ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN-POLONIA
VOL. XLIX, 12 SECTIO A 1995

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Harmonic and Quasiconformal Mappings which Agree on the Boundary

ABSTRACT. In this paper we discuss the deviation between harmonic and quasiconformal mappings of a given simply connected domain in the extended plane bounded by a Jordan curve Γ onto the unit disc. These mappings are assumed to have the same boundary values on Γ, and the deviation is expressed in terms of Euclidean and hyperbolic distances.

0. Introduction. Let $\mathbf{\Delta} := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbf{T} := \{z \in \mathbb{C} : |z| = 1\}$. Suppose $\Omega \subset \hat{\mathbb{C}}$ is an arbitrarily fixed simply connected domain bounded by a Jordan curve $\Gamma = \partial\Omega$. We will write $\mathbb{Q}(K;\Omega,\Omega')$ for the class of all K-quasiconformal mappings of Ω onto the domain $\Omega' \subset \hat{\mathbb{C}}$, $K \geq 1$. For any complex-valued function F on Ω we set $\hat{\partial} F(z) := \lim_{u \to z} F(u)$ if the limit exists as u approaches z in Ω and $\hat{\partial} F(z) := 0$ otherwise. It is well known that every $\varphi \in \mathbb{Q}(\Omega, \mathbf{\Delta}) := \bigcup_{1 \leq K < \infty} \mathbb{Q}(K;\Omega,\mathbf{\Delta})$ has a continuous extension to Γ and $\hat{\partial}\varphi$ is a sense-preserving homeomorphism of Γ onto $\mathbf{T}$; cf. [LV, p. 42]. On the other hand, by the eminent Radó-Kneser-Choquet theorem for convex domains, there exists a unique $\psi \in S_H(\Omega, \mathbf{\Delta})$

1991 *Mathematics Subject Classification.* 30C55, 30C62.

Key words and phrases. Harmonic mappings in the plane, Quasiconformal mappings in the plane, Special functions.

such that $\hat{\partial}\psi = \hat{\partial}\varphi$. Here $S_H(\Omega, \mathbf{\Delta})$ stands for the class of all sense-preserving univalent harmonic mappings of Ω onto $\mathbf{\Delta}$. It seems natural to compare the mappings φ and ψ in Ω. The problem makes sense, since φ and ψ do not differ if $\varphi \in \mathbb{Q}(1;\Omega,\mathbf{\Delta})$. The following topic is discussed: Estimate the distance $\mathrm{dist}(\varphi(\zeta),\psi(\zeta))$ for $\zeta \in \Omega$. In Section 1 we study the case of the Euclidean distance $\mathrm{dist}(z,w) := |z-w|$, $z, w \in \mathbb{C}$. In Section 2 we deal with the case of the hyperbolic distance $\mathrm{dist}(z,w) := \varrho(z,w)$, $z,w \in \mathbf{\Delta}$, defined by the metric density function $(1-|z|^2)^{-1}$ for $z \in \mathbf{\Delta}$. In Section 3 we give complementary remarks on the quantities which appear in two previous sections. These results were presented by the first named author on the conference "Planar harmonic mappings", Technion (Haifa), May 8-15, 1995.

1. An estimate of the Euclidean distance. Assume $\gamma : \Gamma \to \mathbf{T}$ is a sense-preserving homeomorphism of Γ onto $\mathbf{T}$. Then there exists the unique solution H_γ to the Dirichlet problem in Ω for the boundary function γ, i.e. H_γ is a complex-valued harmonic function on Ω satisfying $\hat{\partial}H_\gamma = \gamma$. If ψ is a conformal mapping of $\mathbf{\Delta}$ onto Ω then by the Radó-Kneser-Choquet theorem (cf. [R], [Kn], [C]) $H_\gamma \circ \psi \in S_H(\mathbf{\Delta},\mathbf{\Delta})$ and consequently $H_\gamma \in S_H(\Omega,\mathbf{\Delta})$. If $\Omega = \mathbf{\Delta}$ then γ is a homeomorphic self-mapping of $\mathbf{T}$ and H_γ has a simple form given by the Poisson integral

$$(1.1) \qquad H_\gamma(z) = \frac{1}{2\pi}\int_{\mathbf{T}} \gamma(u)\,\mathrm{Re}\,\frac{u+z}{u-z}\,|du|\,, \quad z \in \mathbf{\Delta}.$$

Lemma 1.1. *Suppose that $K \ge 1$ and $\varphi \in \mathbb{Q}(K;\Omega,\mathbf{\Delta})$. Then for every $\zeta \in \Omega$*

$$(1.2) \qquad |H_\gamma(\zeta) - \varphi(\zeta)| \le r(K) := 2\sin\left(\frac{\pi}{2}M(K)\right),$$

where $\gamma := \hat{\partial}\varphi$,

$$M(K) := 2\Phi^2_{\sqrt{K}}(1/\sqrt{2}) - 1\,, \quad K \ge 1\,,$$

and Φ_K is the Hersch-Pfluger distortion function; for the definition of Φ_K cf. [LV, p. 63]; *also cf.* [HP].

Proof. For $u \in \hat{\mathbb{C}}$ and $z \in \Delta$ write

$$h_z(u) := \frac{u - z}{1 - \bar{z}u} .$$

Given $K \geq 1$ suppose first that $\varphi \in \mathbb{Q}(K; \Delta) := \mathbb{Q}(K; \Delta, \Delta)$. Setting $a := \varphi(0)$ we get from (1.1)

$$\begin{aligned} |H_\gamma(0) - \varphi(0)| &= \left| \frac{1}{2\pi} \int_{\mathbf{T}} \gamma(u)|du| - \frac{1}{2\pi} \int_{\mathbf{T}} h_{-a}(e^{i\theta}u)|du| \right| \\ &\leq \frac{1}{2\pi} \int_{\mathbf{T}} |\gamma(u) - h_{-a}(e^{i\theta}u)||du| \end{aligned}$$

for every $\theta \in \mathbb{R}$. Hence, by [P1, Th. 1.4],

$$\begin{aligned} |H_\gamma(0) - \varphi(0)| &\leq \min_{\theta \in \mathbb{R}} \frac{1}{2\pi} \int_{\mathbf{T}} |\gamma(u) - h_{-a}(e^{i\theta}u)||du| \\ &\leq 2 \sin\left(\frac{\pi}{2} M(K)\right) . \end{aligned} \tag{1.3}$$

Consider now any $\zeta \in \Omega$ and let Φ be a conformal mapping of Ω onto Δ satisfying $\Phi(\zeta) = 0$. If $\varphi \in \mathbb{Q}(K; \Omega, \Delta)$ then $\varphi \circ \Phi^{-1} \in \mathbb{Q}(K; \Delta)$, $\varphi \circ \Phi^{-1}(0) = \varphi(\zeta)$ and $H_{\hat{\partial}(\varphi \circ \Phi^{-1})} = H_\gamma \circ \Phi^{-1}$. Replacing now γ by $\hat{\partial}(\varphi \circ \Phi^{-1})$ and φ by $\varphi \circ \Phi^{-1}$ we conclude from (1.3) that

$$\begin{aligned} |H_\gamma(\zeta) - \varphi(\zeta)| &= |H_\gamma(\Phi^{-1}(0)) - \varphi(\Phi^{-1}(0))| \\ &= |H_{\hat{\partial}(\varphi \circ \Phi^{-1})}(0) - \varphi \circ \Phi^{-1}(0)| \leq 2 \sin\left(\frac{\pi}{2} M(K)\right) , \end{aligned}$$

which completes the proof of (1.2). □

Lemma 1.2. *Suppose that $K \geq 1$ and $\varphi \in \mathbb{Q}(K; \Omega, \Delta)$. Then for every $\zeta \in \Omega$*

$$|H_\gamma(\zeta)| \leq R(K, |\varphi(\zeta)|) , \tag{1.4}$$

where $\gamma := \hat{\partial}\varphi$ *and*

$$(1.5)\qquad \begin{aligned} R(K,t) :&= \cos\left(2\frac{1-t}{1+t}\arccos\Phi_K\left(\frac{1}{\sqrt{2}}\right)\right) \\ &\le 1-2\left(\frac{1-t}{1+t}\right)^2\Phi_{1/K}^2\left(\frac{1}{\sqrt{2}}\right) \end{aligned}$$

for $K \ge 1$ *and* $0 \le t < 1$.

Proof. Fix $K \ge 1$ and assume first that $\varphi \in \mathbb{Q}(K;\mathbf{\Delta})$. Setting $a := \varphi(0)$ we see that $\psi := h_a \circ \varphi \in \mathbb{Q}(K;\mathbf{\Delta})$ and $\psi(0) = 0$. The harmonic measure ω is quasi-invariant in the sense that the inequality

$$(1.6)\qquad \frac{1}{K}\mu\left(\cos\left(\frac{\pi}{2}\omega(0,\mathbf{\Delta})[I]\right)\right)$$

$$\le \mu\left(\cos\left(\frac{\pi}{2}\omega(\psi(0),\mathbf{\Delta})[\hat{\partial}\psi(I)]\right)\right) \le K\mu\left(\cos\left(\frac{\pi}{2}\omega(0,\mathbf{\Delta})[I]\right)\right)$$

holds for every subarc I of $\mathbf{T}$; cf. [H]. Here μ stands for the module of the Grötzsch extremal domain $\mathbf{\Delta} \setminus [0,r]$; cf. [LV, p. 60]. Since $\psi(0) = 0$ and $2\pi\omega(0,\mathbf{\Delta})[I] = |I|_1$ for any arc $I \subset \mathbf{T}$, we conclude from (1.6) and the definition of Φ_K that

$$\Phi_{1/K}\left(\cos\frac{|I|_1}{4}\right) \le \cos\frac{|h_a\circ\gamma(I)|_1}{4} \le \Phi_K\left(\cos\frac{|I|_1}{4}\right).$$

Hence for every arc $I \subset \mathbf{T}$ of length $|I|_1 = \pi$

$$\Phi_{1/K}\left(\frac{1}{\sqrt{2}}\right) \le \cos\frac{|h_a\circ\gamma(I)|_1}{4} \le \Phi_K\left(\frac{1}{\sqrt{2}}\right),$$

and consequently

$$\begin{aligned} |\gamma(I)|_1 = |h_{-a}\circ h_a\circ\gamma(I)|_1 &= \int_{h_a\circ\gamma(I)} |h'_{-a}(z)||dz| \\ &\ge (1-|a|)(1+|a|)^{-1}|h_a\circ\gamma(I)|_1 \\ &\ge 4(1-|a|)(1+|a|)^{-1}\arccos\Phi_K\left(\frac{1}{\sqrt{2}}\right). \end{aligned}$$

By this

$$|\gamma(u)+\gamma(-u)| \le 2R(K,|a|)\ , \quad u\in \mathbf{T}\ ,$$

which implies, by (1.1),

$$\begin{aligned}|H_\gamma(0)| &= \frac{1}{4\pi}\left|\int_{\mathbf{T}}(\gamma(u)+\gamma(-u))|du|\right| \\ &\le \frac{1}{4\pi}\int_{\mathbf{T}}|\gamma(u)+\gamma(-u)||du| \le R(K,|a|)\ .\end{aligned} \tag{1.7}$$

Given $\zeta\in\Omega$ let Φ be a conformal mapping of Ω onto $\mathbf{\Delta}$ such that $\Phi(\zeta)=0$. If $\varphi\in\mathbb{Q}(K;\Omega,\mathbf{\Delta})$ then, as in the proof of Lemma 1.1, we conclude from (1.7) that

$$\begin{aligned}|H_\gamma(\zeta)| &= |H_\gamma(\Phi^{-1}(0))| = |H_{\hat\partial(\varphi\circ\Phi^{-1})}(0)| \\ &\le R(K,|\varphi\circ\Phi^{-1}(0)|) = R(K,|\varphi(\zeta)|)\ ,\end{aligned}$$

and the proof of (1.4) is complete. To prove the inequality in (1.5) we apply the identity

$$\Phi_K^2(t)+\Phi_{1/K}^2(\sqrt{1-t^2}) = 1\ , \quad 0\le t\le 1\ ; \tag{1.8}$$

cf. [AVV, Th. 3.3]. Then for all $0\le t<1$ and $K\ge 1$ we obtain

$$\begin{aligned}&\cos\left(2\frac{1-t}{1+t}\arccos\Phi_K\left(\frac{1}{\sqrt2}\right)\right) = 1-2\sin^2\left(\frac{1-t}{1+t}\arccos\Phi_K\left(\frac{1}{\sqrt2}\right)\right)\\ &\le 1-2\left[\frac{1-t}{1+t}\sin\left(\arccos\Phi_K\left(\frac{1}{\sqrt2}\right)\right)\right]^2 = 1-2\left(\frac{1-t}{1+t}\right)^2\Phi_{1/K}^2\left(\frac{1}{\sqrt2}\right),\end{aligned}$$

which completes the proof. □

For $\zeta\in\mathbb{C}$ and $r\ge 0$, write $\overline{\mathbf{\Delta}}(\zeta,r) := \{z\in\mathbb{C} : |z-\zeta|\le r\}$ and $\mathbf{T}(\zeta,r) := \{z\in\mathbb{C} : |z-\zeta| = r\}$. As an immediate conclusion from Lemmas 1.1 and 1.2 we obtain

Theorem 1.3. *Suppose that $K\ge 1$, $\varphi\in\mathbb{Q}(K;\Omega,\mathbf{\Delta})$ and $\gamma := \hat\partial\varphi$. Then for every $\zeta\in\Omega$*

$$H_\gamma(\zeta)\in\overline{\mathbf{\Delta}}(0,R(K,|\varphi(\zeta)|))\cap\overline{\mathbf{\Delta}}(\varphi(\zeta),r(K))\ .$$

This implies

Corollary 1.4. *Let $K \geq 1$, φ and γ be as in* Theorem 1.3. *Then for every $\zeta \in \Omega$*

$$|H_\gamma(\zeta) - \varphi(\zeta)| \leq \min\{r(K), R(K, |\varphi(\zeta)|) + |\varphi(\zeta)|\} . \tag{1.9}$$

In the case $\Omega = \mathbf{\Delta}$ we have

Corollary 1.5. *Suppose that $K \geq 1$ and that $\varphi \in \mathbb{Q}(K; \mathbf{\Delta})$ satisfies $\varphi(0) = 0$. Let $\gamma := \hat{\partial}\varphi$. Then for every $\zeta \in \mathbf{\Delta}$*

$$|H_\gamma(\zeta) - \varphi(\zeta)| \leq \min\{r(K), R(K, \Phi_K(|\zeta|)) + \Phi_K(|\zeta|)\} . \tag{1.10}$$

Proof. Given $K \geq 1$ assume that $\varphi \in \mathbb{Q}(K; \mathbf{\Delta})$ and $\varphi(0) = 0$. By the counterpart of Schwarz's lemma for quasiconformal self-mappings of the unit disk

$$|\varphi(z)| \leq \Phi_K(|z|) , \quad z \in \mathbf{\Delta} ; \tag{1.11}$$

cf. [LV, p. 64]. Combining (1.11) with (1.9) we obtain (1.10). □

2. An estimate of the hyperbolic distance. We recall that the hyperbolic distance $\varrho(\cdot, \cdot)$ is represented by the formula

$$\varrho(z, w) = \frac{1}{2} \log \frac{|1 - \overline{z}w| + |z - w|}{|1 - \overline{z}w| - |z - w|} , \quad z, w \in \mathbf{\Delta} . \tag{2.1}$$

An easy calculation shows that

$$|1 - \overline{z}w|^2 = (1 - |z|^2)(1 - |w|^2) + |z - w|^2 , \quad z, w \in \mathbf{\Delta} .$$

Thus (2.1) becomes

$$\varrho(z, w) = \log \frac{\sqrt{(1 - |z|^2)(1 - |w|^2) + |z - w|^2} + |z - w|}{\sqrt{1 - |z|^2}\sqrt{1 - |w|^2}} , \quad z, w \in \mathbf{\Delta} . \tag{2.2}$$

We are now in a position to prove

Theorem 2.1. *Suppose that* $K \geq 1$, $\varphi \in \mathbb{Q}(K;\Omega,\Delta)$ *and* $\gamma := \hat{\partial}\varphi$. *Then for every* $\zeta \in \Omega$

$$\begin{aligned}(2.3)\qquad &\varrho(H_\gamma(\zeta),\varphi(\zeta)) \\ &\leq \log \frac{\sqrt{(1-|\varphi(\zeta)|^2)(1-R^2(K,|\varphi(\zeta)|))+r^2(K)}+r(K)}{\sqrt{(1-|\varphi(\zeta)|^2)(1-R^2(K,|\varphi(\zeta)|))}}.\end{aligned}$$

In particular,

$$\begin{aligned}(2.4)\qquad \varrho(H_\gamma(\zeta),\varphi(\zeta)) &\leq \frac{1}{2}\log\frac{1-|\varphi(\zeta)|}{1+|\varphi(\zeta)|}\,\frac{1+|\varphi(\zeta)|+r(K)}{1-|\varphi(\zeta)|-r(K)} \\ &= \varrho(|\varphi(\zeta)|+r(K),0)-\varrho(|\varphi(\zeta)|,0)\end{aligned}$$

if $|\varphi(\zeta)|+r(K)\leq R(K,|\varphi(\zeta)|)$, *and*

$$\begin{aligned}(2.5)\qquad \varrho(H_\gamma(\zeta),\varphi(\zeta)) &\leq \frac{1}{2}\log\frac{1+|\varphi(\zeta)|}{1-|\varphi(\zeta)|}\,\frac{1+R(K,|\varphi(\zeta)|)}{1-R(K,|\varphi(\zeta)|)} \\ &= \varrho(|\varphi(\zeta)|,0)+\varrho(R(K,|\varphi(\zeta)|),0)\end{aligned}$$

if $|\varphi(\zeta)|+R(K,|\varphi(\zeta)|)\leq r(K)$.

Proof. Let $\varphi \in \mathbb{Q}(K;\Omega,\Delta)$, $K \geq 1$, and let $\zeta \in \Omega$. By (2.2) we have

$$\begin{aligned}(2.6)\qquad &\varrho(H_\gamma(\zeta),\varphi(\zeta)) \\ &= \log\frac{\sqrt{(1-|H_\gamma(\zeta)|^2)(1-|\varphi(\zeta)|^2)+|H_\gamma(\zeta)-\varphi(\zeta)|^2}+|H_\gamma(\zeta)-\varphi(\zeta)|}{\sqrt{1-|H_\gamma(\zeta)|^2}\sqrt{1-|\varphi(\zeta)|^2}}.\end{aligned}$$

Lemmas 1.1 and 1.2 now yield the estimate (2.3).

If $|\varphi(\zeta)|+r(K)\leq R(K,|\varphi(\zeta)|)$ then Lemma 1.1 gives $|H_\gamma(\zeta)|\leq |\varphi(\zeta)|+r(K)$, and the estimate (2.4) follows from (2.6) and (2.1).

If $|\varphi(\zeta)|+R(K,|\varphi(\zeta)|)\leq r(K)$ then Lemma 1.2 gives $|H_\gamma(\zeta)-\varphi(\zeta)|\leq|\varphi(\zeta)|+R(K,|\varphi(\zeta)|)$, and the estimate (2.5) follows from (2.6) and (2.1). □

Remark. The estimates (2.3), (2.4) and (2.5) can be deduced in an alternative way from Theorem 1.3 and from the conformal invariance of the hyperbolic distance.

Corollary 2.2. *Let $K \ge 1$, φ and γ be as in* Theorem 2.1. *Then for every $\zeta \in \Omega$*

(2.7)
$$\varrho(H_\gamma(\zeta),\varphi(\zeta)) \le \log\left(1+\frac{2r(K)}{\sqrt{(1-|\varphi(\zeta)|^2)(1-R^2(K,|\varphi(\zeta)|))}}\right)$$
$$\le \log\left(1+\frac{r(K)}{\Phi_K\left(\frac{1}{\sqrt{2}}\right)\Phi_{1/K}\left(\frac{1}{\sqrt{2}}\right)}\frac{(1+|\varphi(\zeta)|)^{1/2}}{(1-|\varphi(\zeta)|)^{3/2}}\right).$$

Proof. The first inequality in (2.7) is a consequence of (2.3) and the trivial inequality $\sqrt{a+b} \le \sqrt{a}+\sqrt{b}$ for $a,b \ge 0$. Combining (1.5) with (1.8) we obtain

$$1-R^2(K,t) = \sin^2\left(2\frac{1-t}{1+t}\arccos\Phi_K\left(\frac{1}{\sqrt{2}}\right)\right)$$
$$\ge \left(\frac{1-t}{1+t}\right)^2 \sin^2\left(2\arccos\Phi_K\left(\frac{1}{\sqrt{2}}\right)\right)$$
$$= \left(2\frac{1-t}{1+t}\Phi_K\left(\frac{1}{\sqrt{2}}\right)\Phi_{1/K}\left(\frac{1}{\sqrt{2}}\right)\right)^2$$

for all $K \ge 1$ and $0 \le t < 1$. This implies the second inequality in (2.7). □

In the special case $\varphi(\zeta) = 0$, Theorem 2.1 is reduced to

Corollary 2.3. *Suppose that $K \ge 1$, $\zeta \in \Omega$ and that $\varphi \in \mathbb{Q}(K;\Omega,\mathbf{\Delta})$ satisfies $\varphi(\zeta) = 0$. Let $\gamma := \partial\varphi$. Then*

$$(2.8)\quad \varrho(H_\gamma(\zeta),0) \le \varrho(R(K,0),0) = \log\frac{\Phi_K(1/\sqrt{2})}{\Phi_{1/K}(1/\sqrt{2})} = \frac{1}{2}\log\lambda(K)\ ;$$

for the definition of the λ-distortion function cf. [LV, p. 81].

Proof. Given $K \ge 1$ and $\zeta \in \Omega$ assume that $\varphi \in \mathbb{Q}(K;\Omega,\mathbf{\Delta})$ and $\varphi(\zeta) = 0$. As observed by J. Zając (oral communication), from [P2, Th. 1.1 and (2.3)] it follows that

$$2M(K) = 2\max_{0\le t\le 1}(\Phi_K^2(\sqrt{t})-t) \ge 2\left(\Phi_K^2\left(\frac{1}{\sqrt{2}}\right)-\frac{1}{2}\right) = M(K^2)\ .$$

Hence

$$R(K,0) = 2\Phi_K^2\left(\frac{1}{\sqrt{2}}\right) - 1 = M(K^2) \le 2M(K)$$
$$\le 2\sin\left(\frac{\pi}{2}M(K)\right) = r(K).$$

This shows, by (2.5) and (1.8), that

$$\varrho(H_\gamma(\zeta),0) \le \varrho(R(K,0),0) = \frac{1}{2}\log\frac{\Phi_K^2(1/\sqrt{2})}{1-\Phi_K^2(1/\sqrt{2})}$$
$$= \frac{1}{2}\log\frac{\Phi_K^2(1/\sqrt{2})}{\Phi_{1/K}^2(1/\sqrt{2})} = \log\frac{\Phi_K(1/\sqrt{2})}{\Phi_{1/K}(1/\sqrt{2})}\,.$$

The identity

$$\lambda(K) = \frac{\Phi_K^2(1/\sqrt{2})}{\Phi_{1/K}^2(1/\sqrt{2})}, \quad K \ge 1\,,$$

(cf. [AVV, (1.8)]) completes the proof of (2.8). □

If $\Omega = \mathbf{\Delta}$ we obtain by Corollary 2.2 the following

Corollary 2.4. *Suppose that $K \ge 1$ and that $\varphi \in \mathbb{Q}(K;\mathbf{\Delta})$ satisfies $\varphi(0) = 0$. Let $\gamma := \hat{\partial}\varphi$. Then for every $\zeta \in \mathbf{\Delta}$*

$$\varrho(H_\gamma(\zeta),\varphi(\zeta)) \le \log\left(1 + \frac{2\sin\left(\frac{\pi}{2}M(K)\right)}{\Phi_K\left(\frac{1}{\sqrt{2}}\right)\Phi_{1/K}\left(\frac{1}{\sqrt{2}}\right)}\frac{(1+\Phi_K(|\zeta|))^{1/2}}{(1-\Phi_K(|\zeta|))^{3/2}}\right)$$

$$\le \log\left(1 + \frac{\pi M(K)}{\Phi_K\left(\frac{1}{\sqrt{2}}\right)\Phi_{1/K}\left(\frac{1}{\sqrt{2}}\right)}\Phi_{1/K}^{-3/2}\left(\frac{1-|\zeta|}{1+|\zeta|}\right)\right) \tag{2.9}$$

$$= \log\left(1 + \frac{\pi M(K)}{\Phi_K\left(\frac{1}{\sqrt{2}}\right)\Phi_{1/K}\left(\frac{1}{\sqrt{2}}\right)}\Phi_{1/K}^{-3/2}\left(e^{-2\varrho(\zeta,0)}\right)\right).$$

Proof. The first inequality in (2.9) follows from (2.7) and (1.11). To obtain the second one we apply the identities (1.8) and

$$\Phi_{1/K}\left(\frac{1-t}{1+t}\right) = \frac{1-\Phi_K(t)}{1+\Phi_K(t)}, \quad 0 \le t \le 1\,;$$

cf. [AVV, Theorem 3.3]. □

3. Complementary remarks. For $K > 0$ and $0 \le x \le 1$, write $h(x) := (1-x)(1+x)^{-1}$ and $\phi_K(x) := \min\{4^{1-1/K} x^{1/K}, 1\}$. Define

$$\begin{aligned} &\Phi_0[K,t](x) := \Phi_t \circ \phi_K \circ \Phi_{1/t}(x)\,, \\ &\Phi_1[K,t](x) := h \circ \Phi_0[1/K,t] \circ h(x)\,, \quad K>0\,, \\ &\Phi[K,t](x) := \begin{cases} \min\{\Phi_0[K,t](x)\,,\ \Phi_1[K,t](x)\} & ,\ K \ge 1 \\ \max\{\Phi_0[K,t](x)\,,\ \Phi_1[K,t](x)\} & ,\ 0 < K \le 1 \end{cases} \end{aligned}$$

for $0 \le x \le 1$, $t > 0$. All estimates obtained so far depend on the function Φ_K which can be approximated by the sequence $\Phi[K, 2^n]$, $n = 0, 1, \cdots$, with arbitrarily preassigned accuracy. Furthermore, for all $K \ge 1$, $0 \le x \le 1$ and $n = 0, 1, 2, \cdots$

$$\Phi_K(x) \le \Phi[K, 2^n](x) \quad \text{and} \quad \Phi_{1/K}(x) \ge \Phi[1/K, 2^n](x)\,, \tag{3.1}$$

and $\Phi[K, 2^n]$ lies closer to Φ_K step by step as $n \to \infty$ for any $K > 0$; cf. [P1, Remark 1]. Note that Φ_t are elementary functions for $t = 2^n$, $n \in \mathbb{Z}$; cf. [LV, p. 64]. Therefore the inequalities (3.1) enable us to express any estimate involving the function Φ_K, $K > 0$, by means of elementary functions $\Phi[K, 2^n]$, $n = 0, 1, \cdots$.

For $K \ge 1$ and $0 \le t \le 1$, write

$$p(K,t) := \begin{cases} \left(\dfrac{1 + 4^{1-1/K} t^{1/K}}{1 - 4^{1-1/K} t^{1/K}} \right)^{3/2} & \text{as } 1 - 4^{1-1/K} t^{1/K} > 0 \\ +\infty & \text{otherwise} \end{cases}.$$

Applying (3.1) with e.g. $n = 0$ we can now rephrase Corollary 2.4 as follows.

Corollary 3.1. *Suppose that $K \ge 1$ and that $\varphi \in \mathbb{Q}(K; \Delta)$ satisfies $\varphi(0) = 0$. Let $\gamma := \hat{\partial}\varphi$. Then for every $\zeta \in \Delta$*

$$\begin{aligned} \varrho(H_\gamma(\zeta), \varphi(\zeta)) &\le \log\left(1 + C(K) \min\left\{p(K, |\zeta|),\ 2^{3(K-1)} p^K(1, |\zeta|)\right\}\right) \\ &\le \log\left(1 + 2^{3(K-1)} C(K) e^{3K\varrho(\zeta,0)}\right)\,, \end{aligned}$$

where

$$C(K) \le \pi 4^{K-1} 2^{K/2} (32^{1-1/\sqrt{K}} - 1) 2^{1/(2K)}\,.$$

Consider finally two examples.

Example 3.2. Given $K \geq 1$ set $k := (K-1)/(K+1)$. For $\zeta \in \Delta$, define $\varphi_k(\zeta) := (\zeta - k)/(1-\overline{\zeta}k)$ and $\psi_K(\zeta) := \zeta\,|\zeta|^{(K-1)}$. An easy calculation shows that φ_k and ψ_K are K-quasiconformal self-mappings of Δ and $\hat{\partial}\varphi_k(z) = \hat{\partial}\psi_K(z) = z$ for $z \in \mathbb{T}$. Setting $a := \varphi_k(\zeta)$ we have

$$\zeta - k = a - ak\overline{\zeta} \quad \text{and} \quad \overline{\zeta} - k = \overline{a} - \overline{a}k\zeta\,.$$

Hence

$$\zeta = \frac{a(1-k^2) + k(1-|a|^2)}{1-|a|^2k^2}\,.$$

Assume that $0 \leq a < 1$. Then

$$\zeta = \frac{k+a}{1+ka} = \frac{(1+a)K - 1 + a}{(1+a)K + 1 - a}\,.$$

On the other hand, $\psi_K(a^{1/K}) = a$. Since $H_{\hat{\partial}\varphi_k}(z) = z$, $z \in \Delta$, we obtain the following lower bound for the function R:

(3.2)
$$\max\left\{|a|^{1/K}\,, \frac{(1+|a|)K - 1 + |a|}{(1+|a|)K + 1 - |a|}\right\}$$
$$\leq \max\{|H_{\hat{\partial}\varphi}(\zeta)| : \zeta \in \Delta\,,\ \varphi \in \mathbb{Q}(K;\Delta)\,,\ \varphi(\zeta) = a\} \leq R(K,|a|)\,.$$

In particular,

$$\frac{K-1}{K+1} \leq \max\{|H_{\hat{\partial}\varphi}(\zeta)| : \zeta \in \Delta\,, \varphi \in \mathbb{Q}(K;\Delta)\,,\ \varphi(\zeta) = 0\} \leq M(K^2)\,.$$

Evaluating the maximal value of $|z - \varphi_k(z)|$ on Δ we obtain

$$2\frac{\sqrt{K}-1}{\sqrt{K}+1} \leq \sup\{|H_{\hat{\partial}\varphi}(\zeta) - \varphi(\zeta)| : \zeta \in \Delta\,, \varphi \in \mathbb{Q}(K;\Delta)\} \leq r(K)\,.$$

Furthermore, by (1.5) and (1.8)

$$R(K,|a|) \leq 1 + \left(\frac{1-|a|}{1+|a|}\right)^2 (R(K,0) - 1)\,.$$

Hence by (3.2)

$$R(K,|a|)-R(K,0)\le\left(1-\left(\frac{1-|a|}{1+|a|}\right)^2\right)(1-R(K,0))$$

$$\le\frac{4|a|}{(1+|a|)^2}\left(1-\frac{K-1}{K+1}\right)=\frac{8|a|}{(1+|a|)^2(K+1)},$$

and consequently we obtain the following upper bound for the function R:

$$R(K,|a|)\le M(K^2)+\frac{8|a|}{(1+|a|)^2(K+1)}.$$

Example 3.3. Fix $0<t<1$. In [T], (also cf. [Kü, p. 59]), Teichmüller constructed a K-quasiconformal mapping φ_t of $\mathbf{\Delta}$ onto itself such that

(i) $\varphi_t(0)=-t$;
(ii) $\hat{\partial}\varphi_t(z)=z$ for $z\in\mathbf{T}$;
(iii) $K=\coth^2(\mu(t)/2)$.

Here μ stands for the module of the Grötzsch extremal domain $\mathbf{\Delta}\setminus[0,r]$. Given $K\ge 1$ we determine $t=\check{\mu}(2\operatorname{arc}\coth\sqrt{K}\,)$, where $\check{\mu}$ denotes the inverse mapping of μ. Since $H_{\hat{\partial}\varphi_t}(0)=0$, we have

$$\check{\mu}(2\operatorname{arc}\coth\sqrt{K}\,)\le\sup\{|H_{\hat{\partial}\varphi}(\zeta)-\varphi(\zeta)|:\zeta\in\mathbf{\Delta}\,,\\ \varphi\in\mathbb{Q}(K;\mathbf{\Delta})\}\le r(K)$$

and

$$\frac{1}{2}\log\frac{1+\check{\mu}(2\operatorname{arc}\coth\sqrt{K}\,)}{1-\check{\mu}(2\operatorname{arc}\coth\sqrt{K}\,)}\\ \le\sup\{\varrho(H_{\hat{\partial}\varphi}(\zeta),\varphi(\zeta)):\zeta\in\mathbf{\Delta}\,,\,\varphi\in\mathbb{Q}(K;\mathbf{\Delta})\}\,.$$

Since $h_{-t}\circ\varphi_t\in\mathbb{Q}(K;\mathbf{\Delta})$, $h_{-t}\circ\varphi_t(0)=0$ and $H_{\hat{\partial}(h_{-t}\circ\varphi_t)}(0)=h_{-t}(0)=t$, we obtain

$$\check{\mu}(2\operatorname{arc}\coth\sqrt{K}\,)\le\max\{|H_{\hat{\partial}\varphi}(\zeta)|:\zeta\in\mathbf{\Delta}\,,\,\varphi\in\mathbb{Q}(K;\mathbf{\Delta}),\\ \varphi(\zeta)=0\}\le M(K^2)\,.$$

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