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Regularity and Almost Sure Convergence

ABSTRACT. We give a sufficient condition for almost sure convergence in the sense of [1] to be equivalent to almost uniform convergence.

Preliminaries. In the section we collect basic facts on the theory of non-commuta- tive L^p-spaces associated with an arbitrary von Newumann algebra. For details cf. [4].

Let M be a σ- finite von Neumann algebra with a faithful normal state ω_0. The Hilbert space $H = L^2(M, \omega_0)$ is the completion of M under the norm $x \mapsto \omega_0(x^*x)^{1/2} = ||x||_2$. In the sequel, we assume that M acts in a standard way on the Hilbert space H with a cyclic and separating vector ζ_0 such that $\omega_0(x) = (x\zeta_0, \zeta_0)$ for $x \in M$. We identify M with the subset $M\zeta_0 = \{x\zeta_0 : x \in M\}$ of H. We denote by N the crossed product $R(M, \sigma^{\omega_0})$ of M by the modular automorphism group σ^{ω_0} associated with ω_0 (see [2]). Then N admits the dual action $\{\Theta_s\}, s \in \mathbb{R}$, and the faithful normal semifinite trace τ satisfying $\tau \circ \Theta_s = e^{-s}\tau$, $s \in \mathbb{R}$ (see [2]). The topological $^*-$ algebra of all closed densely defined (affiliated with N) locally measurable operators with respect to N is denoted by $\overline{N}$ (see [6]). The dual actions $\{\Theta_s\}$, $s \in \mathbb{R}$, are extended to continuous $^*-$ automorphisms of $\overline{N}$. Let $L^p(M)$, $0 < p < \infty$, denote the Haagerup spaces associated with ω_0 and M [4]. It is known that M acts in a standard way in $H = L^2(M) \subset \overline{N}$ and

$\omega_0(x) = (xh_0^{1/2}, h_0^{1/2})_H = \text{tr}(h_0 x) = \text{tr}(xh_0) = \text{tr}(h_0^{1/2}\, xh_0^{1/2})$ (M is identified with its isomorphic image in N). Note that $h_0 \in L^1(M)_+$ (the set of positive self- adjoint operators from $L^1(M)$).

For $1 \le p < \infty$, $L^p(M)$ is a Banach space and its dual space is $L^q(M)$, where $1/p + 1/q = 1$. The duality is given by the following bilinear form:

$$(h, g) \mapsto \text{tr}(hg) = \text{tr}(gh)\,, \quad h \in L^p(M)\;,\; g \in L^q(M)\;.$$

The (quasi-)norm of $L^p(M)$ for $0 < p < \infty$ is defined by $||h||_p = \text{tr}(|h|^p)^{1/p}$, $h \in L^p(M)$. The space $L^p(M)$ is independet of the choice of a faithful normal state on M up to an isomorphism. Furthermore, if M has a faithful normal semifinite trace τ_0 , then $L^p(M)$ can be identified with the non–commutative $L^p(M, \tau_0)$ - space introduced in [3]. Let $\hat{\omega}_0 = \tau(h_0 \cdot)$ stand for the weight dual to ω_0 (see [2], [4]).

2. Regularity and almost sure convergence.

Definition 2.1 ([5]). Assume that N is a von Neuman algebra, whereas τ is a faithful normal semifinite trace on N . We say that a weight $\varphi = \tau(h\cdot)$ defined on N_+ (the set of positive self–adjoint operators from N) is regular if the operator h^{-1} is locally measurable with respect to N (that is, xh^{-1} is closable for each $x \in N$).

For some $\zeta \in H = L^2(M, \omega_0)$ and an orthogonal projection $p \in M$ we set

$$S_{\zeta,p} = \Big\{ (x_k) \subset M : \sum_{k=1}^{\infty} x_k \zeta_0 = \zeta \text{ in } H \Big\},$$

where $\sum_{k=1}^{\infty} x_k p$ converges in norm in M and

$$||\zeta||_p = \inf \Big\{ \| \sum_{k=1}^{\infty} x_k p \| \;:\; (x_k) \in S_{\zeta,p} \Big\} \ \text{(see [1])}\ .$$

Definition 2.2 ([1]). A sequence (ζ_n) in $H = L^2(M, \omega_0)$ is said to be almost surely (a.s.) convergent to $\zeta \in H$ if, for each $\varepsilon > 0$, there exists a projection p in M such that $\omega_0(1-p) < \varepsilon$ and $||\zeta_n - \zeta||_p \to 0$ as $n \to \infty$.

Definition 2.3. Let $x_n, x \in M$. A sequence (x_n) tends to x almost uniformly (a.u.) if, for any $\varepsilon > 0$, there is a projection $p \in M$, $\omega_0(1-p) < \varepsilon$, such that $||(x_n - x)p|| \to 0$.

Theorem 2.1. *Assume that the dual weight* $\hat{\omega}_0 = \tau(h_0 \cdot)$ *is regular. Then, for any* $h \in L^2(M)$ *and a projection* $p \in M$, $||h||_p = ||hh_0^{-1/2}p||$, *that is* $hh_0^{-1/2}p \in M$.

Proof. $h_0^{-1/2}$ is locally measurable with respect to N (see [5]), and

$$\Theta_s(hh_0^{-1/2}) = \Theta_s(h)\Theta_s(h_0^{-1/2}) = e^{-s/2}h\,e^{s/2}h_0^{-1/2} = hh_0^{-1/2},$$

that is, $hh_0^{-1/2}$ is affiliated with M. If $(\sum_{k=1}^n x_k)\,h_0^{1/2} \to h$ in $L^2(M)$, then $(\sum_{k=1}^n x_k)\,h_0^{1/2} \to h$ locally in measure in the sense of [6] and $(\sum_{k=1}^n x_k) \to hh_0^{-1/2}$ locally in measure. At the same time, $(\sum_{k=1}^n x_k)p \to x = xp$ in M for some $x \in M$. Hence $(\sum_{k=1}^n x_k)\,p \to hh_0^{-1/2}p$ locally in measure, which implies $xp = hh_0^{-1/2}p$ and $||h||_p = ||hh_0^{-1/2}p||$. This ends the proof.

Corollary 2.1. *Assume that the dual weight* $\hat{\omega}_0$ *is regular. Then*

(i) *Let* $h, h_n \in L^2(M)$, $\varepsilon > 0$. $h_n \to h$ *(a.s.) if and only if* $h_nh_0^{-1/2}p \to hh_0^{-1/2}p$ *in* M *for a projection* $p \in M$ *such that* $\omega_0(1-p) < \varepsilon$;

(ii) $x_n \to x$ *(a.s.) if and only if* $x_n \to x$ *(a.u.),* $x, x_n \in M$.

Proof. (ii) Identifying x_n, x with $x_nh_0^{1/2}$, $xh_0^{1/2} \in L^2(M)$, respectively, we have $||x_n - x||_p = ||(x_n - x)h_0^{1/2}h_0^{-1/2}p|| = ||(x_n - x)p||$.

Remark 2.1. Assume that the dual weight is faithful normal semifinite. Let A be the set of all elements x in M such that the function $s \to \sigma_s^{\omega_0}(x) = h_0^{is}xh_0^{-is}$ is extended to an M-valued entire function. Then A is a σ-weakly dense $*$-subalgebra of M

and $\left[ah_0^{1/p}\right]_p = L^p(M)$, gdzie $[\cdot]_p$ denotes the closure in $L^p(M)$ (see [7]). Suppose that

$$\left(\sum_{k=1}^{n} x_k\right) h_0^{1/2} \to h \in L^2(M)\,, \quad (x_k) \subset A\,,$$

and

$$\left(\sum_{k=1}^{n} x_k\right) p \to x = xp \in M \text{ in } M$$

for a projection $p \in M$. Then

$$\left(\sum_{k=1}^{n} x_k\right) h_0^{1/2} = h_0^{1/4}\, \sigma_{i/4}^{\omega_0} \left(\sum_{k=1}^{n} x_k\right) h_0^{1/4} \to h\,.$$

By continuity of the involution in $\overline{N}$

$$\left(\sum_{k=1}^{n} x_k^*\right) h_0^{1/2} = h_0^{1/4}\, \sigma_{i/4}^{\omega_0} \left(\sum_{k=1}^{n} x_k^*\right) h_0^{1/4} \to h^* \in \overline{N}\,.$$

Because of the continuity of the product in $\overline{N}$, we have

$$p\left(\sum_{k=1}^{n} x_k^*\right) h_0^{1/2} \to ph^* = px^* h_0^{1/2} = x^* h_0^{1/2}\,.$$

Thus we have $hp = h_0^{1/2} xp$ and $\|xp\| = \|h_0^{-1/2} hp\|$ for any sequence (x_k) in A .

In the sequel, we shall interpret $R(M, \sigma^{\omega_0})$ as an implemented continuous crossed product (see [2], Def. 13.2.6). In the case when the algebra M is semifite, while τ_0 is a faithful normal semifinite trace on M , let $\omega_0(x) = \tau_0(a_0 x) = \tau_0(x a_0)$, $x \in M$, for some $a_0 \in L^1(M, \tau_0)$.

Proposition 2.1.

(i) *If the algebra* M *is semifinite then the dual weight* $\hat{\omega}_0$ *is regular, if and only if* M *is finite.*

(ii) *If* M *is an algebra of type* III *then* h_0^{-1} *is locally measurable provided that the operator* Δ_0^{-1}, *where* Δ_0^{-1} *is the modular operator, is locally measurable with respect to the von Neumann algebra* N_1 *generated by the operators* x, Δ_0^{is} $(x \in M,\ s \in \mathbb{R})$.

Proof.

(i) In this case, we have $R(M,\sigma^{\omega_0}) \simeq M \otimes L^\infty(R)$, $\Theta_s(x \otimes f) \simeq a_0^{is} x a_0^{-is} \otimes l(s)(f)$ where $l(s)$ denotes the translation by s in $L^2(R)$. Finally, $h_0 \simeq a_0 \otimes 1$ where $l(s) = 1^{is}$. Consequently, h_0^{-1} is locally measurable with respect to $R(M,\sigma^{\omega_0})$ if and only if a_0^{-1} is locally measurable with respect to M. Since $1 = \omega_0(1) = \tau_0(a_0)$, M is a finite algebra (see [5]). Conversely, if M is a finite algebra then obviously a_0^{-1} is locally measurable with respect to M (affiliated with M), that is, h_0^{-1} is locally measurable with respect to $R(M,\sigma^{\omega_0})$.

(ii) In this case, $h_0 \simeq \Delta_0 \otimes 1$ and there exists $a*$ - isomorphism from $R(M,\sigma^{\omega_0})$ onto a von Neumann subalgebra of $N_1 \otimes L^\infty(R)$.

3. Concluding remarks.

Remark 3.1. It is known that N is a factor of type II_∞ if and only if M is a factor of type III_1. Let $h_0 = \int_0^\infty \mu \, de_\mu$ be the spectral decomposition of h_0. Then $\tau(1 - e_\mu) = 1/\mu\omega_0(1) = 1/\mu$ (see [4]). If $h_0^{-1} = \int_0^\infty \mu \, df_\mu$ then $1 - f_\mu = e_{1/\mu}$. So, $\tau(1 - f_\mu) = \infty$. Consequently, h_0^{-1} is not locally measurable if N is acting on a separable Hilbert space.

Example 3.1. The following example shows that there exist sequences of operators from the algebra M, such that $x_n \to 0$ in H and $\|x_n p\| = 1$, $n = 1, 2, \ldots$, for some projection $p \in M$.

Assume that M is properly infinite von Neumann algebra, whereas ω_0 a faithful normal state on M. Moreover, let $1 = \oplus_{i=1}^\infty p_i$, $p_i \sim p_j \sim 1$, $\sum_{i=1}^\infty \varepsilon_i^2 = 1$, $\varepsilon_{i+1} < \varepsilon_i$, $\omega_0(p_{i_n})/\varepsilon_n^2 \le 1/n$, $u_i^* u_i = p_i$,

$u_i u_i^* = 1$. Put $x_n = 1/\varepsilon_n u_{i_n}$. We have

$$\omega_0(|x_n|^2) = \omega_0(p_{i_n})/\varepsilon_n^2 \leq 1/n \to 0$$

, i.e. $x_n \to 0$ in H. Let now $v_n^* v_n = p_{i_1}$, $v_n v_n^* = p_{i_n}$. Put $v = \sum_{n=1}^{\infty} \varepsilon_n v_n$. Then $v^* v = p_{i_1}$, $vv^* = p$ for some projection $p \in M$. For $\xi, \zeta \in H$, $||\xi|| = 1$, $p\xi = \xi$, $\xi = v\zeta$, we have

$$x_j \xi = x_j \sum_{n=1}^{\infty} \varepsilon_n v_n \zeta = 1/\varepsilon_j u_{i_j} \sum_{n=1}^{\infty} \varepsilon_n v_n \zeta = u_{i_j} v_j \zeta .$$

Hence

$$||x_j \xi|| = ||u_{i_j} v_j \zeta|| = ||v_j \zeta|| = ||\zeta|| = 1 ,$$

that is, $||x_j p|| = 1$, $j = 1, 2, \ldots$.

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