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V. K. JAIN (Kharagpur, India)

On Cauchy's Type Bounds for Zeros of a Polynomial

ABSTRACT. In this paper we have obtained two bounds of Cauchy's type for zeros of a polynomial. In many cases these bounds are much better than some other known bounds.

1. Introduction and statement of results. The following results due to Cauchy [1] are well known in the theory of distribution of zeros of polynomials.

Theorem A. *All the zeros of the polynomial* $P(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n$ *, satisfy*

$$|z| \le R\,, \tag{1.1}$$

where R is the positive root of the equation

$$z^n - |a_{n-1}| z^{n-1} - |a_{n-2}| z^{n-2} - \cdots - |a_1| z - |a_0| = 0\,.$$

Theorem B. *All the zeros of the polynomial* $P(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n$ *lie in the disk*

$$|z| < 1 + Q^n\,, \tag{1.3}$$

where

$$(1.4) \qquad Q = \left\{ \max_{1 \le k \le n-1} |a_k| \right\}^{1/n} .$$

In the literature, there already exist ([4, Theorem 1], [2, Theorem 1]) many improvements of the bound (1.3). In [3], we improved the bound (1.3) considerably for the polynomial $P(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n$, with $a_{n-1} = 0$, $a_{n-2} = 0$, and obtained the following result.

Theorem C. *All the zeros of the polynomial* $P(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n$ *with* $a_{n-1} = 0$, $a_{n-2} = 0$ *lie in the disk*

$$(1.5) \qquad |z| < R_1$$

where R_1 *is the positive root of the equation* $x^3 - x^2 - Q^n = 0$.

In this paper, we obtain two bounds of Cauchy's type, which, in many cases, are much better than other known bounds. First bound is similar to the bound (1.1) and involves positive roots of three different equations of smaller degree than that of (1.1) – obviously a great advantage. Second bound is for the polynomial $P(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n$, with $a_{n-1} = a_{n-2} = 0$ and is similar to the bound (1.3) – an advantage over the bound (1.5). More precisely, we prove

Theorem 1. *All the zeros of the polynomial*

$$P(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n \qquad (n \ge 3),$$

lie in the disk

$$(1.6) \qquad |z| \le \max\{ R_{n-k-2} , R'_{k-l-1} , r_1 \} ,$$

where $k \in \{1, 2, \ldots, n-2\}$, $l \in \{0, 1, \ldots, k-1\}$, R_{n-k-2} *is the unique positive root of the equation*

$$z^{n-k-1} - |a_{n-1}| z^{n-k-2} - |a_{n-2}| z^{n-k-3} - \cdots - (1 + |a_{k+1}|) = 0 ,$$

R'_{k-l-1} is the unique positive root of the equation

$$z^{k-1} - |a_k|z^{k-l-1} - |a_{k-1}|z^{k-l-2} - \cdots - (1 + |a_{l+1}|) = 0 \,,$$

and r_1 is the unique positive root of the equation

$$z^{l+1} - |a_l|z^l - |a_{l-1}|z^{l-1} - \cdots - |a_0| = 0 \,.$$

Theorem 2. *All the zeros of the polynomial*

$$P(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n$$

with $a_{n-1} = a_{n-2} = 0$, lie in the disk

$$|z| < 2^{2/9}(1 + Q^n)^{1/3} \,, \tag{1.7}$$

except for $Q > 1$, $|a_j| = Q^n$ for some j, $0 \leq j \leq n-3$, and $|a_i| \leq \alpha = 2^{1/3} - 1$, $\forall i \,(\neq j) \in \{0, 1, 2, \ldots, n-3\}$, in which case, all the zeros of $P(z)$ lie in the disk

$$|z| < (1 + Q^n)^{1/3} \,. \tag{1.8}$$

Remark 1. In many cases, one obtains better bound by Theorem 1, than those given by other results. Consider the polynomial

$$P_5(z) = z^5 + a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0 \,,$$

with $|a_4| = 0.8$, $|a_3| = 0.1$, $|a_2| = 0.5$, $|a_1| = 0.3$, $|a_0| = 0.7$. Let $k = 2$ and $l = 1$. Then

$$R_{5-2-2} = R_1 < 1.525$$
$$R'_{2-1-1} = R'_0 = 1.5$$
$$r_1 = 1$$

and so, by Theorem 1, all the zeros of $P_5(z)$ lie in

$$|z| < 1.525 \,,$$

where as all the zeros of $P_5(z)$ lie in

$$|z| < 1.8\,, \quad \text{(by Theorem B)},$$
$$|z| < 1.75\,, \quad \text{(by [4, Theorem 1])},$$
$$|z| < 1.76\,, \quad \text{(by [2, Theorem 1])}\,.$$

Remark 2. A great advantage of Theorem 1 is that one can repeat the process suggested in the theorem, as many times as one wishes.

Remark 3. In many cases, one gets better bound by Theorem 2, than those given by other results. Consider the polynomial

$$P_5(z) = z^5 + a_2 z^2 + a_1 z + a_0\,,$$

with $|a_2| = 6$, $|a_1| = 2$, $|a_0| = 8$. By Theorem 2, all the zeros of $P_5(z)$ lie in

$$|z| < 2.426\,,$$

where as all the zeros of $P_5(z)$ lie in the disks

$$|z| < 9\,, \quad \text{(by Theorem B)}\,,$$
$$|z| < 3.37\,, \quad \text{(by [4, Theorem 1])}\,,$$
$$|z| < R \approx 9\,, \quad \text{(by [2, Theorem 1])}\,.$$

2. Lemma.

Lemma 1. *Let* α *be as in* Theorem 2 *and* $Q > 1$. *Then for* $|z| \geq (1 + Q^n)^{1/3}$,

$$\frac{\alpha}{|z|^n} + \frac{\alpha}{|z|^{n-1}} + \cdots + \frac{\alpha}{|z|^4} + \frac{Q^n}{|z|^3} < 1\,. \tag{2.1}$$

Proof. For $|z| = (1 + Q^n)^{1/3}$ the inequality (2.1) can be equivalently written as

$$\frac{\alpha}{|z|^n} + \frac{\alpha}{|z|^{n-1}} + \cdots + \frac{\alpha}{|z|^4} + \frac{|z|^3 - 1}{|z|^3} < 1\,,$$

i.e.

$$\frac{\alpha}{|z|^{n-3}} + \cdots + \frac{\alpha}{|z|} < 1 \,. \tag{2.2}$$

Now, for $|z| = (1+Q^n)^{1/3}$,

$$\frac{\alpha}{|z|^{n-3}} + \cdots + \frac{\alpha}{|z|} < \alpha\left\{\left(\frac{1}{2^{1/3}}\right)^{n-3} + \cdots + \frac{1}{2^{1/3}}\right\}$$

$$= \alpha\, \frac{\frac{1}{2^{1/3}}\left[1 - \left(\frac{1}{2^{1/3}}\right)^{n-3}\right]}{1-\frac{1}{2^{1/3}}} = 1 - \left(\frac{1}{2^{1/3}}\right)^{n-3} < 1 \,,$$

and so, this completes the proof of inequality (2.1) for $|z| = (1+Q^n)^{1/3}$. Furthermore, since

$$\frac{\alpha}{|z|^n} + \cdots + \frac{\alpha}{|z|^4} + \frac{Q^n}{|z|^3} < 1$$

is a monotonically decreasing function of $|z|$, we get inequalits (2.1) for $|z| \geq (1+Q^n)^{1/3}$.

3. Proofs of the Theorems.

Proof of Theorem 1. We have

$$|P(z)| \geq |z|^n - \sum_{j=0}^{n-1} |a_j||z|^j$$

$$= \begin{cases} |z|^n - |z|^{k+1} + \left(|z|^{k+1} - \sum_{j=0}^{k} |a_j||z|^j\right) \\ \quad -\left(\sum_{j=k+1}^{n-1} |a_j||z|^j\right), \quad 0 < k \leq n-2, \\ |z|^{k+1}\Big\{|z|^{n-k-1} - |a_{n-1}||z|^{n-k-2} - |a_{n-2}||z|^{n-k-3} \\ \quad - \cdots - (1+|a_{k+1}|)\Big\} \\ \quad + \Big\{|z|^{k+1} - |z|^{l+1} + \left(|z|^{l+1} - \sum_{j=0}^{l} |a_j||z|^j\right) \\ \quad - \sum_{j=l+1}^{k} |a_j||z|^j\Big\}, \quad 0 \leq l < k \,. \end{cases}$$

Thus

$$P(z) \geq |z|^{k+1}\Big\{|z|^{n-k-1} - |a_{n-1}||z|^{n-k-2} - |a_{n-2}||z|^{n-k-3}$$
$$- \cdots - (1 + |a_{k+1}|)\Big\} + |z|^{l+1}\Big\{|z|^{k-l} - |a_k||z|^{k-l-1}$$
$$- |a_{k-1}||z|^{k-l-2} - \cdots - (1 + |a_{l+1}|)\Big\}$$
$$+ \Big(|z|^{l+1} - \sum_{j=0}^{l} |a_j||z|^j\Big\} > 0,$$

if

$$|z| > \max\{R_{n-k-2}\,,\, R'_{k-l-1}\,,\, r_1\}\,.$$

This ends the proof of Theorem 1.

Proof of Theorem 2. We first consider the exceptional case. Let z_0 be a zero of $P(z)$ such that $|z_0| > 1$. So, we have

(3.1)
$$|z_0|^n \leq |a_0| + |a_1||z_0| + \cdots + |a_{j-1}||z_0|^{j-1}$$
$$+ |a_j||z_0|^j + \cdots + |a_{n-3}||z_0|^{n-3}$$
$$\leq \alpha + \alpha|z_0| + \cdots + Q^n|z_0|^j + \cdots + \alpha|z_0|^{n-3}$$
$$\leq \alpha + \alpha|z_0| + \cdots + \alpha|z_0|^j + \cdots + \alpha|z_0|^{n-4} + Q^n|z_0|^{n-3}\,,$$

as $Q^n > \alpha$ and $|z_0|^{n-3} \geq |z_0|^j$. Now, by (3.1), we get

$$1 \leq \frac{\alpha}{|z_0|^n} + \frac{\alpha}{|z_0|^{n-1}} + \cdots + \frac{\alpha}{|z_0|^4} + \frac{Q^n}{|z_0|^3}\,. \tag{3.2}$$

By (3.2) and Lemma 1 we can say that

$$|z_0| < (1 + Q^n)^{1/3}\,.$$

Hence, all the zeros of $P(z)$ lie in the disk

$$|z| < (1 + Q^n)^{1/3}\,.$$

In general case we consider

$$f(x) = x^3 - Q^n\,, \qquad h(x) = x^2\,,$$

and the expression

$$\begin{aligned}&\left\{f(2^{2/9}(1+Q^n)^{1/3})\right\}^3-\left\{h(2^{2/9}(1+Q^n)^{1/3})\right\}^3\\&=\left[3+2\cdot 2^{2/3}-4\cdot 2^{1/3}\right]Q^{3n}+\left[12+3\cdot 2^{2/3}-14\cdot 2^{1/3}\right]Q^{2n}\\&+\left[12-10\cdot 2^{1/3}\right]Q^n+\left[4-2\cdot 2^{1/3}\right]\\&=b_3y^3+b_2y^2+b_1y+b_0=g(y)\,,\end{aligned}$$

where $y=Q^n$. We easily obtain

$$g''(y)>0 \quad \text{for} \quad y\geq 1\,, \qquad g'(1)>0\,, \quad g(1)>0\,,$$

and so, we get

$$g(y)>0\,, \quad \text{for} \quad y\geq 1\,. \tag{3.3}$$

Further, we observe that

$$b_2<0\,, \quad b_1<0\,, \quad b_0>0\,, \quad b_3>0\,.$$

Hence, for $0<y<1$,

$$b_0+b_1y+b_2y^2>b_0+b_1+b_2>0\,,$$

which implies

$$g(y)>0\,, \quad \text{for} \quad 0<y<1\,. \tag{3.4}$$

Hence, by (3.3) and (3.4),

$$g(y)>0\,, \quad \text{for} \quad y>0\,,$$

which implies that

$$\left\{f(2^{2/9}(1+Q^n)^{1/3})\right\}^3-\left\{h(2^{2/9}(1+Q^n)^{1/3})\right\}^3>0$$

i.e.

$$(x^3-Q^n)^3>(x^2)^3\,, \quad \text{for} \quad x=2^{2/9}(1+Q^n)^{1/3}\,,$$

or

$$x^3-Q^n>x^2\,, \qquad \text{for} \qquad x=2^{2/9}(1+Q^n)^{1/3}. \tag{3.5}$$

Now, by (3.5) and Theorem C, we can say that all the zeros of $P(z)$ lie in the disk

$$|z|<2^{2/9}(1+Q^n)^{1/3}.$$

This completes the proof of Theorem 2.

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Mathematics Department
Indian Institute of Technology
Kharagpur, India

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