)+k\overline{h(z)},\qquad h(z)=\frac{\pm i}{2\alpha(1+k)}\left[\left(\frac{1\pm iz}{1\mp iz}\right)^\alpha-1\right]$$

we have by Theorem 2:

$$f(\pm ri)=\frac{\mp i}{2\alpha K}\left[1-\left(\frac{1-r}{1+r}\right)^\alpha\right].$$

Thus, the radius given in Theorem 3 is sharp for the family $H(\alpha,K)$. □

The Koebe domain of the family $H(\alpha,K)$ is a maximal univalent domain containing $w=0$ and contained in the set $\bigcap_{f\in H(\alpha,K)}F_f$.

**Corollary 3.** *The Koebe domain of the family $h(\alpha,K)$ contains a disc of center 0 and radius $1/(2\alpha K)$. The radius is maximal.*

Let us observe, that if the function

$$f_\phi(z)=\frac{e^{i\phi}}{2\alpha(1+k)}\left[\left(\frac{1-ze^{-i\phi}}{1+ze^{-i\phi}}\right)^\alpha-1\right]+\frac{ke^{-i\phi}}{2\alpha(1+k)}\overline{\left[\left(\frac{1-ze^{-i\phi}}{1+ze^{-i\phi}}\right)^\alpha-1\right]}$$

from Theorem 2 belongs to $H(\alpha,K)$ and $\gamma(\phi)=f_\phi(e^{i\phi})$ then the Koebe domain of the family $H(\alpha,K)$ is contained in a domain bounded by the curve

$$\gamma(\phi)=-\frac{e^{i\phi}+ke^{-i\phi}}{2\alpha(1+k)},\qquad\phi\in[0,2\pi].$$

**Theorem 4.** *For a function $f \in H(\alpha, K)$ the following sharp inequalities are true:*

(9)
$$\frac{1}{2\alpha K}\left[1-\left(\frac{1-r}{1+r}\right)^{\alpha}\right] \le d(0, f(z)) \le l(0, f(z)) \le \frac{K}{2\alpha}\left[\left(\frac{1+r}{1-r}\right)^{\alpha}-1\right].$$

*On the right hand side the equality for $d(0, f(z))$ and $l(0, f(z))$ is attained for the function (7) with $\phi = \pm\frac{\pi}{2}$ and $z = \pm ri$; whereas on the left hand side for the function (8) with $z = \pm ri$.*

**Proof.** The left inequalities in (9) for $d$ and $l$ follow from Theorem 3 with equality for the function (8) and $z = \pm ri$.

Let $z = re^{i\phi}$. From Corollary 2 we have

$$l(0, f(z)) \le \int_0^r |f_t'(te^{i\phi})|dt \le K\int_0^r \frac{(1+t)^{\alpha-1}}{(1-t)^{\alpha+1}}dt = \frac{K}{2\alpha}\left[\left(\frac{1+r}{1-r}\right)^{\alpha}-1\right].$$

Here, on the left part of the inequality we have the sign of equality for the function (7) with $\phi = \pm\frac{\pi}{2}$ and $z = \pm ri$. Indeed, for this function we have

$$f(\pm ri) = \frac{\pm iK}{2\alpha}\left[\left(\frac{1+r}{1-r}\right)^{\alpha} - 1\right],$$

and

$$l(0, f(\pm ri)) = |f(\pm ri)|.$$

From the definition of $d(w_1, w_2)$ it follows that for this function, with $z = \pm ri$,

$$d(0, f(z)) = l(0, f(z)).$$

Thus, the upper estimate in (9) is sharp, too. □

**Corollary 4.** $f \in H(\alpha, K) \Rightarrow |f(z)| \le \frac{K}{2\alpha}\left[\left(\frac{1+|z|}{1-|z|}\right)^{\alpha} - 1\right]$ *in $\Delta$. The inequality is sharp and the sign of equality is attained for the function (7) with $\phi = \pm\frac{\pi}{2}$ and $z = \pm i|z|$.*

**Corollary 5.** *For every $b, c \in \Delta$ and $\theta \in R$*

$$\frac{1}{2\alpha K}\left[1-\left(\frac{1-r}{1+r}\right)^{\alpha}\right] \leq \frac{d_F(f(b), f(c))}{(1-|c|^2)|\partial_\theta f(c)|} \leq$$

$$\frac{l_F(f(b), f(c))}{(1-|c|^2)|\partial_\theta f(c)|} \leq \frac{K}{2\alpha}\left[\left(\frac{1+r}{1-r}\right)^{\alpha}-1\right],$$

*where $r = |c-b|/|1-\bar{c}b|$. The inequality is sharp in the sense that for every $c \in \Delta$ and $\theta \in R$ for the left and right side there exist $b \in \Delta$ and $f \in H(\alpha, K)$ such that inequalities become equalities for $b, f$ suitably chosen. In this sense the inequality*

$$\frac{|f(b)-f(c)|}{(1-|c|^2)|\partial_\theta f(c)|} \leq \frac{K}{2\alpha}\left[\left(\frac{|\bar{c}b-1|+|c-b|}{|\bar{c}b-1|-|c-b|}\right)^{\alpha}-1\right]$$

*is sharp, too.*

Indeed, let us denote by $F_1 = f_\theta(\Delta, a)$ a manifold corresponding to the function $f_\theta(z, a)$, where $f \in H(\alpha, K)$. Next, if $c = ae^{i\theta}$, then

$$f_\theta(e^{-i\theta}\frac{c-b}{\bar{c}b-1}, ce^{-i\theta}) = \frac{f(b)-f(c)}{(1-|c|^2)\partial_\theta f(c)}$$

and

$$l_{F_1}(f_\theta(e^{-i\theta}\frac{c-b}{\bar{c}b-1}, ce^{-i\theta}), 0) = \frac{l_F(f(b), f(c))}{(1-|c|^2)|\partial_\theta f(c)|};$$

This is true if $l$ is replaced by $d$. Thus, applying Theorem 4 to the function $f_\theta(z, ce^{-i\theta})$ with $z = e^{-i\theta}(c-b)(\bar{c}b-1)^{-1}$ we get our result.

In [1] the following estimate for $\phi \in \mathcal{U}_\alpha$ was given

$$|\mathrm{Arg}\phi'(z)| \leq 2\alpha\Xi\left(|z|, \frac{1}{\alpha}\right) \leq \sqrt{\alpha^2-1}\log\frac{1+|z|}{1-|z|} + 2\arcsin|z|,$$

where

$$\Xi\left(|z|, \frac{1}{\alpha}\right) = \frac{1}{\alpha}\arcsin\frac{|z|}{\alpha} + \frac{1}{2}\sqrt{1-\frac{1}{\alpha^2}}\log\frac{\sqrt{1-\frac{|z|^2}{\alpha^2}}+|z|\sqrt{1-\frac{1}{\alpha^2}}}{\sqrt{1-\frac{|z|^2}{\alpha^2}}-|z|\sqrt{1-\frac{1}{\alpha^2}}},$$

$\text{Arg}\phi'(0) = 0$ and $\text{Arg}\phi'(z)$ is continuous function of $z$. Since, for $f \in H(\alpha, K)$, $\theta \in (-\pi, \pi]$

$$\partial_\theta f(z) = a_1(f)\phi'(z)e^{i\theta} + k\omega(z)\overline{a_1(f)\phi'(z)e^{i\theta}},$$

$\phi \in U_\alpha$, $|\omega(z)| < 1$, $\omega(0) = a_{-1}(f)/(ka_1(f))$ (see the proof of Theorem 2), we get

$$|\text{Arg}\,\partial_\theta f(z)| \le |\theta| + |\text{Arg}\phi'(z)|$$

$$+|\arg a_1(f)| + |\arg(1 + k\overline{\omega(z)}\frac{\overline{a_1(f)\phi'(z)e^{i\theta}}}{a_1(f)\phi'(z)e^{i\theta}})|.$$

From (5) we see that

$$\left|\frac{1 - a_1(f)}{a_1(f)}\right| = \left|\frac{a_{-1}(f)}{a_1(f)}\right| = \left|\frac{g'(0)}{h'(0)}\right| \le k.$$

Hence the set of values of $a_1(f)$ is the disc with the center $C = 1/(1-k^2)$ and the radius $R = k/(1-k^2)$. Thus $|\arg a_1(f)| \le \arcsin(R/C) = \arcsin k$ and

$$\left|\arg(1 + k\overline{\omega(z)}\frac{\overline{a_1(f)\phi'(z)e^{i\theta}}}{a_1(f)\phi'(z)e^{i\theta}})\right| \le \arcsin k.$$

In this way we have proved

**Theorem 5.** *If $f \in H(\alpha, K)$, $z \in \Delta$, $\theta \in (-\pi, \pi]$, then*

$$|\,\text{Arg}\,\partial_\theta f(z)| \le |\theta| + 2\arcsin k + 2\alpha\Xi(|z|, \frac{1}{\alpha})$$

$$\le |\theta| + 2\arcsin k + \sqrt{\alpha^2 - 1}\log\frac{1 + |z|}{1 - |z|} + 2\arcsin|z|;$$

*where $\text{Arg}\,\partial_0 f(0) = 0$ and $\text{Arg}\,\partial_\theta f(z)$ is a continuous function of $z$ and $\theta$.*

This theorem, as well as the previous results imply the known results of Ch. Pommerenke ([1]) for $U_\alpha$ $(k = 0)$.

The definition of the order of a linearly-invariant family given by Ch. Pommerenke suggests the following

**Definition 4.** The order of a linearly-invariant family $\mathcal{H}$ of harmonic functions is defined as the number

$$\text{ord}\,\mathcal{H} = \sup_{f \in \mathcal{H}} \frac{1}{2}(|\partial\,\partial f(0) + \overline{\partial}\,\overline{\partial} f(0)|) = \sup_{f \in \mathcal{H}} |a_2(f) + a_{-2}(f)|$$

**Theorem 6.** $\operatorname{ord} H(\alpha, K) = \alpha K$.

**Proof.** Let $f \in H(\alpha, K)$, $f(z) = h(z) + \overline{g(z)}$, $h(z) = a_1\phi(z)$, $\phi(z) = z + \sum_{n=2}^{\infty} c_n z^n \in \mathcal{U}_\alpha$, $(a_1 = a_1(f))$; $g'(z)$ can be given in the form (see the proof of Theorem 2) $g'(z) = k\omega(z)h'(z)$. Thus

$$a_2 = a_2(f) = \frac{1}{2}h''(0) = a_1 c_2,$$

$$a_{-2} = a_{-2}(f) = \frac{1}{2}\overline{g''(0)} = \frac{1}{2}k \cdot (\overline{h''(0)} \cdot \overline{\omega(0)} + \overline{h'(0)} \cdot \overline{\omega'(0)})$$
$$= \frac{k}{2}\left(\overline{a_1} \cdot 2\overline{c_2} \cdot \frac{a_{-1}}{k\overline{a_1}} + \overline{a_1}\overline{\beta_1}(1 - |\omega(0)|^2)\right),$$

where $\beta_1$ is a complex number and $|\beta_1| \leq 1$. For our function $\omega$ there exists a regular function $\omega_0(z) = \beta_1 z + \cdots$, $|\omega_0(z)| \leq 1$, such that

$$\omega(z) = \frac{\omega(0) + \omega_0(z)}{1 + \overline{\omega(0)}\omega_0(z)} = \omega(0) + z\beta_1(1 - |\omega(0)|^2) + \cdots .$$

Thus

$$|a_2 + a_{-2}| = \left|a_1 c_2 + k \cdot \left(\frac{\overline{c_2}a_{-1}}{k} + \overline{a_1} \cdot \overline{\beta_1}\frac{1 - |\omega(0)|^2}{2}\right)\right|$$

$$\leq |a_1 c_2 + a_{-1}\overline{c_2}| + \frac{k}{2}|a_1|(1 - |\omega(0)|^2)$$

$$= \alpha(|a_1| + |a_{-1}|) + \frac{k}{2}|a_1|(1 - |\omega(0)|^2),$$

because $\phi \in \mathcal{U}_\alpha$ and $|c_2| \leq \alpha$. Since $|\omega(0)| = |a_{-1}|/|ka_1|$, we get

$$|a_2 + a_{-2}| \leq |a_1|\left[\alpha + \alpha \cdot k \cdot |\omega(0)| + \frac{k}{2}(1 - |\omega(0)|)^2\right].$$

Observe that the function $q(x) = \alpha + \alpha k x + \frac{k}{2}(1 - x^2)$ is increasing in $[0, 1]$. Using (4), we obtain

$$|a_2 + a_{-2}| \leq \frac{1}{1-k}q(1) = \alpha K.$$

Now, let us observe that for the function (7), with $\phi = \pm\frac{\pi}{2}$ we have

$$|a_2 + a_{-2}| = \left|\frac{\mp i\alpha}{1-k} - \frac{k(\pm i\alpha)}{1-k}\right| = \alpha K.$$

□

**Corollary 6.** *For all $f \in H(\alpha, K)$ and for all real $\theta$:*

$$\left|\frac{\partial_\theta \partial_\theta f(z)}{\partial_\theta f(z)}\right| \le \frac{2K(\alpha + |z|)}{1-|z|^2}.$$

*The inequality is sharp and the equality is attained for the function*

$$(10)\quad f(z) = h(z) + k\overline{h(z)}, \qquad h(z) = \frac{1}{2\alpha(1+k)}\left[\left(\frac{1+z}{1-z}\right)^\alpha - 1\right],$$

*and $z = r$, $\theta = \pm\frac{\pi}{2}$.*

**Proof.** Let $f \in H(\alpha, K)$ and let us consider

$$\psi(z) = f_\theta(z, a) = \sum_{n=1}^{\infty} b_n z^n + \sum_{n=1}^{\infty} b_{-n}\bar{z}^n \in H(\alpha, K).$$

Then

$$\partial\psi(z) = \frac{\partial f(e^{i\theta}\frac{z+a}{1+\bar{a}z})e^{i\theta}}{\partial_\theta f(ae^{i\theta})(1+\bar{a}z)^2},$$

$$\bar{\partial}\psi(z) = \frac{\bar{\partial} f(e^{i\theta}\frac{z+a}{1+\bar{a}z})\bar{e}^{i\theta}}{\partial_\theta f(ae^{i\theta})(1+a\bar{z})^2};$$

$$2b_2 = \partial\,\partial\psi(0) = \frac{e^{i\theta}}{\partial_\theta f(ae^{i\theta})}[\partial\partial f(ae^{i\theta})e^{i\theta}(1-|a|^2) + \partial f(ae^{i\theta})(-2\bar{a})],$$

$$2b_{-2} = \bar{\partial}\,\bar{\partial}\psi(0) = \frac{e^{-i\theta}}{\partial_\theta f(ae^{i\theta})}[\bar{\partial}\bar{\partial} f(ae^{i\theta})e^{-i\theta}(1-|a|^2) + \bar{\partial} f(ae^{i\theta})(-2a)],$$

$$|b_2 + b_{-2}| = \left|\frac{\partial_\theta\partial_\theta f(ae^{i\theta})}{\partial_\theta f(ae^{i\theta})}\frac{1-|a|^2}{2} - \frac{\partial f(ae^{i\theta})\bar{a}e^{i\theta} + \bar{\partial} f(ae^{i\theta})ae^{-i\theta}}{\partial f(ae^{i\theta})e^{i\theta} + \bar{\partial} f(ae^{i\theta})e^{-i\theta}}\right|.$$

Since $f_\theta(z, a) \in H(\alpha, K)$, we have $|b_2 + b_{-2}| \le \alpha K$ by Theorem 6. Thus

$$(11)\quad \left|\frac{\partial_\theta\partial_\theta f(ae^{i\theta})}{\partial_\theta f(ae^{i\theta})}\right|\frac{1-|a|^2}{2} \le \alpha K + |a|\left|\frac{1 + \frac{\bar{\partial} f(ae^{i\theta})}{\partial f(ae^{i\theta})}\frac{a}{\bar{a}}e^{-2i\theta}}{1 + \frac{\bar{\partial} f(ae^{i\theta})}{\partial f(ae^{i\theta})}e^{-2i\theta}}\right|$$
$$\le \alpha K + |a|K,$$

and from the above we get our inequality. For the function (10) we have

$$\frac{\partial_{\pm\pi/2}\partial_{\pm\pi/2}f(r)}{\partial_{\pm\pi/2}f(r)} = \frac{(\pm i)2K(\alpha+r)}{1-r^2}.$$

This ends our proof. □

Let us observe that for $a = |a|$ $(= r)$ we get from (11) the following sharp inequality

$$\left|\frac{\partial^2 f(re^{i\theta})/\partial r^2}{\partial f(re^{i\theta})/\partial r}\right| \leq 2\frac{\alpha K + r}{1-r^2},$$

which is better (in this case $\theta = \arg z$) then the inequality in Corollary 6. The estimates given in this paper are true for K-quasiconformal functions from $K_H$ for $\alpha = 2$ and from $C_H$ for $\alpha = 3$ (with the normalization (5) in these classes). But in the case of these classes sharpness of the estimaties is an open problem.

Some known results for $\mathcal{U}_\alpha$ have no counterparts for $H(\alpha, K)$. For example for all $\phi \in \mathcal{U}_\alpha$ and $\theta \in [0, 2\pi)$ the function

$$|\phi'(re^{i\theta})|\frac{(1-r)^{\alpha+1}}{(1+r)^{\alpha-1}}$$

is decreasing with respect to $r \in [0,1)$ (see [4]). But in $H(\alpha, K)$ we have no analogous result for $f'_r(z)$. The function $f(z) = h(z) + \overline{g(z)}$ with

$$h(z) = \frac{1}{2\alpha}\left[\left(\frac{1+z}{1-z}\right)^\alpha - 1\right], \qquad g'(z) = kzh'(z)e^{i\theta}$$

belongs to $H(\alpha, K)$, but the function

$$\left|\frac{\partial f}{\partial r}(r)\right| \frac{(1-r)^{\alpha+1}}{(1+r)^{\alpha-1}} = |1 + kre^{-i\theta}|$$

is not monotonic with respect to $r$ on some set of $\theta$. One can show that for almost all $\theta$ there exists a limit

$$\lim_{r\to 1^-}\left|\frac{\partial f}{\partial r}(re^{i\theta})\right| \frac{(1-r)^{\alpha+1}}{(1+r)^{\alpha-1}} = \delta_\theta \in [0, K],$$

if $f \in H(\alpha, K)$. One can show that for all $f \in H(\alpha, K)$ and real $\theta$ there exists a sequence $r_n \uparrow 1^-$ such that there exists a limit

$$\lim_{n\to\infty}\left|\frac{\partial f}{\partial r}(r_n e^{i\theta})\right| \frac{(1-r_n)^{\alpha+1}}{(1+r_n)^{\alpha-1}} \in [0, K].$$

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