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Applications of the Idea of Möbius Invariance to Obtain Equivalent Definitions of Bloch Function

ABSTRACT. In this paper we investigate the Bloch class $\mathcal{B}$ of functions ($|g'(z)| = \mathcal{O}((1-|z|^2)^{-1})$, for $z \in D$) and "the little" Bloch class $\mathcal{B}_0$ ($\max_{|z|=r} |g'(z)| = o((1-|z|^2)^{-1}$, for $|z| \to 1^-$). Applying the idea of Möbius invariance we give a few conditions equivalent to the definitions of $\mathcal{B}$ and $\mathcal{B}_0$.

1. Introduction. A function g holomorphic in the unit disc D is called a Bloch function if

$$|g'(z)| = \mathcal{O}\Big(\frac{1}{1-|z|^2}\Big),$$

for $z \in D$. The space of all Bloch functions is denoted by $\mathcal{B}$ and

$$\|g\|_{\mathcal{B}} := |g(0)| + \sup_{|z|<1} (1-|z|^2)|g'(z)| < \infty.$$

Moreover, let $\mathcal{B}(0) = \{g \in \mathcal{B} : g(0) = 0\}$ and $\mathcal{B}_0$ be the family of functions g holomorphic in D such that

$$\max_{|z|=r} |g'(z)| = o\Big(\frac{1}{1-|z|^2}\Big),$$

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for $|z| \to 1^-$. Obviously $\mathcal{B}_0 \subset \mathcal{B}$.

For $a \in D$ let the Möbius function $\phi_a : D \longrightarrow D$ be defined by

$$\phi_a(z) = \frac{a+z}{1+\overline{a}z}, \text{ for } z \in D.$$

If f is a function locally univalent in D then the order of f is defined as follows

$$\operatorname{ord} f = \sup_{a\in D} \left| \left\{ \frac{f(\phi_a(z)) - f(a)}{f'(a)(1-|a|^2)} \right\}_2 \right|,$$

where $\{h(z)\}_2$ denotes the second Taylor coefficient of the function $h(z) = z + \dots$.

The universal linearly invariant (or universal Möbius invariant) family U_α (see [5]) is the class of all functions $f(z) = z + \dots$ holomorphic in D such that:

1^0 $f'(z) \neq 0$ in D,
2^0 $\operatorname{ord} f \leq \alpha$.

In this paper we give equivalent definitions of the Bloch classes. In Section 2 we prove necessary and sufficient conditions for a holomorphic function in D to be in $\mathcal{B}$. In Section 3 we deal with the class $\mathcal{B}_0$.

We start with a result which gives us the relationship between the Bloch space and the universal Möbius invariant family.

Lemma 1.1. *The folowing equality holds:* $\mathcal{B}(0) = \{\log f' : f \in \bigcup_{\alpha<\infty} U_\alpha\}$. *Moreover, if* $g = \log f' \in \mathcal{B}(0)$ *and ord* $f = \alpha$, *then* $2(\alpha - 1) \leq \|g\|_{\mathcal{B}} \leq 2(\alpha+1)$.

Proof. Let $g(z) = \log f'(z)$ and $\operatorname{ord} f = \alpha$. Then $\left| \frac{f''(z)}{f'(z)} \frac{1-|z|^2}{2} \right| \leq \alpha + 1$, ([5; Lemma 1.2]). Thus $(1-|z|^2)|g'(z)| \leq 2(\alpha+1)$, and this proves that $g \in \mathcal{B}(0)$ and $\|g\|_{\mathcal{B}} \leq 2(\alpha+1)$.

Assume now that $g \in \mathcal{B}(0)$. Let

$$f(z) = \int_0^z \exp g(s) ds.$$

Note that $f(0)=0$, $f'(0)=1$ and consequently

$$\alpha = \operatorname{ord} f = \sup_{z\in D}\left|\frac{1-|z|^2}{2}\frac{f''(z)}{f'(z)} - \overline{z}\right|$$
$$= \sup_{z\in D}\left|\frac{1-|z|^2}{2}g'(z) - \overline{z}\right| \leq \frac{\|g\|_{\mathcal{B}}}{2} + 1 .$$

This proves our Lemma. □

Now, let us give a few examples extending some known results on U_α to the Bloch class.

Let us denote by $\mathcal{I}_\alpha$ a family of all complex-valued functions μ of bounded variation on $[0,2\pi]$ such that:

(i) $$\int_0^{2\pi} d\mu(t) = 1,$$

(ii) $$\sup_{|z|<1} \left|\int_0^{2\pi} \frac{z+e^{it}}{1+\overline{z}e^{it}} d\mu(t)\right| \leq \alpha, \quad \text{where } \alpha \geq 1.$$

Corollary 1.1. *Let g be a function holomorphic in D. Then $g \in \mathcal{B}$ if and only if there exist $\alpha \geq 1$ and a sequence $\mu_n \in \mathcal{I}_\alpha$ such that*

$$g(z) - g(0) = \lim_{n\to\infty} \int_0^{2\pi} (-2\log(1-ze^{it}))d\mu_n(t) ,$$

where

$$\alpha = \operatorname{ord} \int_0^z \exp(g(s)-g(0))ds .$$

Proof. It was shown in [8] that $f \in U_\alpha$ if and only if there exists a sequence $\mu_n \in \mathcal{I}_\alpha$ such that

$$f'(z) = \lim_{n\to\infty} \exp[-2\int_0^{2\pi} \log(1-ze^{it})d\mu_n(t)].$$

Using Lemma 1.1 we get our result. □

Now, let for $x \in [0,1)$, $q \in [-1,1]$

$$\Xi(x,q) = \int_0^x \frac{\sqrt{1-q^2t^2}}{1-t^2}dt$$

$$= \frac{1}{2}\sqrt{1-q^2}\log\frac{\sqrt{1-q^2x^2}+x\sqrt{1-q^2}}{\sqrt{1-q^2x^2}-x\sqrt{1-q^2}} + q\arcsin x$$

$$\leq \frac{1}{2}\sqrt{1-q^2}\log\frac{1+x}{1-x} + \arcsin x\,.$$

In [5] Ch. Pommerenke proved the following inequality for a function $f \in U_\alpha$:

$$\left|\operatorname{Re}\left\{e^{i\lambda}\log((1-|z|^2)f'(z))\right\}\right| \leq 2\alpha\,\Xi\left(|z|, \frac{\sin\lambda}{\alpha}\right).$$

Observe that the function $\alpha\Xi(|z|, \frac{\sin\lambda}{\alpha})$ is increasing with respect to α. Hence using Lemma 1.1, we obtain the following

Corollary 1.2. *Let $g \in \mathcal{B}$, λ be a real number and $\alpha = \operatorname{ord}\int_0^z \exp(g(s)$ $-g(0))ds$. Then for $z \in D$*

$$|\operatorname{Re}\{(g(z)-g(0))e^{-i\lambda}\} + \cos\lambda Log(1-|z|^2)| \leq 2\Xi\big(|z|, \frac{\sin\lambda}{\alpha}\big)$$

$$\leq \sqrt{1-\frac{\sin^2\lambda}{\alpha^2}}\log\frac{1+|z|}{1-|z|} + 2\arcsin|z|\,.$$

Moreover, the function $g(z)-g(0)$ maps the disc $\{z : |z| \leq r\}$ into a domain with the boundary $2\alpha e^{i\lambda}\Xi(r, \frac{\sin\lambda}{\alpha}) - \log(1-|z|^2)$, where $\lambda \in [0, 2\pi]$.

From Corollary 2.2 in [5] and Lemma 1.1 we obtain the following result.

Corollary 1.3. *Let $g \in \mathcal{B}$, λ-a real number and $\alpha = \operatorname{ord} \int_0^z \exp(g(s)$ $-g(0))ds$. Then for $z_1, z_2 \in D$, we have*

$$\left|\operatorname{Re}\left\{e^{-i\lambda}\left[g(z_2) - g(z_1) + \log\frac{1-|z_2|^2}{1-|z_1|^2} + 2i\arg(1-\overline{z_1}z_2)\right]\right\}\right|$$

$$\leq 2\alpha\, \Xi\left(\left|\frac{z_2 - z_1}{1-\overline{z_1}z_2}\right|, \frac{\sin\lambda}{\alpha}\right).$$

In [8] it was shown that for all $f \in U_\alpha$ the functions

$$|f'(re^{i\theta})|\frac{(1-r)^{\alpha+1}}{(1+r)^{\alpha-1}} \quad \text{and} \quad \max_{|z|=r}|f'(z)|\frac{(1-|z|)^{\alpha+1}}{(1+|z|)^{\alpha-1}}$$

with every $\theta \in [0, 2\pi]$, are decreasing for r, $r \in (0,1)$ and they have limits (as $r \to 1^-$) belonging to $[0,1]$. Thus, after differentiation and applying Lemma 1.1 we get the following

Corollary 1.4. *Let $f \in \mathcal{B}$. Then, for every $\theta \in [0, 2\pi]$ the following functions*

$$\operatorname{Re}[g(re^{i\theta}) - g(0)] + (\alpha+1)\log(1-r) - (\alpha-1)\log(1+r)$$

and

$$\max_\theta \operatorname{Re}[g(re^{i\theta}) - g(0)] + (\alpha+1)\log(1-r) - (\alpha-1)\log(1+r)$$

are decreasing for $r \in (0,1)$ and have limits (as $r \to 1^-$) which are less than or equal to 0. Moreover, the limits can be equal 0 for a fixed α only for the function

$$g(z) = g(0) - (\alpha+1)\log(1 - ze^{i\theta}) + (\alpha-1)\log(1+ze^{i\theta}).$$

Here $\alpha = \operatorname{ord} \int_0^z \exp[g(s) - g(0)]ds$.

2. The class $\mathcal{B}$.

Theorem 2.1. *Let g be a function holomorphic in D. Then $g \in \mathcal{B}$ if and only if there exists a positive constant $C(g)$ such that for all $z \in D$*

$$\sup_{a\in D} |g(\phi_a(z)) - g(a) - 2\log(1+\overline{a}z)| \leq C(g)\log\frac{1+r}{1-r} - \log(1-r^2)\,, \tag{2.1}$$

where $r = |z|$. Here $C(g) = \operatorname{ord}\int_0^z \exp[g(s) - g(0)]ds$ and this constant is the best.

Proof. It is enough to prove the theorem for the class $\mathcal{B}(0)$.

1^0. Let $g \in \mathcal{B}(0)$. By Lemma 1.1 there exists a function $f \in U_\alpha$ and $\alpha = \operatorname{ord} f$ such that $g(z) = \log f'(z)$. Since the family U_α is Möbius invariant, we have

$$F(z) = \int_0^z \frac{f'(\phi_a(s))}{f'(a)(1+\overline{a}s)^2}ds \in U_\alpha$$

for all $a \in D$. For the function F we have ([5]) the following inequality:

$$\left|\log(F'(z)(1-|z|^2))\right| \leq \alpha\log\frac{1+|z|}{1-|z|}.$$

Thus

$$\left|\log\frac{f'(\phi_a(z))\,(1-|z|^2)}{f'(a)(1+\overline{a}z)^2}\right| \leq \alpha\log\frac{1+|z|}{1-|z|}$$

and consequently

$$\left|g\left(\phi_a(z)\right) - g(a) - 2\log(1+\overline{a}z) + \log(1-|z|^2)\right| \leq \alpha\log\frac{1+|z|}{1-|z|}.$$

2^0. Now, let a holomorphic function g satisfy (2.1) (with $g(0) = 0$). If we put $f(z) = \int_0^z \exp g(s)ds$, then by (2.1) we have

$$\left|\log\frac{f'(\phi_a(z))}{f'(a)(1+\overline{a}z)^2}\right| \leq C(g)\log\frac{1+r}{1-r} - \log(1-r^2),$$

for $|z| = r$. Thus

$$|\mathrm{Re}\{\log f'(\phi_a(z)) - \log f'(a) - 2\log(1+\overline{a}z)\}|$$
$$\leq C(g)\log\frac{1+r}{1-r} - \log(1-r^2)\,.$$

From the last inequality (after differenting with respect to r at the point $z = re^{i\eta} = 0$) we get

$$-C(g) \leq \mathrm{Re}\{(\frac{f''(a)}{f'(a)}\frac{1-|a|^2}{2} - \overline{a})e^{i\eta}\} \leq C(g)$$

and consequently

$$\left|\frac{f''(a)}{f'(a)}\frac{1-|a|^2}{2} - \overline{a}\right| \leq C(g)\,.$$

Hence $\mathrm{ord} f = \alpha$. By Lemma 1.1 we get $g(z) = \log f'(z) \in \mathcal{B}(0)$. □

It is clear that the inequality in Theorem 2.1 can be written in an equivalent form

$$\sup_{a\in D} |g(\phi_a(z)) - g(a)| \leq K_g \log\frac{1}{1-|z|}\,,$$

where the best constant $K_g = \|g(z) - g(0)\|_{\mathcal{B}}$, (see [2], p.230).

For $z, a \in D$, the pseudo-hyperbolic distance $d(a,z)$ between a and z is defined by

$$d(a,z) = |\phi_a(z)|\,.$$

For $a \in D$ and $r \in (0,1)$, the pseudo-hyperbolic disc $D(a,r)$ with (pseudo-hyperbolic centre a and pseudo-hyperbolic radius r) is defined by

$$D(a,r) = \{z \in D : d(a,r) < r\}\,.$$

Since ϕ_a is a fractional linear transformation, the pseudo-hyperbolic disc $D(a,r)$ is also an Euclidean disc. Except for the special case $D(a,r) = rD$, the Euclidean and pseudo-hyperbolic radii and centres do not coincide. The measure of $D(a,r)$ with respect to the normalized area $\frac{dA}{\pi}$ will be denoted by $|D(a,r)|$.

Corollary 2.1. *Let $0 < r < 1$. Then, for functions g holomorphic in D, the following conditions are equivalent:*
i) $g \in \mathcal{B}$,
ii) *There exists a constant C_g such for all $a \in D$ and all $z \in D(a,r)$*

$$|g(z) - g(a)| \le C_g \ .$$

Let us observe that for $p \ge 1$,

$$\sup_{a\in D}\left[\frac{1}{|D(a,r)|}\int_{D(a,r)}|g(z)-g(a)|^p dA(z)\right]^{1/p}$$
$$\le \sup_{a\in D}\left[\frac{C_g^p}{|D(a,r)|}|D(a,r)|\right]^{1/p} = C_g \ .$$

Thus by Theorem 1 from [2] we get $g \in \mathcal{B}$.

Our next result depends upon the following

Lemma 2.1. *Let $\phi(\zeta)$ and $\zeta = \Phi(z)$ be functions holomorphic in their domains and let the function $\phi(\Phi(z))$ be defined. Moreover, let $j_1, \dots, j_l, m_1, \dots, m_l$ be nonnegative integers and $k = m_1 + \dots + m_l$. Then all components of the derivative*

$$\frac{d^n}{dz^n}\left[\phi^{(k)}(\Phi(z))\left(\Phi^{(j_1)}(z)\right)^{m_1}\left(\phi^{(j_2)}(z)\right)^{m_2}\cdots\left(\Phi^{(j_l)}(z)\right)^{m_l}\right]$$

have the form (up to constants)

$$\phi^{(k+p)}(\Phi(z))\left(\Phi^{(i_1)}(z)\right)^{n_1}\left(\Phi^{(i_2)}(z)\right)^{n_2}\cdots\left(\Phi^{(i_q)}(z)\right)^{n_q},$$

where $p = 0, 1, \dots, n$ and $n_1 + \dots + n_q = k + p$.

Proof. It is enough to prove the Lemma in the case $n = 1$. For $n \ge 2$ the result follows by induction.

We have

$$\frac{d}{dz}[\phi^{(k)}(\Phi(z))\prod_{\alpha=1}^{l}(\Phi^{(j_\alpha)}(z))^{m_\alpha}]$$

$$= \phi^{(k+1)}(\Phi(z))\Phi'(z)\prod_{\alpha=1}^{l}(\Phi^{(j_\alpha)}(z))^{m_\alpha}$$

$$+\phi^{(k)}(\Phi(z))\sum_{\beta=1}^{l}\Big[\Big(\prod_{\substack{\alpha=1\\ \alpha\neq\beta}}^{l}(\Phi^{(j_\alpha)}(z))^{m_\alpha} m_\beta(\Phi^{(j_\beta)}(z))^{m_\beta-1}\Phi^{(j_\beta+1)}(z)\Big]\,.$$

□

Now, for a positive integer n, let us define $\mathcal{B}^{(n)}$ as the family of all holomorphic functions g in D such that

$$|g^{(n)}(z)| = \mathcal{O}\Big(\frac{1}{(1-|z|^2)^n}\Big)\,.$$

The following result is well known ([2],[10]), but we give an alternative proof.

Theorem 2.2. *$\mathcal{B}^{(n)} = \mathcal{B}$, for each positive integer n.*

Proof.

1^0. First, we show by induction that $\mathcal{B}^{(n)} \subset \mathcal{B}$. For $n = 1$ the theorem is true by the definition.

Now, let us assume that we have proved the inclusion for a positive integer n and we will prove it for $n+1$.

Let

$$|g^{(n+1)}(z)| \le \frac{K_{n+1}(g)}{(1-|z|^2)^{n+1}}\,,$$

for $z \in D$. Then, for $z = re^{i\phi}$

$$|g^{(n)}(z) - g^{(n)}(0)| \le \int_0^r |g^{(n+1)}(te^{i\phi})|dt \le \int_0^r \frac{K_{n+1}(g)}{(1-t^2)^{n+1}}dt$$

$$= K_{n+1}(g)\Big(\int_0^r \frac{t}{(1-t^2)^{n+1}}dt + \int_0^r \frac{1-t}{(1-t^2)^{n+1}}dt\Big)$$

$$\le 2K_{n+1}(g)\frac{1}{(1-r^2)^n}\,.$$

Thus there exists a constant $K_n(g)$ such that

$$|g^{(n)}(z)| \le \frac{K_n(g)}{(1-|z|^2)^n}$$

in D and by assumption $g \in \mathcal{B}$.

2^0. Let $g(z) = \sum_{n=0}^{\infty} c_n z^n \in \mathcal{B}$. It is known ([6]) that $|c_n| \le 2\,\|g\|_{\mathcal{B}}$. We will again use induction.

For $n = 1$ our result is true. Let us assume that for some positive n

$$(1-|z|^2)^n |g^{(n)}(z)| \le K(g,n) ,$$

for $z \in D$. From the proof of Theorem 2.1 it follows that the function

$$\psi_a(z) = g(\phi_a(z)) - g(a) - 2\log(1+\overline{a}z)$$

belongs to $\mathcal{B}$.

We have

$$\psi_a'(z) = g'(\phi_a(z))\phi_a'(z) - \frac{2\overline{a}}{1+\overline{a}z}$$

and

$$|\phi_a^{(k)}(0)| = k!(1-|a|^2) .$$

By Lemma 2.1 the following expression

$$\psi_a^{(n+1)}(z) + \left(\frac{2\overline{a}}{1+\overline{a}z}\right)^{(n)} - g^{(n+1)}(\phi_a(z))(\phi_a'(z))^{n+1}$$

$$= \psi_a^{(n+1)}(z) + 2(\overline{a})^{n+1} n!(-1)^n (1+\overline{a}z)^{-n-1} - g^{(n+1)}(\phi_a(z))(\phi_a'(z))^{n+1}$$

is a sum (with $N(n)$ components) of functions of the form (up to constants)

$$g^{(1+p)}(\phi_a(z)) \prod_{\alpha=1}^{q} (\phi_a^{(j_\alpha)}(z))^{m_\alpha} ,$$

where $\sum_{\alpha=1}^{q} m_\alpha = 1+p$ and p is an integer satisfying $0 \le p \le n-1$. By our assumption we get

$$|g^{(1+p)}(\phi_a(0)) \prod_{\alpha=1}^{q} (\phi_a^{(j_\alpha)}(0))^{m_\alpha}|$$

$$= |g^{(1+p)}(a)| \prod_{\alpha=1}^{q} ((j_\alpha)!(1-|a|^2))^{m_\alpha}$$

$$\le ((n+1)!)^{n+1} |g^{(1+p)}(a)|(1-|a|^2)^{1+p}$$

$$\le ((n+1)!)^{n+1} K_{p+1}(g) .$$

Thus

$$|g^{(n+1)}(\phi_a(0))(\phi_a'(0))^{n+1}| \le |\psi_a^{(n+1)}(0)| + 2|a|^{n+1} n!$$

$$+N(n)((n+1)!)^{n+1} \max_{p\in\{0,1,\dots,n-1\}} K_{p+1}(g)$$

which is equivalent to

$$|g^{(n+1)}(a)|(1-|a|^2)^{n+1}$$

$$\le 2\,\|g\|_{\mathcal{B}}\,(n+1)! + 2n! + N(n)((n+1)!)^{n+1}$$

$$\times \max_{p\in\{0,1,\dots,n-1\}} K_{p+1}(g) := K_{n+1}(g)\,.$$

This proves that $g \in \mathcal{B}^{(n)}$. □

3. The class $\mathcal{B}_0$. In this section we give some results concerning the class $\mathcal{B}_0$.

Theorem 3.1. *Let g be a function holomorphic in D. Then $g \in \mathcal{B}_0$ if and only if there exists a function $e(r,|a|)$ defined on $[0,1)\times[0,1)$ such that*

(i) $e(0,|a|) = 0$,

(ii) *there exists the right hand side derivative $\partial e/\partial r(0,|a|)$ and $\partial e/\partial r(0,|a|) \to 0$, as $|a| \to 1^-$, and*

$$|g(\phi_a(z)) - g(a)| \le e(r,|a|)\,, \tag{3.1}$$

for all z, $|z| = r < 1$.

Proof.

1^0. Assume $g \in \mathcal{B}_0$. Let us denote

$$e(|z|) = \max_{|z|=r} |g'(z)|(1-|z|)\,.$$

By our assumptions $e(|z|) \to 0$ as $|z| \to 1^-$. Using the Schwarz lemma we get

$$|\frac{d}{dz}[g(\phi_a(z)) - g(a)]|(1-|z|) \le 2e(|\phi_a(z)|)$$

for all $a \in D$. Thus

$$|g(\phi_a(z)) - g(a)| = \left|\int_0^z \frac{d}{ds} g(\phi_a(s))ds\right|$$
$$\le \int_0^r \frac{2e(|\phi_a(s)|)}{1-|s|} d|s| \le 2\max_{|z|\le r} e(|\phi_a(s)|)(-\log(1-r))$$
$$= 2\max_{|z|\le r} e(|\phi_{|a|}(z)|)(-\log(1-r)).$$

We denote by $e(r,|a|)$ the last term and we put $e(0,|a|) = 0$. Then we have

$$\left|\frac{\partial}{\partial r} e(0,|a|)\right| = 2\lim_{r\to 0^+}\left[\frac{-\log(1-r)}{r}\max_{|z|\le r} e(|\phi_{|a|}(z)|)\right] = 2e(|a|).$$

Moreover, $e(|a|) \to 0$ as $|a| \to 1^-$.

2^0. Now let us assume that there exists a function $e(r,|a|)$ as in the theorem. For $r = 0$ we have the equality in (3.1). Thus

$$\frac{d}{dr}_{|r=0} |g(\phi_a(z)) - g(a)| \le \frac{\partial}{\partial r}_{|r=0} e(r,|a|). \tag{3.2}$$

Since

$$\frac{d}{dr}|g(\phi_a(z)) - g(a)| = |g(\phi_a(z)) - g(a)|\frac{d}{dr}\operatorname{Re}\log(g(\phi_a(z)) - g(a)),$$

we get

$$\frac{d}{dr}_{|r=o} |g(\phi_a(z)) - g(a)| = |g'(a)|(1-|a|^2)$$

if $g'(a) \neq 0$. If $g'(a) = 0$, the result is true, too. Thus (1.2) is equivalent to

$$|g'(a)|(1-|a|^2) \le \frac{\partial}{\partial r}_{|r=0} e(r,|a|)\ .$$

Moreover the last expression tends to 0 as $|a| \to 1$. This proves that $g \in \mathcal{B}_0$. □

Remark 3.1. If $g \in \mathcal{B}_0$ and $e(r) = \max_{0\le r<1}|g'(r)|(1-r)$, then we can choose

$$e(r,|a|) = -2\log(1-r)\sup_{|z|<r} e(|\phi_{|a|}(z)|)\ .$$

Corollary 3.1. *Let $0 < r < 1$. Then, for functions g holomorphic in D, the following conditions are equivalent:*
(i) $g \in \mathcal{B}_0$,
(ii) $\max_{z\in D(a,r)} |g'(z)|(1-|z|^2) \to 0$, as $|a| \to 1^-$,
iii) $\max_{z\in D(a,r)} |g(z)-g(a)| \to 0$, as $|a| \to 1^-$.

Proof. We show forst that (i) and (ii) are equivalent.

Let $g \in \mathcal{B}_0$ and let $r_a = \max_{z\in D(a,r)} |z|$. Let us observe that $r_a \to 1^-$ as $|a| \to 1^-$. Thus

$$\max_{z\in D(a,r)} [(1-|z|^2)|g'(z)|] \le \max_{|z|\le r_a} [(1-r^2)|g'(z)|] \to 0\,,$$

as $|a| \to 1^-$.

Now, assume that $g \notin \mathcal{B}_0$. Then there exists a sequence $z_n \in D$, $|z_n| \to 1^-$, such that

$$(1-|z_n|^2)|g'(z_n)| \to K > 0\,.$$

However,

$$\max_{z\in D(a,r))} [(1-|z|^2)|g'(z)|] \le (1-|z_n|^2)|g'(z_n)|$$

and this contradicts (ii).

Now, we show that (i) and (iii) are equivalent. Let us first assume that $g \in \mathcal{B}_0$. By Theorem 3.1 there exists $e(r,|a|)$ such that

$$\max_{z\in D(a,r)} |g(z)-g(a)| \le e(r,|a|) \to 0$$

as $|a| \to 1^-$.

Suppose (iii) holds and $e(|a|) = \max_{z\in D(a,r)} |g(z)-g(a)| \to 0$ as $|a| \to 1^-$. Then for p, $p \ge 1$:

$$\left[\frac{1}{|D(a,r)|}\int_{D(a,r)} |g(z)-g(a)|^p dA(z)\right]^{1/p}$$

$$\le \left[\frac{e^p(|a|)}{|D(a,r)|}|D(a,r)|\right]^{1/p} = e(|a|) \to 0$$

as $|a| \to 1^-$. Thus by [2; Th.2] we get $g \in \mathcal{B}_0$. □

We will need the following lemmas.

Lemma 3.1. *Let g be a holomorphic function in D. Then $g \in \mathcal{B}_0$ if and only if $g(\phi_a(z) \in \mathcal{B}_0$ for each $a \in D$.*

Proof. Let $g \in \mathcal{B}_0$. Let us denote $G_a(z) = g(\phi_a(z))$, and

$$\epsilon(r) = \max_{|z|=r} |g'(z)|(1-r^2) .$$

Then

$$|G_a'(z)|(1-|z|^2) = |g'(\phi_a(z))| \frac{1-|a|^2}{|1+\overline{a}z|^2}(1-|z|^2)$$

$$= |g'(\phi_a(z))|(1-|\phi_a(z)|^2) \leq \epsilon(|\phi_a(z)|) .$$

Thus

$$\max_{|z|=r} |G_a'(z)|(1-r^2) \leq \max_{|z|=r} \epsilon(|\phi_a(z)|)$$

$$= \max\left\{ \epsilon(r) : r \in \left[\left| \frac{|a|-r}{1-|a|r} \right|, \frac{|a|+r}{1+|a|r} \right] \right\} .$$

Since $\lim_{r\to 1^-} \epsilon(r) = 0$ we have proved that $G_a \in \mathcal{B}_0$.

The rest of the proof is trivial. □

Lemma 3.2. *Let $g \in \mathcal{B}_0$, $a \in D$ and $G_a(z) := g(\phi_a(z)) = \sum_{n=0}^{\infty} c_n(a) z^n$. Then for each nonnegative integer n we have*

$$\max_{|a|=\rho} |c_n(a)| \to 0$$

as $\rho \to 1^-$.

Proof. We have

$$n|c_n(a)| = \frac{1}{2\pi} \left| \int_{|z|=r} \frac{g'(\phi_a(z)) \frac{1-|a|^2}{(1+\overline{a}z)^2}}{z^n} dz \right|$$

$$\leq \frac{\max_t |g'(\phi_a(re^{it}))|(1-|a|^2)}{r^{n-1}(1-|\overline{a}|r)^2} .$$

Since $\max_t |\frac{re^{it}+a}{1+\overline{a}re^{it}}| = \frac{r+|a|}{1+|a|r}$, we get

$$\max_t |g'(\frac{re^{it}+a}{1+\overline{a}re^{it}})| \leq \max_t |g'(\frac{r+|a|}{1+|a|r} e^{it})| .$$

Let $\epsilon(r)$ will be as in the proof of Lemma 3.1. Then, for fixed $r \in (0,1)$, we obtain

$$n|c_n(a)| \leq \frac{\epsilon(\frac{r+|a|}{1+|a|r})}{r^{n-1}(1-r)^2}\frac{(1+r|a|)^2}{1-r^2} \leq \frac{1+r}{r^{n-1}(1-r)^3}\epsilon(\frac{r+|a|}{1+|a|r}) .$$

Letting $|a| \to 1$ we get $\epsilon(\frac{r+|a|}{1+|a|r}) \to 0$ and this completes the proof of the Lemma. □

Now, for a positive integer n, let us introduce the class $\mathcal{B}_0^{(n)}$ as a family of all holomorphic functions g in D such that

$$\max_{|z|=r} |g^{(n)}(z)| = o(\frac{1}{(1-|z|^2)^n})$$

as $|z| \to 1^-$. The next result , similarly as Theorem 2.2, is well known ([2],[10]), but we give an alternative proof.

Theorem 3.2. *$\mathcal{B}_0^{(n)} = \mathcal{B}_0$, for each positive integer n.*

Proof.

1^0 Using induction we show that $\mathcal{B}_0^{(n)} \subset \mathcal{B}_0$. For $n=1$ this is true by definition. Now, let us assume that

$$\max_{|z|=r} |g^{(n+1)}(z)|(1-|z|^2)^{n+1} \to 0 ,$$

for $|z| \to 1^-$ which is equivalent to

$$\max_{|z|=r} |g^{(n+1)}(z)|(1-|z|)^{n+1} := \epsilon(r) \to 0$$

for $r=|z| \to 1^-$. The function $\epsilon(r)$ is continuous in $[0,1)$. Let us choose a sequence (r_m) such that $\lim_{m\to\infty} r_m = 1$, $r_m \in (0,1)$. Moreover, let $r_m < r < 1$ for a fixed m and $\epsilon_m = \max_{t\in[r_m,r]} \epsilon(t)$, $z = re^{i\phi}$, $z_m = r_m e^{i\phi}$ for a fixed ϕ. Then

$$|g^{(n)}(z) - g^{(n)}(z_m)| \leq \int_{r_m}^{r} |g^{(n+1)}(te^{i\phi})|dt$$

$$\leq \int_{r_m}^{r} \frac{\epsilon(t)}{(1-t)^{n+1}}dt \leq \epsilon_m \int_{r_m}^{r} \frac{dt}{(1-t)^{n+1}} \leq \frac{\epsilon_m}{n(1-r)^n}.$$

Thus

$$|g^{(n)}(re^{i\phi})|(1-r)^n \leq |g^{(n)}(r_m e^{i\phi})|(1-r)^n + \frac{\epsilon_m}{n}$$

and

$$\max_{|z|=r} |g^{(n)}(z)|(1-r)^n \leq \max_{\phi\in[0,2\pi]} |g^{(n)}(r_m e^{i\phi})|(1-r)^n + \frac{\epsilon_m}{n}.$$

Hence

$$\limsup_{r\to 1^-}(\max_{|z|=r} |g^{(n)}(z)|(1-r)^n) \leq \frac{\epsilon_m}{n}.$$

Since $\lim_{m\to\infty} \epsilon_m = 0$, we have $g \in \mathcal{B}_0$.

2^0. Let $g \in B_0$. We use induction. For $n = 1$ the result is true. Let us assume that for some positive integer n

$$\max_{|z|=r}(1-|z|^2)^n|g^{(n)}(z)| = o(1)$$

as $|z| \to 1$. By Lemma 3.1 the function $G_a(z) = g(\phi_a(z)) \in B_0$ for each $a \in D$. Similarly, as in the proof of Theorem 2.2, let us observe that

$$G_a^{(n+1)}(0) - g^{(n+1)}(\phi_a(0))(\phi_a'(0))^{n+1}$$

is a sum of terms of the form (up to constants)

$$g^{(1+p)}(\phi_a(0)) \prod_{\alpha=1}^{q} (\phi_a^{(j_\alpha)}(0))^{m_\alpha},$$

where $\sum_{\alpha=1}^{q} m_\alpha = 1+p$. By assumption we get

$$|g^{(1+p)}(\phi_a(0))| \prod_{\alpha=1}^{q} |g^{(j_\alpha)}(0)|^{m_\alpha} \leq ((n+1)!)^{n+1}|g^{(1+p)}(a)|(1-|a|^2)^{1+p}$$

$$\leq ((n+1)!)^{n+1}\epsilon_{p+1}(|a|),$$

where $\epsilon_{p+1}(|a|) \to 0$ for $|a| \to 1$.

Thus

$$|g^{(n+1)}(\phi_a(0))||\phi_a'(0)|^{n+1} = |g^{(n+1)}(a)|(1-|a|^2)^{n+1}$$

$$\leq |G_a^{(n+1)}(0)| + N(n)((n+1)!)^{n+1}\epsilon(|a|) ,$$

where

$$\epsilon(|a|) = \max_{p\in\{1,\dots,n\}} \epsilon_p(|a|) \to 0$$

as $|a| \to 1$. From Lemma 3.2 we have

$$\max_{|a|=\rho} |G_a^{(n+1)}(0)| \to 0$$

as $\rho \to 1$.

Thus

$$\max_{|a|=\rho} |g^{(n+1)}(a)|(1-|a|^2)^{n+1} = o(1)$$

as $\rho \to 1$. □

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