ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN-POLONIA
VOL. L, 20 SECTIO A 1996

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Mean Value Properties of Solutions of Cauchy-Riemann systems

ABSTRACT. It is shown here that certain contour integral formulae for solutions of Cauchy-Riemann systems lead to area integral representations. These represent generalizations of (area) mean value theorems for analytic functions and are valid under essentially weaker conditions on the coefficients of the Cauchy-Riemann systems than the corresponding contour integral formulae.

I. Let a (generalized) Cauchy-Riemann system, i. e. a uniformly elliptic system of the form

$$f_{\bar z} = \nu(z) f_z + \mu(z)\overline{f_z}\,, \tag{1}$$

be given. Uniform ellipticity, in the most general case of (1), means that the coefficients ν, μ have to satisfy

$$\nu, \mu \in L_\infty, \quad \|\, |\nu| + |\mu|\, \|_{L_\infty} =: k < 1\,. \tag{2}$$

Here $L_\infty := L_\infty(\mathbb{C})$, which of course is no loss of generality in (2).

The solutions of (1) possess certain contour integral representations [7], [4], [5] which, however, require additional, incisive or lengthy, conditions

1991 *Mathematics Subject Classification*. Primary 30G20, secondary 30C62, 35B05.

on the coefficients ν, μ. Therefore we ask for corresponding area integral formulae which do not have this disadvantage.

Since we here consider solutions of (1) only in bounded domains, we may assume that

$$\nu(z) = \mu(z) = 0 \quad \text{if} \quad |z| > R_\infty \tag{3}$$

with any fixed positive R_∞.

The conditions (2) in general admit only generalized solutions, i.e. functions continuous in a domain Ω and possessing weak derivatives from $L_{2,loc}(\Omega)$ which satisfy (1) almost everywhere in Ω. Thus, if we want classical derivatives of solutions to exist at prescribed points, further conditions on ν, μ are necessary. Such a condition rather convenient but of course not the weakest one possible, is Bojarski's condition [2]

$$\frac{\nu(z) - \nu(z_0)}{z - z_0}, \ \frac{\mu(z) - \mu(z_0)}{z - z_0} \in L_p := L_p(\mathbb{C}), \quad p > 2. \tag{4}$$

The contour integral formulae in [7], [4], [5] require certain special solutions (of adjoint systems), namely certain fundamental solutions and generalized powers, respectively. Concerning the existence of global fundamental solutions under rather weak conditions on ν, μ we have by [4, chap. IV],

Proposition 1. *Let ν, μ satisfy (2) and (3). For every fixed $z_0 \in \mathbb{C}$ there exists a solution $H(z, z_0, \nu, \mu)$ of (1) in $\mathbb{C} \setminus \{z_0\}$ unique up to the branch of the logarithm, with a representation*

(I) *$H(z, z_0, \nu, \mu) = log(z - z_0) + r_\infty(z, z_0)$, where $r_\infty(z, z_0)$ is single–valued and continuous in $\mathbb{C} \setminus \{z_0\}$,*

(II) *$r_\infty(z, z_0) \in L_s(E)$ for any compact set $E \subset \mathbb{C}$ and any $s \in [1, \infty)$*

(III) *$\lim_{z \to \infty} r_\infty(z, z_0) = 0$.*

We shall call this $H(z, z_0, \nu, \mu)$ *the* fundamental solution of (1).

Further, by [3] we have

Proposition 2. *Let ν, μ satisfy (2), (3) and (4) with a fixed $z_0 \in \mathbb{C}$. For every nonzero integer n and any nonzero complex constant a there exists a unique solution $w(z) =: [a(z - z_0)^n]_{\nu,\mu}$ of (1) in $\mathbb{C} \setminus \{z_0\}$ (even in $\mathbb{C}$ if $n > 0$, resp.) with the following properties*

(I) *$w(z) = (\chi(z))^n$ where $\chi(z)$ is a K–quasiconformal mapping of $\mathbb{C}$ onto itself with $\chi(z_0) = 0$, $K = \frac{1+k}{1-k}$,*

(II) *$w(z)$ admits the asymptotic expansion*

$$w(z) = a(z - z_0 + \mathfrak{b}\overline{(z - z_0)})^n - b\overline{a}(\overline{z - z_0} + \overline{\mathfrak{b}}(z - z_0))^n + O(|z - z_0|^{n+\alpha})$$

at z_0 where α is a positive constant and $\mathfrak{b} = \mathfrak{b}(z_0)$, $b = b(z_0)$ are algebraic expressions of $\nu(z_0), \mu(z_0)$ (e.g. cf. [4, p. 70]).

Because of their topological equivalence with ordinary powers we call these functions generalized powers.

The area integral formulae in question require certain weight functions $P(\cdot)$. We restrict ourselves here to two kinds of $P(\cdot)$, namely $P(r) = r^m =: P_m(r)$ where m is a real constant > -2 (which has to satisfy further restrictions) and $P(r) = P_\infty(r)$, where

$$P_\infty(r) > 0 \ \ \forall r \in (0,\infty), \quad P_\infty(|z|) \in L_\infty,$$
$$\lim_{r\to 0} P_\infty(r) r^l = 0 \quad \text{for each integer } l\,.$$

The modifications to be made for other kinds of weight functions then are obvious.

For brevity we call a system (1) also a (ν,μ)–system and a solution of (1) also a (ν,μ)–solution.

Without loss of generality we put $z_0 = 0$ and consider solutions of (1) in $B(R_0) = \{z : |z| < R_0\}, R_0 \in (0,\infty)$. Since for $K = 1$, i.e. $k = 0$, there is nothing to prove here, we assume that $K > 1$. Let

$$(5) \qquad M(z) = H(z,0,\nu,\overline{\mu}), \quad N(z) = H(z,0,\nu,-\overline{\mu}),$$

further

$$(6) \qquad G_1(z) = [d_1 z^{-1}]_{\nu,\overline{\mu}}, \quad G_2(z) = [d_2 z^{-1}]_{\nu,-\overline{\mu}}$$

where

$$d_1 = -(1-\overline{b}\mathfrak{b})/(1-|b|^2),\ d_2 = -(1+\overline{b}\mathfrak{b})/(1-|b|^2),\ b = b(0),\ \mathfrak{b} = \mathfrak{b}(0),$$

cf. Proposition 2(II) above,

$$F_1(z) = [cz^{-j}]_{\nu,\overline{\mu}}, \quad F_2(z) = [cz^{-j}]_{\nu,-\overline{\mu}}\,,$$

c any constant and j any fixed nonzero integer , and

$$(7)\ \Lambda(z;f) := (1 - \frac{\overline{z}}{z}\nu(z))f(z) - \mu(z)\overline{f(z)}, \quad I = I(R;P) = \int_0^R P(r) r\,dr.$$

Then we have

Theorem 1. *Let ν, μ satisfy (2), (3), and let f be a solution of (1) in $B(R_0)$. Then*

$$\text{(I)}\quad f(0) = \frac{1}{2\pi I}\int\limits_{B(R)} [\operatorname{Re}(M_z(z)z\Lambda(z;f)) + i\operatorname{Im}(N_z(z)z\Lambda(z;f))]P(|z|)d\sigma_z$$

$\forall R \in (0, R_0)$ and for every $P(\cdot) = P_m(\cdot)$ with
(II) $m > -\frac{2}{s} + K - \frac{1}{K}$, *where s is fixed but arbitrary $\in [1, 2K/(K-1))$.*
Moreover, then
(III) $M_z(z)zP(|z|), N_z(z)zP(|z|) \in L_{s,loc}$.

Note that the condition (3) does not restrict the generality of ν, μ and f in $B(R_0)$.

Amongst other things, Theorem 1 embodies a (further) possibility for estimating the supremum norm of a (ν,μ)-solution f in a domain Ω over a compact subdomain of Ω by the norm of f in $L_q(\Omega)$ for each $q > 2K/(K+1)$.

Theorem 2. *Let ν, μ satisfy (2), (3) and (4), and let f be a solution of (1) in $B(R_0)$. Then*

$$\text{(I)}\quad f_z(0) = \frac{1}{2\pi I}\int\limits_{B(R)} [\operatorname{Re}(G_{1z}(z)z\Lambda(z;f)) + i\operatorname{Im}(G_{2z}(z)z\Lambda(z;f))]P(|z|)d\sigma_z$$

$\forall R \in (0, R_0)$ and for every $P(\cdot) = P_m(\cdot)$ with
(II) $m > -\frac{2}{s} + 2 - \frac{1}{K}$, *with any fixed s from $[1, 2K/(K-1)$.*
Moreover, then
(III) $G_{lz}(z)zP(|z|) \in L_{s,loc}, \quad l = 1, 2.$

This in particular implies the equivalence of certain supremum norms of derivatives with certain L_q-norms of the function itself, where q is to be choosen as with Theorem 1. A further consequence is that a series of (ν,μ)-solutions may be differentiated term by term at any point z_0 from the (interior of the) domain of convergence if ν, μ satisfy a condition (4) at z_0.

Of course, Theorem 1 and 2 express mean value properties of f and f_z which reduce to the (area) mean value theorem for analytic functions if

$\nu = \mu \equiv 0$ in $\mathbb{C}$. The concrete upper bounds of s in both theorems are consequences of the results in [1].

Let

$$I(R_1, R_2) = I(R_1, R_2; P) = \int_{R_1}^{R_2} P(r)r\,dr, \quad 0 \leq R_1 < R_2 < \infty,$$

where $P(\cdot)$ is as above. For generalized powers we then have

Theorem 3. *Let ν, μ satisfy (2), (3) and (4), and let $f(z) = [az^n]_{\nu,\mu}$, n an integer. Then*

(I)

$$ac(1-|b|^2)(-j)\delta_{n,j} = \frac{1}{2\pi I(R_1, R_2)} \int_{\{R_1<|z|<R_2\}} [\mathrm{Re}(F_{1z}(z)z\Lambda(z;f)) + i\mathrm{Im}(F_{2z}(z)z\Lambda(z;f))]P(|z|)d\sigma_z,$$

for every $P = P_m$ with

(II) $\quad m > -n + j - 1 - \dfrac{1}{K} \quad$ *(m arbitrary if $R_1 > 0$),*

where $b = b(0)$ is the constant mentioned in Proposition 2, *and $\delta_{n,j}$ means the Kronecker symbol.*

II. We now come to the proof of Theorem 1–3, which rests upon certain contour integral formulae [4], [5]. These read with the above notations and specifications

$$f(0) = \mathrm{Re}\frac{1}{2\pi i}\oint_{|z|=r} f(z)dM(z) + i\mathrm{Im}\frac{1}{2\pi i}\oint_{|z|=r} f(z)dN(z) \tag{8}$$

$$f_z(0) = \mathrm{Re}\frac{1}{2\pi i}\oint_{|z|=r} f(z)dG_1(z) + i\mathrm{Im}\frac{1}{2\pi i}\oint_{|z|=r} f(z)dG_2(z) \tag{9}$$

$$\begin{aligned} ac(1-|b|^2)(-j)\delta_{n,j} = {} & \mathrm{Re}\frac{1}{2\pi i}\oint_{|z|=r} [az^n]_{\nu,\mu} d[cz^{-j}]_{\nu,\overline{\mu}} \\ & + i\mathrm{Im}\frac{1}{2\pi i}\oint_{|z|=r} [az^n]_{\nu,\mu} d[cz^{-j}]_{\nu,-\overline{\mu}} \end{aligned} \tag{10}$$

Here ν, μ have to satisfy additional conditions. Therefore we assume, for the time being, that

$$(11) \qquad \textit{Hölder continuity in } \mathbb{C} \textit{ and } (2),(3)$$

are satisfied for ν, μ.

To start with, we observe that every expression on the right-hand side of (8)-(10) is of the form

$$(12) \qquad \mathrm{Re}\frac{1}{2\pi i}\oint_{|z|=r} f(z)dh_1(z) + i\mathrm{Im}\frac{1}{2\pi i}\oint_{|z|=r} f(z)dh_2(z) =: A$$

where f, h_1, h_2 are $(\nu,\mu)-, (\nu,\overline{\mu})-, (\nu,-\overline{\mu})$-solutions in $B(R_0), \mathbb{C}\setminus\{0\}$, respectively. Because of (11) these functions are continuously differentiable in their respective domain of definition.

A simple calculation yields

$$A = \frac{1}{2\pi}\int_0^{2\pi}[\mathrm{Re}(h_{1z}(z)z\Lambda(z;f)) + i\mathrm{Im}(h_{2z}(z)z\Lambda(z;f))]d\phi,$$

with $\Lambda(z;f)$ defined in (7), and $z = re^{i\phi}$. Multiplication by $P(r)r$ and integration give

$$AI(R;P) = \frac{1}{2\pi}\int_0^R\left(\int_0^{2\pi}[\mathrm{Re}(h_{1z}z\Lambda) + i\mathrm{Im}(h_{2z}z\Lambda)]P(r)rd\phi\right)dr.$$

Thus

$$(13) \qquad AI(R;P) = \frac{1}{2\pi}\int_{B(R)}[\mathrm{Re}(h_{1z}z\Lambda) + i\mathrm{Im}(h_{2z}z\Lambda)]P(|z|)d\sigma_z$$

if

$$(14) \qquad h_{lz}(z)zP(|z|)\Lambda(z;f) \in L_1(B(R)), \quad l = 1,2.$$

In the cases of (8) and (9), where we have $\Lambda(z;f) \in L_\infty(B(R))$, this leads to the (sufficient) condition

$$(15) \qquad h_{lz}(z)zP(|z|) \in L_{1,loc} \; (:= L_{1,loc}(\mathbb{C})),$$

whereas in case of (10) we are led (in view of Proposition 2(II)) to

$$h_{lz}(z)z^{n+1}P(|z|) \in L_{1,loc}. \tag{16}$$

For the sake of convenience we now drop the index l in h_{lz}. Because of the Bers–Nirenberg representation theorem, every such $h_z(z)$ is of the form

$$h_z(z) = \frac{\chi_z(z)}{\chi(z)^{j+1}}c(j) \tag{17}$$

with $c(j) = -j$ if $j \neq 0$ and $c(0) = 1$, where $\chi(z)$ is a K- quasiconformal mapping of $\mathbb{C}$ onto itself and $\chi(0) = 0$.

By [1]

$$\chi_z \in L_{p,loc} \quad \forall p \in [1, 2K/(K-1)). \tag{18}$$

In the case of (10) with $j \neq 0$ we obtain

$$h_z z^{n+1}P|z|) = c(j)\chi_z \frac{1}{\chi(z)}\frac{1}{\chi(z)^j}z^{n+1}P(|z|). \tag{19}$$

Further, for each χ from (17) we have

$$\frac{1}{|\chi(z)|} = O(|z|^{-1}) \quad \text{or} \quad = O(|z|^{-K}) \quad \text{if } j \neq 0,\ j = 0, \text{ respectively}\ , \tag{20}$$

by Proposition 2(II) and the Theorem of Mori, respectively.

Thus, in the situation of Theorem 3,

$$h_z z^{n+1}P(|z|) = \chi_z \cdot P(|z|) \cdot O(|z|^{-j+n}). \tag{21}$$

By (18) this is $\in L_{1,loc}$ if

$$P(|z|)|z|^{-j+n} \in L_{q,loc} \tag{22}$$

with a q satisfying $1 > \frac{K-1}{2K} + \frac{1}{q}$, i.e. with a q satisfying

$$q > \frac{2K}{K+1}. \tag{23}$$

If $P(\cdot) = P_\infty(\cdot)$ then (22) is satisfied even for every $q \in [1, \infty)$. If $P(\cdot) = P_m(\cdot)$ with m finite, then (22), (23) lead to

$$m - j + n > -\frac{2}{q} > -2\frac{K+1}{2K}.$$

Hence, there exists a q such that (22) and (23) are satisfied if

$$m > -n + j - 1 - \frac{1}{K}.$$

This proves Theorem 3 for smooth ν, μ and $R_1 = 0$. The case $R_1 > 0$ is obvious.

As to Theorem 1 and 2, we replace the condition (15) by the more general condition

$$h_z z P(|z|) \in L_{s,loc} \tag{24}$$

with an s to specify. Using (17) and (20) we obtain

$$h_z = \chi_z \cdot O(|z|^{-K(1-j)-2j}), \quad j = 0, 1. \tag{25}$$

Because of (18) and the Hölder inequality, (24) holds if

$$|z|^{-K(1-j)-2j+1} \cdot P(|z|) \in L_{q,loc} \tag{26}$$

with a $q \geq 1$ satisfying

$$\frac{1}{s} > \frac{K-1}{2K} + \frac{1}{q}. \tag{27}$$

Of course, this is possible only for

$$s \in [1, 2K/(K-1)) = [1, 1 + \frac{1}{k}).$$

For these s, (27) is equivalent to

$$q > \frac{s(1+k)}{k+1-sk} = \frac{2sK}{2K - s(K-1)}. \tag{28}$$

Again, if $P(\cdot) = P_\infty(\cdot)$, then (26) holds for every $q \geq 1$, in particular for those which satisfy (28). If $P = P_m$ with m finite then (26), (28) hold simultaneously if

$$-K(1-j) - 2j + 1 + m > -\frac{2}{q} > -\frac{2}{s} + 1 - \frac{1}{K},$$

i.e. if

$$m > -\frac{2}{s} + K(1-j) + 2j - \frac{1}{K}, \quad j = 0, 1.$$

This proves Theorem 1 and 2 for ν, μ satisfying (11).

Now we want to come to the actual ν, μ mentioned in Theorem 1–3. First note that the validity of (III) under the condition (II) in Theorem 1 and 2 does not depend on any additional smoothness conditions on ν, μ. Thus we only have to show that (I) in Theorem 1-3 holds for the corresponding ν, μ under the respective condition (II).

Let ν, μ satisfy (2) and (3). By [6] there exist sequences of $C_0^\infty(\mathbb{C})$-functions ν_l, μ_l such that the following conditions are satisfied simultaneously:

(a) $\| \, |\nu_l| + |\mu_l| \, \|_{L_\infty} \le k$, $\nu_l(z) \to \nu(z)$, $\mu_l(z) \to \mu(z)$ a.e. in $\mathbb{C}$, $\nu_l(z) = \mu_l(z) \equiv 0$ for $|z| > R_\infty + 1$,

(b) for the corresponding fundamental solutions $M_l(z), N_l(z)$ holds $M_l(z) \to M(z)$, $N_l(z) \to N(z)$ locally uniformly in $\mathbb{C} \setminus \{0\}$, $M_{lz} \to M_z$, $N_{lz} \to N_z$ in $L_{r,loc}\, \forall r \in [1, 2)$,

(c) for any (ν, μ)-solution f in $B(R_0)$ and any fixed positive $R < R_0$, there exits a sequence of (ν_l, μ_l)- solutions f_l in $B(R)$ such that $f_l(z) \to f(z)$ uniformly in $B(R)$

as $l \to \infty$.

As proved already we have

$$f_l(0) = \frac{1}{2\pi I} \int\limits_{B(R)} [\mathrm{Re}(M_{lz} z \Lambda_l(z; f_l)) + i\mathrm{Im}(N_{lz} z \Lambda_l(z; f_l))] P(|z|) d\sigma_z ,$$

where $\Lambda_l(\,\cdot\,;\,\cdot)$ belongs to ν_l, μ_l according to (7). Taking limits on both sides and observing that in any case $m > -2$, we obtain (I) of Theorem 1. If we resrict s in Theorem 1, then condition (II) on m can be weakened essentially. Namely, using (b) above in place of (18), (20), we obtain by considerations analogous to those which led to condition (II) of Theorem 1 the following completion.

Corollary 1. (I) *and* (III) *of* Theorem 1 *remain valid if* (II) *is replaced by*

(II′) $\quad m > -\dfrac{2}{s}$, *where s is fixed but arbitrary* $\in [1, 2)$.

We shall say that a sequence g_l satisfies a uniform Bojarski condition at $z_0 = 0$ if

$$\left\| \frac{g_l(z) - g_l(0)}{z} \right\|_{L_p} < C \quad \forall\, l$$

with a fixed positive constant C and a fixed $p > 2$. By [6] we have the following statements.

Proposition 3. *Let ν, μ satisfy (2), (3) and (4), let j be any fixed nonzero integer, c any fixed constant, f any (ν,μ)-solution in $B(R_0)$ and R be fixed $\in (0, R_0)$.*

Then there exist sequences ν_l, μ_l, satisfying (a) above as well as a uniform Bojarski condition at $z_0 = 0$, and (ν_l, μ_l)-solutions $f_l(z)$ and $D_l(z) := [cz^j]_{\nu_l,\mu_l}$ in $B(R)$, $\mathbb{C} \setminus \{0\}$, respectively, such that (simultaneously) holds

(I) $D_l(z) \to D(z) := [cz^j]_{\nu,\mu}$ *locally uniformly in* $\mathbb{C} \setminus \{0\}$,

(II) $D_{lz}(z) z^{-j+1} \rightharpoonup D_z(z) z^{-j+1}$ *weakly in* $L_{p,loc}$ *for every* $p \in \left[1, \frac{2K}{K-1}\right)$,

(III) $D_l(z) z^{-j}$ *are locally uniformly bounded in* $\mathbb{C}$,

(IV) $f_l(z) \to f(z)$ *uniformly in* $B(R)$ *and* $f_{lz}(0) \to f_z(0)$

as $l \to \infty$.

Proposition 3 applied to G_1, G_2 from (6) means that, for corresponding G_{1l}, G_{2l},

$$G_{\iota l z} z^2 \rightharpoonup G_{\iota z} z^2 \quad \text{weakly in} \quad L_p(B(R)), \quad \iota = 1, 2,$$

for every $p \in [1, 2K/(K-1))$. Since m from Theorem 2 in any case has to satisfy

$$m - 1 > -1 - \frac{1}{K} \tag{29}$$

(and since $K > 1$), we find a $q > 1$ such that

$$q(m-1) > -2 > -(1 + \frac{1}{K})q.$$

For each such q

$$\frac{P(|z|)}{|z|} \in L_{q,loc}, \quad q > \frac{2K}{K+1}.$$

The inequality for q means that there exists a $p \in (1, 2K/(K-1))$ such that $p^{-1} + q^{-1} = 1$. Moreover, because of Proposition 3(IV),

$$\frac{P(|z|)}{|z|} \Lambda_l(z; f_l) \to \frac{P(|z|)}{|z|} \Lambda(z; f) \text{ (strongly) in } L_q(B(R)) \text{ as } l \to \infty.$$

Thus, using the smooth version of Theorem 2(I) and letting there l tend to infinity we obtain Theorem 2 completely. In a quite analogous way Theorem 3 follows.

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received October 4, 1996