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DARIUSZ PARTYKA (Lublin)

Some Extremal Problems Concerning the Operator B_γ

ABSTRACT. Following [P2] we assign to each quasisymmetric automorphism γ of the unit circle $\mathbf{T}$ a linear homeomorphic self-mapping B_γ of a Hilbert space $(H, \|\cdot\|_H)$. A complete solution to the following extremal problem is found: For which quasisymmetric automorphisms γ of $\mathbf{T}$, $\|B_\gamma(f)\|_H = \sqrt{K(\gamma)}$ for some $f \in H$ with $\|f\|_H = 1$? Here $K(\gamma)$ stands for the maximal dilatation of an extremal quasiconformal extension of γ to the unit disk. As an application a relation between the Schober constant $\lambda(\Gamma)$ of a quasicircle $\Gamma \subset \mathbb{C}$ and an extremal quasiconformal reflection in Γ is established.

0. Introduction. Given a domain Ω in the extended complex plain $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ we denote by $H(\Omega)$ ($A(\Omega)$) the class of all complex-valued harmonic (analytic) functions on Ω. If a function $F : \Omega \to \mathbb{C}$ has partial derivatives for almost every (a.e. for short) $z = x + iy \in \Omega$ then the Dirichlet integral $\mathcal{D}_\Omega[F]$ of F is defined by

$$(0.1) \qquad \mathcal{D}_\Omega[F] := \int_\Omega (|\partial_x F|^2 + |\partial_y F|^2)dS = 2\int_\Omega (|\partial F|^2 + |\bar{\partial} F|^2)dS\,,$$

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where $dS := dxdy$ and

$$\partial_x F := \frac{\partial F}{\partial x}, \partial_y F := \frac{\partial F}{\partial y},$$

(0.2)

$$\partial F := \frac{1}{2}(\partial_x F - i\partial_y F), \bar{\partial} F := \frac{1}{2}(\partial_x F + i\partial_y F).$$

The class $\dot{A}^2(\Omega) := \{F \in A(\Omega) : \mathcal{D}_\Omega[F] < \infty\}$ is a closed subspace of the space $\dot{H}^2(\Omega) := \{F \in H(\Omega) : \mathcal{D}_\Omega[F] < \infty\}$ in the pseudo-norm $\|F\|_X := \sqrt{\frac{1}{2}\mathcal{D}_\Omega[F]}$, $F \in X := \dot{H}^2(\Omega)$.

Suppose Ω is bounded by a Jordan curve $\Gamma = \partial\Omega$. Given a function $F : \Omega \to \mathbb{C}$ we define for every $z \in \Gamma$, $\hat{\partial} F(z) := \lim_{\Omega \ni u \to z} F(u)$ provided the limit exists, while $\hat{\partial} F(z) := 0$ otherwise. Write $\mathbf{\Delta} := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbf{T} := \{z \in \mathbb{C} : |z| = 1\}$ for the unit disk and its boundary, respectively.

In case $\Omega = \mathbf{\Delta}$ we will use the symbol $\hat{\partial}_r F$ for the radial limiting values function of F, i.e. for every $z \in \mathbf{T}$, $\hat{\partial} F_r(z) := \lim_{t \to 1^-} F(tz)$ if the limit exists, while $\hat{\partial}_r F(z) := 0$ otherwise.

Given $K \geq 1$ we denote by $\mathbb{Q}_{\mathbf{\Delta}}(K)$ the class of all K-quasiconformal (K-qc. for brevity) self-mappings of $\mathbf{\Delta}$, and let $\mathbb{Q}_{\mathbf{\Delta}} := \bigcup_{K \geq 1} \mathbb{Q}_{\mathbf{\Delta}}(K)$. It is well known that every $\varphi \in \mathbb{Q}_{\mathbf{\Delta}}$ has a continuous extension to $\mathbf{T}$ and $\hat{\partial}\varphi$ is a sense-preserving homeomorphic self-mapping of $\mathbf{T}$; cf. [LV, p. 42]. Due to Krzyż the class $\mathbb{Q}_{\mathbf{T}} := \{\hat{\partial}\varphi : \varphi \in \mathbb{Q}_{\mathbf{\Delta}}\}$ has a very simple characterization by means of quasisymmetric automorphisms of $\mathbf{T}$; cf. [K1] and [K2].

Another interesting characterization of the class $\mathbb{Q}_{\mathbf{T}}$ by quasihomographies was introduced by Zając; cf. [Z], see also [K3]. For $K \geq 1$, define $\mathbb{Q}_{\mathbf{T}}(K) := \{\hat{\partial}\varphi : \varphi \in \mathbb{Q}_{\mathbf{\Delta}}(K)\}$. Thus $\mathbb{Q}_{\mathbf{T}}(K)$ is the class of all quasisymmetric automorphisms of $\mathbf{T}$ which admit a K-qc. extension to $\mathbf{\Delta}$. The functional $K[\varphi] := \inf\{K \geq 1 : \varphi \in \mathbb{Q}_{\mathbf{\Delta}}(K)\}$ is the maximal dilatation of φ.

Analogously, for $\gamma \in \mathbb{Q}_{\mathbf{T}}$ we set $K(\gamma) := \inf\{K \geq 1 : \gamma \in \mathbb{Q}_{\mathbf{T}}(K)\}$. In both definitions inf may be replaced by min because of the compactness of the class $\{\varphi \in \mathbb{Q}_{\mathbf{\Delta}}(K) : \hat{\partial}\varphi = \gamma\}$ in the uniform convergence topology on $\mathbf{\Delta}$; cf. [LV, p. 73]. Thus $K(\gamma)$ is the maximal dilatation of an extremal qc. extension φ of $\gamma \in \mathbb{Q}_{\mathbf{T}}$ to $\mathbf{\Delta}$; extremal means that $\varphi \in \mathbb{Q}_{\mathbf{\Delta}}(K[\gamma])$. For $p \geq 1$, we adopt the usual notation $L^p(\mathbf{T})$ for the class of all functions $f : \mathbf{T} \to \mathbb{C}$, p-integrable on Γ with respect to the Lebesgue arc-length measure, i.e. $\|f\|_p := (\int_\Gamma |f(z)|^p |dz|)^{1/p} < \infty$.

The notation $f \doteq g$, $f, g \in L^1(\mathbf{T})$, means that $f - g$ equals a constant almost everywhere (a.e. for brevity) on $\mathbf{T}$. It is clear that $\doteq$ is an equivalence relation in $L^1(\mathbf{T})$, and let $L^1(\mathbf{T}) := \{[f/\doteq] : f \in L^1(\mathbf{T})\}$ stand for the quotient space $L^1(\mathbf{T})/\doteq$. Recall that for every $f \in L^1(\mathbf{T})$ and $z \in \mathbf{\Delta}$ the

Schwarz and Poisson formulas read as

$$(0.3)\qquad f_{\Delta}(z) := \frac{1}{2\pi}\int_{\mathbf{T}} f(u)\frac{u+z}{u-z}|du| = a_0(f) + \sum_{n=1}^{\infty} a_n(f)z^n,$$

$$(0.4)\qquad \mathcal{P}[f](z) := \frac{1}{2\pi}\int_{\mathbf{T}} f(u)\,\mathrm{Re}\,\frac{u+z}{u-z}|du|,$$

where

$$(0.5)\qquad \begin{aligned} a_0(f) &:= \frac{1}{2\pi}\int_{\mathbf{T}} f(u)|du|, \\ a_n(f) &:= \frac{1}{\pi}\int_{\mathbf{T}} \overline{u}^n f(u)|du|, \quad n = 1, 2, \dots . \end{aligned}$$

Obviously, for every $f \in L^1(\mathbf{T})$, $f_{\Delta} \in A(\Delta)$ and $\mathcal{P}[f] \in H(\Delta)$. According to the Poisson integral properties, for each $f \in L^1(\mathbf{T})$ we have $\hat{\partial}_r \mathcal{P}[f] = f$ a.e. on $\mathbf{T}$; cf. [Du, p. 5], [R, Sect. 11.12]. Therefore, the operator $\mathcal{S}_0 : L^1(\mathbf{T}) \to L^1(\mathbf{T})$,

$$\mathcal{S}_0([f/\doteqdot]) := \hat{\partial}_r \mathcal{P}[f] - \mathcal{P}[f](0), \quad f \in L^1(\mathbf{T}),$$

is a selector on the quotient space $L^1(\mathbf{T})$. We call it the Poisson selector.

Consider the class $\dot{H}^2(\partial\Delta) := \{f \in L^1(\mathbf{T}) : \mathcal{P}[f] \in \dot{H}^2(\Delta)\}$, and define the quotient space

$$\boldsymbol{H} := \mathrm{Re}\,\dot{H}^2(\partial\Delta)/\doteqdot = \{\boldsymbol{f} \in L^1(\mathbf{T}) : \mathcal{S}_0(\boldsymbol{f}) \in \mathrm{Re}\,\dot{H}^2(\partial\Delta)\}.$$

Here and subsequently, $\mathrm{Re}\,X := \{\mathrm{Re}\,f : f \in X\}$ for any space X of complex-valued functions. If $f \in \mathrm{Re}\,\dot{H}^2(\partial\Delta)$ then, by (0.3) and (0.5), we get

$$\|f\|_2^2 = 2\pi|f_{\Delta}(0)|^2 + \pi\sum_{n=1}^{\infty}|a_n(f)|^2 \le 2\pi|a_0(f)|^2 + \pi\sum_{n=1}^{\infty} n|a_n(f)|^2$$

$$= 2\pi|a_0(f)|^2 + \int_{\Delta}|(f_{\Delta})'|^2 dS < \infty,$$

so that $f \in L^2(\mathbf{T})$. Therefore, $(\boldsymbol{H}, \|\cdot\|_{\boldsymbol{H}})$ is a real Hilbert space, where

$$(0.6)\qquad 2\|\boldsymbol{f}\|_{\boldsymbol{H}}^2 := \mathcal{D}[\mathcal{P}[\mathcal{S}_0(\boldsymbol{f})]] = \int_{\Delta}|(\mathcal{S}_0(\boldsymbol{f})_{\Delta})'|^2 dS .$$

For brevity we shall write $\mathcal{D}[F]$ for the Dirichlet integral $\mathcal{D}_{\Delta}[F]$. We denote by $\mathbb{P}$ the set of all complex polynomials. For a non-empty set $K \subset \mathbb{C}$, let $\mathbb{P}(K) := \{P_{|K} : P \in \mathbb{P}\}$. From (0.3), (0.5) and (0.6) it follows, in the standard way, that

(0.7) $\mathcal{S}_0(\boldsymbol{H}) \subset \mathrm{Re}\,L^2(\mathbf{T})$;

(0.8) $\|\mathcal{S}_0(\boldsymbol{f})\|_2 \le \sqrt{2}\|\boldsymbol{f}\|_{\boldsymbol{H}}$, $\boldsymbol{f} \in \boldsymbol{H}$;

(0.9) $\{\boldsymbol{f} : \mathcal{S}_0(\boldsymbol{f}) \in \mathrm{Re}\,\mathbb{P}(\mathbf{T})\}$ is a dense subspace of $\boldsymbol{H}$;

cf. [P6, Thm. 2.4.8] and [P5, Thm. 1.2]. Moreover, we can show that for every $F \in \dot{A}^2(\Delta)$, F belongs to the Hardy class H^2, and so

(0.10) $F = (\mathrm{Re}\,\hat{\partial}_r F)_\Delta + i\,\mathrm{Im}\,F(0)$ and $\mathrm{Re}\,\hat{\partial}_r F \in \mathrm{Re}\,\dot{H}^2(\partial\Delta)$;

(0.11) $2\|[\mathrm{Re}\,\hat{\partial}_r F/\doteqdot]\|_{\boldsymbol{H}}^2 = \mathcal{D}[\mathcal{P}[\mathrm{Re}\,\hat{\partial}_r F]] = \int_\Delta |((\mathrm{Re}\,\hat{\partial} F)_\Delta)'|^2 dS$;

cf. [P6, Thm. 2.4.4]. We adopt the usual notation $C(K)$ for the class of all complex-valued continuous functions on a set $K \neq \emptyset$. From Lemma 1.1 and (0.9) it follows that there exists a unique linear bounded operator $\boldsymbol{B}_\gamma : \boldsymbol{H} \to \boldsymbol{H}$ satisfying

$$\boldsymbol{B}_\gamma([f/\doteqdot]) = [f \circ \gamma/\doteqdot]\,, \quad f \in \mathrm{Re}\,C(\mathbf{T}) \cap \dot{H}^2(\partial\Delta)\,. \tag{0.12}$$

Let $\|\boldsymbol{T}\|$ stand for the supremum norm of a linear operator $\boldsymbol{T} : \boldsymbol{H} \to \boldsymbol{H}$, i.e. $\|\boldsymbol{T}\| := \sup\{\|\boldsymbol{T}(\boldsymbol{f})\|_{\boldsymbol{H}} : \boldsymbol{f} \in \boldsymbol{H}$ and $\|\boldsymbol{f}\|_{\boldsymbol{H}} \leq 1\}$. Following [BS] we will use the notation $\check{\varphi}$ for the inverse mapping to a complex-valued mapping φ if it exists. By definition and by Theorem 1.2 we easily find that for any $\gamma, \sigma \in \mathbb{Q}_\mathbf{T}$

(0.13) $\|\boldsymbol{B}_\gamma\| \leq \sqrt{K(\gamma)}$;

(0.14) $\boldsymbol{B}_{\gamma\circ\sigma} = \boldsymbol{B}_\sigma \boldsymbol{B}_\gamma$;

(0.15) $\boldsymbol{B}_{\check{\gamma}} = \boldsymbol{B}_\gamma^{-1}$;

(0.16) $\boldsymbol{B}_{\mathrm{id}_\mathbf{T}} = \mathrm{I}$,

where $\mathrm{id}_\mathbf{T} : \mathbf{T} \to \mathbf{T}$ and $\mathrm{I} : \boldsymbol{H} \to \boldsymbol{H}$ are identity mappings; cf. [P6, Corollary 2.5.4] and [P2, Lemma 1.1]. The properties (0.13) and (0.15) say that the operator $\boldsymbol{B}_\gamma$ is a linear homeomorphism of $\boldsymbol{H}$ onto itself. Moreover, it turns out that

$$\boldsymbol{B}_\gamma(\boldsymbol{f}) = [\mathcal{S}_0(\boldsymbol{f}) \circ \gamma/\doteqdot]\,, \quad \boldsymbol{f} \in \boldsymbol{H}\,, \gamma \in \mathbb{Q}_\mathbf{T}\,;$$

cf. [P6, formula (2.5.8)]. However, we will not use this fact in the sequel. In what follows we list four natural questions involving the supremum norm of the operator $\boldsymbol{B}_\gamma$.

Question 0.1. For which $\gamma \in \mathbb{Q}_\mathbf{T}$, $\|\boldsymbol{B}_\gamma\| = \sqrt{K(\gamma)}$?

Question 0.2. For which $\gamma \in \mathbb{Q}_\mathbf{T}$, does there exist $\boldsymbol{f} \in \boldsymbol{H}$ with $\|\boldsymbol{f}\|_{\boldsymbol{H}} = 1$ such that $\|\boldsymbol{B}_\gamma(\boldsymbol{f})\| = \|\boldsymbol{B}_\gamma\|$? This question may be formulated equivalently: When $\|\boldsymbol{B}_\gamma\| = \max\{\|\boldsymbol{B}_\gamma(\boldsymbol{f})\|_{\boldsymbol{H}} : \boldsymbol{f} \in \boldsymbol{H}$ and $\|\boldsymbol{f}\|_{\boldsymbol{H}} \leq 1\}$?

Question 0.3. For which $\gamma \in \mathbb{Q}_\mathbf{T}$, $\|\boldsymbol{B}_\gamma(\boldsymbol{f})\|_{\boldsymbol{H}} = \sqrt{K(\gamma)}$ for some $\boldsymbol{f} \in \boldsymbol{H}$ with $\|\boldsymbol{f}\|_{\boldsymbol{H}} = 1$?

Question 0.4. Does there exist a constant $c > 0$ such that for every $\gamma \in \mathbb{Q}_\mathbf{T}$, $\|\boldsymbol{B}_\gamma\| - 1 \geq c(\sqrt{K(\gamma)} - 1)$?

In the next section we give a complete answer to the Question 0.3. In Section 2 we show that for some $\gamma \in \mathbb{Q}_\mathbf{T}$, $\|\boldsymbol{B}_\gamma\|$ may be expressed by the smallest positive eigenvalue $\lambda_*(\gamma)$ of γ. The results obtained there are helpful in the next section. It turns out that the supremum norms $\|B_\gamma\|$ and $\|B_\gamma^{-1}\|$ are related to the Schober constant $\lambda(\Gamma)$ of a certain quasicircle $\Gamma \subset \mathbb{C}$ whose welding homeomorphism is $\gamma \in \mathbb{Q}_{\mathbf{T}}$; cf. Lemma 3.1. Thus the study of the Schober constant $\lambda(\Gamma)$ can be reduced to the study of norms $\|B_\gamma\|$ and $\|B_\gamma^{-1}\|$, which seems to be easier in some cases. As applications we present a few results in Section 3. The norm $\|B_\gamma\|$ is also closely related to the Grunsky-Kühnau constant κ (cf. [Kü1, p. 383]) for a respective Grunsky matrix associated with γ. However, this topic will be studied in a forthcoming publication. This justifies studying the norm $\|B_\gamma\|$. In the last section we give some comments to our subject.

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1. The main result. It is easily verified that

$$|a_1b_1 + a_2b_2|^2 + |a_1\overline{b_2} + a_2\overline{b_1}|^2 \le (|a_1|^2 + |a_2|^2)(|b_1| + |b_2|)^2$$

for any $a_1, a_2, b_1, b_2 \in \mathbb{C}$. The change of variables formula now shows that for all $F \in \dot{H}^2(\Delta)$, $K \ge 1$ and $\varphi \in \mathbb{Q}_K(\Delta)$

$$\begin{aligned}
(1.1) \qquad \mathcal{D}_\Delta[F \circ \varphi] &= 2\int_\Delta (|\partial(F\circ\varphi)|^2 + |\bar\partial(F\circ\varphi)|^2)dS \\
&\le 2\int_\Delta (|\partial F\circ\varphi|^2 + |\bar\partial F\circ\varphi|^2)(|\partial\varphi| + |\bar\partial\varphi|)^2 dS \\
&\le 2K[\varphi]\int_\Delta (|\partial F\circ\varphi|^2 + |\bar\partial F\circ\varphi|^2)(|\partial\varphi|^2 - |\bar\partial\varphi|^2)dS \\
&= 2K[\varphi]\int_\Delta (|\partial F|^2 + |\bar\partial F|^2)dS = K[\varphi]\mathcal{D}_\Delta[F]\,.
\end{aligned}$$

This means that the Dirichlet integral is quasi-invariant; cf. e.g. [A1, p. 18].

Lemma 1.1. *Given $K \ge 1$ assume that $\varphi \in \mathbb{Q}_K(\Delta)$. Then for all functions $F \in \operatorname{Re}\mathbb{P}$ and $P \in \operatorname{Re}\dot{H}^2(\Delta)$, $G := \mathcal{P}[\hat\partial(F\circ\varphi)] \in \operatorname{Re}\dot{H}^2(\Delta)$ and*

$$(1.2) \qquad \mathcal{D}[F\circ\varphi - G + P] = \mathcal{D}[F\circ\varphi - G] + \mathcal{D}[P]\,.$$

In particular,

$$(1.3) \qquad \mathcal{D}[G] = \mathcal{D}[F\circ\varphi] - \mathcal{D}[F\circ\varphi - G] \le K\mathcal{D}[F]\,.$$

Proof. Suppose K, φ, F and P satisfy the assumptions of our lemma and set $\gamma := \hat\partial\varphi$. The proof will be divided into two parts.

Part I. We first prove the lemma under the assumption that $G \in \dot{H}^2(\Delta)$. Since the class $\operatorname{Re}\mathbb{P}(\Delta)$ is dense in $\operatorname{Re}\dot{H}^2(\Delta)$, there exists a sequence $P_n \in \operatorname{Re}\mathbb{P}$, $n \in \mathbb{N}$, such that

$$(1.4) \qquad \mathcal{D}[P_n - P] \to 0 \,, \quad n \to \infty \,.$$

For $z \in \hat{\mathbb{C}}$ define $\tilde{\varphi}(z) := \varphi(z)$ if $z \in \Delta$, $\tilde{\varphi}(z) := \gamma(z)$ if $z \in \mathbf{T}$ and $\tilde{\varphi}(z) := 1/\overline{\varphi(1/\overline{z})}$ if $z \in \hat{\mathbb{C}} \setminus \overline{\Delta}$. By the reflection principle for qc. mappings (see for instance [LV, p. 47]), $\tilde{\varphi}$ is a qc. self-mapping of the extended complex plane $\hat{\mathbb{C}}$. For every $t \in \mathbb{R}$ the set

$$\ell_\xi(t) := \{z \in \mathbb{C} : \operatorname{Re}(z - t\xi)\overline{\xi} = 0\} \,, \quad \xi \in \mathbb{C} \setminus \{0\} \,,$$

is the straight line passing through the point $t\xi$ and orthogonal to the straight line $\{s\xi : s \in \mathbb{R}\}$. Since $\tilde{\varphi}$ has the ACL-property (for the definition cf. e.g. [LV, p. 127 and 162]), it is absolutely continuous on almost every chord parallel to either of the coordinate axes, i.e. $\tilde{\varphi}$ is absolutely continuous on $\ell_\xi(t) \cap \Delta$ for a.e. $t \in [-1, 1]$, $\xi = 1, i$. By definition, $\hat{\partial} G = F \circ \gamma$, and so $\hat{\partial}(F \circ \varphi - G) = 0$ on $\mathbf{T}$. Moreover, by our assumption, $\mathcal{D}[G] < \infty$, so that for almost every $y \in [-1, 1]$ and $x \in [-1, 1]$

$$\int_{\Delta \cap \ell_i(y)} |\partial_x(F \circ \varphi - G)| < \infty \quad \text{and} \quad \int_{\Delta \cap \ell_1(x)} |\partial_y(F \circ \varphi - G)| < \infty \,.$$

Fix $n \in \mathbb{N}$. We may now integrate by parts to conclude that for a. e. $y \in [-1, 1]$

$$\int_{\Delta \cap \ell_i(y)} \partial_x(F \circ \varphi - G)\partial_x P_n dx = -\int_{\Delta \cap \ell_i(y)} (F \circ \varphi - G)\partial^2_{xx} P_n dx$$

and for a.e. $x \in [-1, 1]$

$$\int_{\Delta \cap \ell_1(x)} \partial_y(F \circ \varphi - G)\partial_y P_n dx = -\int_{\Delta \cap \ell_1(x)} (F \circ \varphi - G)\partial^2_{yy} P_n dx \,,$$

where $\partial^2_{xx} := \partial_x\partial_x$ and $\partial^2_{yy} := \partial_y\partial_y$. Fubini's theorem then implies

$$\int_{\Delta}(\partial_x(F\circ\varphi - G)\partial_x P_n + \partial_y(F\circ\varphi - G)\partial_y P_n)dS$$
$$= \int_{-1}^{1}(\int_{\Delta\cap\ell_1(y)} \partial_x(F\circ\varphi - G)\partial_x P_n dx)dy$$
$$+ \int_{-1}^{1}(\int_{\Delta\cap\ell_1(x)} \partial_y(F\circ\varphi - G)\partial_y P_n dy)dx$$
$$= -\int_{-1}^{1}(\int_{\Delta\cap\ell_1(y)} (F\circ\varphi - G)\partial^2_{xx} P_n dx)dy$$
$$- \int_{-1}^{1}(\int_{\Delta\cap\ell_1(x)} (F\circ\varphi - G)\partial^2_{yy} P_n dy)dx$$
$$= -\int_{\Delta}(F\circ\varphi - G)(\partial^2_{xx}P_n + \partial^2_{yy}P_n)dS = 0\ ,$$

because P_n is a harmonic function on Δ. Hence

$$\mathcal{D}[F\circ\varphi - G + P_n] = \mathcal{D}[F\circ\varphi - G] + \mathcal{D}[P_n]$$
$$+ 2\int_{\Delta}(\partial_x(F\circ\varphi - G)\partial_x P_n + \partial_y(F\circ\varphi - G)\partial_y P_n)dS$$
$$= \mathcal{D}[F\circ\varphi - G] + \mathcal{D}[P_n]\ .$$

A passage to the limit now implies, by (1.4), that

$$\mathcal{D}[F\circ\varphi - G + P] = \lim_{n\to\infty}\mathcal{D}[F\circ\varphi - G + P_n]$$
$$= \mathcal{D}[F\circ\varphi - G] + \lim_{n\to\infty}\mathcal{D}[P_n] = \mathcal{D}[F\circ\varphi - G] + \mathcal{D}[P]\ ,$$

and this is precisely the equality (1.2). Setting $P := G$ in (1.2) we obtain the equality in (1.3). The inequality in (1.3) follows from (1.1).

Part II. We complete the proof by showing the first part of our assertion, i.e. we prove that G always belongs to $H^2(\Delta)$. Let $\mathbb{A}_{\mathbf{T}}$ be the class of all homeomorphisms $\sigma : \mathbf{T}\to\mathbf{T}$ which have a conformal extension to some open annulus containing $\mathbf{T}$. It is easy to check that each $\sigma\in\mathbb{A}_{\mathbf{T}}$ is a quasisymmetric automorphism of $\mathbf{T}$, so that $\mathbb{A}_{\mathbf{T}}\subset\mathbb{Q}_{\mathbf{T}}$. The inclusion follows immediately also from the Fehlmann characterization of the class $\mathbb{Q}_{\mathbf{T}}$; cf. [F1, Thm. 3.1] and [F2]. It turns out that there exist a constant $K^*\geq 1$ and a sequence $\gamma_n\in\mathbb{A}_{\mathbf{T}}\cap\mathbb{Q}_{\mathbf{T}}(K^*)$, $n\in\mathbb{N}$, satisfying

$$\lim_{n\to\infty}\gamma_n(z) = \gamma(z)\ ,\quad z\in\mathbf{T}; \tag{1.5}$$

cf. [P6, Lemma 3.1.3] and [P5, Thm. 2.1]. For $n \in \mathbb{N}$ define $G_n := \mathcal{P}[F \circ \gamma_n]$. Fix $n \in \mathbb{N}$. It is easily seen from the Douglas formula that $\mathcal{D}[G_n] < \infty$; cf. [D] and [A2, Thm. 2-5, p. 32]. However, this can be obtained in a more direct way as below. Integrating by parts we have for each $k \in \mathbb{N}$

$$a_k := \frac{1}{\pi} \int_{\mathbf{T}} F \circ \gamma_n(u) \overline{u}^k |du| = \frac{1}{\pi} \int_0^{2\pi} F \circ \gamma_n(e^{it}) e^{-ikt} dt$$
$$= -\frac{1}{\pi k^2} \int_0^{2\pi} e^{-ikt} \frac{d^2}{dt^2} F \circ \gamma_n(e^{it}) dt .$$

Hence

$$\mathcal{D}[G_n] = \int_{\mathbf{\Delta}} |((F \circ \gamma_n)_{\mathbf{\Delta}})'|^2 dS = \pi \sum_{k=1}^{\infty} k |a_k|^2$$
$$\leq 4\pi \left(\max_{0 \leq t \leq 2\pi} \left| \frac{d^2}{dt^2} F \circ \gamma_n(e^{it}) \right| \right)^2 \sum_{k=1}^{\infty} \frac{1}{k^3} < \infty ,$$

and we can use Part I to obtain

$$\mathcal{D}[G_n] \leq K^* \mathcal{D}[F] . \tag{1.6}$$

From (1.5) it follows that γ_n is uniformly convergent to γ, and consequently for every $z \in \mathbf{\Delta}$, $\lim_{n\to\infty} \partial G_n(z) = \partial G(z)$ and $\lim_{n\to\infty} \bar{\partial} G_n(z) = \bar{\partial} G(z)$. Then (1.6) shows, by Fatou's lemma, that $\mathcal{D}[G] \leq \liminf_{n\to\infty} \mathcal{D}[G_n] \leq K^* \mathcal{D}[F] < \infty$, which is our claim. Combining Parts I and II yields the assertion of the lemma. □

Theorem 1.2. *Given $\gamma \in \mathbb{Q}_{\mathbf{T}}$ assume that φ is its qc. extension to $\mathbf{\Delta}$. Then for every $f \in H$*

$$2\|B_\gamma(f)\|_H^2 = \mathcal{D}[F \circ \varphi] - \mathcal{D}[F \circ \varphi - G] , \tag{1.7}$$

where $F := \mathcal{P}[\mathcal{S}_0(f)]$ and $G := \mathcal{P}[\mathcal{S}_0 B_\gamma(f)]$. In particular,

$$\|B_\gamma(f)\|_H \leq \sqrt{K(\gamma)} \|f\|_H . \tag{1.8}$$

Proof. Suppose γ and φ are as in the assumption and fix $f \in H$. By (0.9) there exists a sequence $f_n := [f_n / \dot{=}]$, $f_n \in \operatorname{Re} \mathbb{P}(\mathbf{T})$, $n \in \mathbb{N}$, such that

$$\|f - f_n\|_H \to 0 \quad \text{as } n \to \infty . \tag{1.9}$$

Then, by continuity,

$$\|B_\gamma(f) - B_\gamma(f_n)\|_H \to 0 \quad \text{as } n \to \infty . \tag{1.10}$$

For $n \in \mathbb{N}$ set $F_n := \mathcal{P}[f_n]$ and $G_n := \mathcal{P}[f_n \circ \gamma]$. From (1.10) and (0.12) it follows that

$$2\|B_\gamma(f)\|_H^2 = 2 \lim_{n\to\infty} \|B_\gamma(f_n)\|_H^2 = \lim_{n\to\infty} \mathcal{D}[G_n] . \tag{1.11}$$

By (1.1), (1.9), (0.6) and the Minkowski inequality we have

$$\begin{aligned} |\mathcal{D}[F \circ \varphi]^{1/2} - \mathcal{D}[F_n \circ \varphi]^{1/2}| &\le \mathcal{D}[F \circ \varphi - F_n \circ \varphi]^{1/2} = \mathcal{D}[(F - F_n) \circ \varphi]^{1/2} \\ &\le \sqrt{K[\varphi]}\mathcal{D}[(F - F_n)]^{1/2} = \sqrt{2K[\varphi]}\|f - f_n\|_H \to 0 \quad \text{as } n \to \infty , \end{aligned}$$

and consequently

$$\lim_{n\to\infty} \mathcal{D}[F_n \circ \varphi] = \mathcal{D}[F \circ \varphi] . \tag{1.12}$$

In the similar way we show that

$$\begin{aligned} |\mathcal{D}[F \circ \varphi - G]^{1/2} - \mathcal{D}[F_n \circ \varphi - G_n]^{1/2}| &\le \mathcal{D}[F \circ \varphi - F_n \circ \varphi + G_n - G]^{1/2} \\ &\le \mathcal{D}[(F - F_n) \circ \varphi]^{1/2} + \mathcal{D}[G - G_n]^{1/2} \\ &\le \sqrt{2K[\varphi]}\|f - f_n\|_H + \sqrt{2}\|B_\gamma(f) - B_\gamma(f_n)\|_H \to 0 \quad \text{as } n \to \infty , \end{aligned}$$

and so

$$\lim_{n\to\infty} \mathcal{D}[F_n \circ \varphi - G_n] = \mathcal{D}[F \circ \varphi - G] . \tag{1.13}$$

From Lemma 1.1 we conclude that for every $n \in \mathbb{N}$, $\mathcal{D}[G_n] = \mathcal{D}[F_n \circ \varphi] - \mathcal{D}[F_n \circ \varphi - G_n]$. Combining this with (1.11), (1.12) and (1.13) we obtain (1.7). It is a well known fact that for any $K \ge 1$ the class $\{\varphi \in \mathbb{Q}_\Delta(K) : \partial\varphi = \gamma\}$ is compact in the uniform convergence topology on Δ; cf. e.g. [LV, p. 73]. Therefore there exists an extremal $K(\gamma)$-qc. extension ψ of γ to Δ. Setting $\varphi := \psi$ in (1.7) yields (1.8). □

We recall that a qc. self-mapping ψ of Δ is said to be a regular Teichmüller mapping if there exists a non-zero function $F \in A(\Delta)$ and a constant k, $0 \le k < 1$, such that the complex dilatation of ψ is of the form

$$\frac{\bar\partial\psi}{\partial\psi} = k\frac{\overline{F}}{|F|} \quad \text{a.e. on } \Delta . \tag{1.14}$$

We are now in a position to answer the Question 0.3.

Theorem 1.3. *Let $\gamma \in \mathbb{Q}_{\mathbf{T}}$ and let $0 \le k < 1$. If $\boldsymbol{f} \in \boldsymbol{H}$ satisfies $\|\boldsymbol{f}\|_{\boldsymbol{H}} > 0$ and if γ admits a regular qc. Teichmüller extension ψ to $\boldsymbol{\Delta}$ with the complex dilatation*

$$\frac{\bar{\partial}\psi}{\partial\psi} = k\frac{\overline{(\mathcal{S}_0(\boldsymbol{f})_{\boldsymbol{\Delta}})'}}{(\mathcal{S}_0(\boldsymbol{f})_{\boldsymbol{\Delta}})'} \quad \text{a.e. on } \boldsymbol{\Delta}\ , \tag{1.15}$$

then there exists $\boldsymbol{g} \in \boldsymbol{H}$ such that $\|\boldsymbol{g}\|_{\boldsymbol{H}} = 1$ and

$$\|\boldsymbol{B}_\gamma(\boldsymbol{g})\|_{\boldsymbol{H}}^2 = K(\gamma) = \frac{1+k}{1-k}\ . \tag{1.16}$$

In particular, $\|B_\gamma\| = \sqrt{K(\gamma)}$.

Conversely, if $\boldsymbol{g} \in \boldsymbol{H}$ and k satisfy $\|\boldsymbol{g}\|_{\boldsymbol{H}} = 1$ and (1.16), then γ admits a regular qc. Teichmüller extension ψ to $\boldsymbol{\Delta}$ with the complex dilatation (1.15), where $\boldsymbol{f} := \boldsymbol{B}_\gamma(\boldsymbol{g})$. Moreover, ψ is uniquely extremal.

Proof. Assume $\gamma \in \mathbb{Q}_{\mathbf{T}}$ admits a qc. extension ψ to $\boldsymbol{\Delta}$ with the complex dilatation (1.15). Let $F := \mathcal{S}_0(\boldsymbol{f})_{\boldsymbol{\Delta}}$ and $G : \boldsymbol{\Delta} \to \mathbb{C}$ be a function satisfying the equation

$$(1-k)G \circ \psi = k\overline{F} + F\ . \tag{1.17}$$

Differentiating both sides of this equality we get

$$(\partial G) \circ \psi\, \partial\psi + (\bar{\partial} G) \circ \psi\, \partial\,\overline{\psi} = (1-k)^{-1} F'\ ,$$
$$(\partial G) \circ \psi\, \bar{\partial}\psi + (\bar{\partial} G) \circ \psi\, \bar{\partial}\,\overline{\psi} = (1-k)^{-1} k\overline{F'}\ .$$

Since $\partial\psi\,\bar{\partial}\,\overline{\psi} - \bar{\partial}\psi\,\partial\,\overline{\psi} = \partial\psi\,\overline{\partial\psi} - \bar{\partial}\psi\,\overline{\bar{\partial}\psi} = |\partial\psi|^2 - |\bar{\partial}\psi|^2 > 0$ a.e. on $\boldsymbol{\Delta}$, (1.15) shows that $\bar{\partial}G = 0$ a.e. on $\boldsymbol{\Delta}$. In this way the function G is analytic on $\boldsymbol{\Delta}$; cf. [A1, p. 33]. Moreover, by (1.1) we have

$$2(1-k)^2 \int_{\boldsymbol{\Delta}} |G'|^2 dS = \mathcal{D}[(1-k)G] = \mathcal{D}[(k\overline{F} + F) \circ \check{\psi}]$$
$$\le K[\check{\psi}]\mathcal{D}[k\overline{F} + F] = 2K[\psi](1+k^2) \int_{\boldsymbol{\Delta}} |F'|^2 dS < \infty\ .$$

Thus $G \in \dot{A}^2(\boldsymbol{\Delta})$ and, by the definition of $\boldsymbol{H}$, there exists $\boldsymbol{g} \in \boldsymbol{H}$ such that $G - G(0) = \mathcal{S}_0(\boldsymbol{g})_{\boldsymbol{\Delta}}$. By (0.9) there exists a sequence $\boldsymbol{g}_n \in \boldsymbol{H}$, $n \in \mathbb{N}$, such that $g_n := \mathcal{S}_0(\boldsymbol{g}_n) \in \operatorname{Re}\mathbb{P}(\mathbf{T})$ and

$$\|\boldsymbol{g} - \boldsymbol{g}_n\|_{\boldsymbol{H}} \to 0 \quad \text{as } n \to \infty\ . \tag{1.18}$$

From (1.17) we see that $G \circ \psi \in \dot{H}^2(\Delta)$, so that

$$\operatorname{Re} G \circ \psi = \mathcal{P}[\mathcal{S}_0(h)] + c \tag{1.19}$$

for some $h \in H$ and $c \in \mathbb{R}$. Moreover, from Lemma 1.1, (1.1), (0.6) and (1.18) it follows that

(1.20)

$$\begin{aligned} 2\|h - B_\gamma(g_n)\|_H^2 &= \mathcal{D}[\operatorname{Re} G \circ \psi - \mathcal{P}[g_n \circ \gamma]] \le \mathcal{D}[\operatorname{Re} G \circ \psi - \mathcal{P}[g_n] \circ \psi] \\ &\le K[\psi]\mathcal{D}[\operatorname{Re} G - \mathcal{P}[g_n]] = 2K[\psi]\|g - g_n\|_H^2 \to 0 \quad \text{as } n \to \infty . \end{aligned}$$

On the other hand, by the definition of the operator B_γ and by (1.18), we obtain

$$\|B_\gamma(g_n) - B_\gamma(g)\|_H \to 0 \quad \text{as } n \to \infty .$$

Combining this with (1.20) we conclude that

$$B_\gamma(g) = h . \tag{1.21}$$

Theorem 1.2 now shows, by (1.19), (1.21), (0.6) and (1.17), that

$$\begin{aligned} 2\|B_\gamma(g)\|_H^2 &= \mathcal{D}[\operatorname{Re} G \circ \psi] - \mathcal{D}[\operatorname{Re} G \circ \psi - \mathcal{P}[\mathcal{S}_0 B_\gamma(g)]] \\ &= \mathcal{D}[\operatorname{Re} G \circ \psi] - \mathcal{D}[\operatorname{Re} G \circ \psi - \mathcal{P}[\mathcal{S}_0(h)]] = \mathcal{D}[\operatorname{Re} G \circ \psi] \\ &= \left(\frac{1+k}{1-k}\right)^2 \mathcal{D}[\operatorname{Re} F] = 2\left(\frac{1+k}{1-k}\right)^2 \|f\|_H^2 . \end{aligned} \tag{1.22}$$

Since G is analytic, we see that

$$G' \circ \psi\, \partial\psi = (1-k)^{-1} F' \quad \text{and} \quad G' \circ \psi\, \bar{\partial}\psi = (1-k)^{-1} k \overline{F'},$$

and hence, by (0.6), that

$$\begin{aligned} 2\frac{1+k}{1-k}\|f\|_H^2 &= \frac{1+k}{1-k}\int_\Delta |F'|^2 dS \\ &= \frac{1}{(1-k)^2}\int_\Delta |F'|^2 dS - \frac{k^2}{(1-k)^2}\int_\Delta |\overline{F'}|^2 dS \\ &= \int_\Delta (|G' \circ \psi\, \partial\psi|^2 - |G' \circ \psi\, \bar{\partial}\psi|^2) dS \\ &= \int_\Delta |G' \circ \psi|^2 (|\partial\psi|^2 - |\bar{\partial}\psi|^2) dS = \int_\Delta |G'|^2 dS = 2\|g\|_H^2 . \end{aligned} \tag{1.23}$$

Combining this with (1.8) and (1.22) we obtain

$$\|B_\gamma(g)\|_H^2 \le K(\gamma)\|g\|_H^2 \le \left(\frac{1+k}{1-k}\right)^2 \|f\|_H^2 = \|B_\gamma(g)\|_H^2 . \tag{1.24}$$

From (1.23) and (1.24) it follows that

$$\|B^*_\gamma(g)\|^2_H = K(\gamma)\|g\|^2_H = (1+k)(1-k)^{-1}\|g\|^2_H .$$

Replacing now g by $g/\|g\|_H$ we obtain (1.16). That $\|B_\gamma\| = \sqrt{K(\gamma)}$ follows from Theorem 1.2 and (1.16), which proves the first part of Theorem 1.3.

Suppose now that $g \in H$ and k satisfy $\|g\|_H = 1$ and (1.16). Since for any $K \geq 1$ the class $\{\varphi \in \mathbb{Q}_\Delta(K) : \hat{\partial}\varphi = \gamma\}$ is compact in the uniform convergence topology on Δ (cf. e.g. [LV, p. 73]), there exists an extremal $K(\gamma)$-qc. extension ψ of γ to Δ. Set $F := (\mathcal{S}_0 B_\gamma(g))_\Delta$ and $G := (\mathcal{S}_0(g))_\Delta$. Then Theorem 1.2 shows, by (1.1) and (0.6), that

$$\begin{aligned}\mathcal{D}[\operatorname{Re} G\circ\psi - \operatorname{Re} F] &= \mathcal{D}[\operatorname{Re} G\circ\psi] - 2\|B_\gamma(g)\|^2_H \leq \\ K[\psi]\mathcal{D}[\operatorname{Re} G] &- 2K(\gamma) = 2K(\gamma)(\|g\|^2_H - 1) = 0 .\end{aligned}$$

Hence the equality $\operatorname{Re} G\circ\psi = \operatorname{Re} F + c$ holds a.e. on Δ for some constant $c \in \mathbb{R}$. Differentiating both sides of this equality we get

$$G'\circ\psi\,\partial\psi + \overline{G'}\circ\psi\,\partial\,\overline{\psi} = F' . \tag{1.25}$$

Hence,

$$\begin{aligned}|F'|^2 &= |G'\circ\psi|^2|\partial\psi|^2 + |G'\circ\psi|^2|\partial\overline{\psi}|^2 + (G'\circ\psi)^2\partial\psi\,\overline{\partial\overline{\psi}} + (\overline{G'}\circ\psi)^2\overline{\partial\psi}\,\partial\overline{\psi} \\ &= |G'\circ\psi|^2(|\partial\psi| + |\bar{\partial}\psi|)^2 - Q = |G'\circ\psi|^2\frac{|\partial\psi| + |\bar{\partial}\psi|}{|\partial\psi| - |\bar{\partial}\psi|}(|\partial\psi|^2 - |\bar{\partial}\psi|^2) - Q\end{aligned} \tag{1.26}$$

a.e. on Δ, where $Q := 2|G'\circ\psi|^2|\partial\psi||\bar{\partial}\psi| - (G'\circ\psi)^2\partial\psi\,\bar{\partial}\psi - (\overline{G'}\circ\psi)^2\overline{\bar{\partial}\psi}\,\overline{\partial\psi}$ a.e. on Δ. Since ψ is $K(\gamma)$-qc., $(|\partial\psi| + |\bar{\partial}\psi|)(|\partial\psi| - |\bar{\partial}\psi|)^{-1} \leq K(\gamma)$ a.e. on Δ. Combining this with (1.16) and (1.26) we obtain by (0.6)

$$\begin{aligned}2K(\gamma) = 2\|B_\gamma(g)\|^2_H &= \int_\Delta |F'|^2 dS \\ &\leq K(\gamma)\int_\Delta |G'\circ\psi|^2(|\partial\psi|^2 - |\bar{\partial}\psi|^2)dS - \int_\Delta Q\,dS \\ &= K(\gamma)\int_\Delta |G'|^2 dS - \int_\Delta Q\,dS = 2K(\gamma)\|g\|^2_H - \int_\Delta Q\,dS \\ &= 2K(\gamma) - \int_\Delta Q\,dS \leq 2K(\gamma) .\end{aligned}$$

The inequality is possible iff the equalities

$$|G'\circ\psi|^2|\partial\psi||\bar{\partial}\psi| = (G'\circ\psi)^2\partial\psi\,\bar{\partial}\psi \quad \text{and} \quad \frac{|\bar{\partial}\psi|}{|\partial\psi|} = k$$

hold a.e. on Δ. Therefore,

$$\frac{\bar{\partial}\psi}{\partial\psi} = k\frac{\overline{G'\circ\psi\,\partial\psi}}{G'\circ\psi\,\partial\psi} \quad \text{a.e. on } \Delta \,. \tag{1.27}$$

Let $f := B_\gamma(g)$. Then $F = (\mathcal{S}_0(f))_\Delta$. We conclude from (1.27) and (1.25) that $G'\circ\psi\,\partial\psi = (1+k)^{-1}F'$, hence that

$$\frac{\bar{\partial}\psi}{\partial\psi} = k\frac{\overline{F'}}{F'} \quad \text{a.e. on } \Delta \,, \tag{1.28}$$

and finally that (1.15) holds. This ends the proof of the converse statement.

We now prove the uniqueness of the extremal extension ψ. Suppose $\tilde{\psi}$ is another extremal $K(\gamma)$-qc. extension of γ to Δ. Then the Beltrami equation (1.28) holds with ψ replaced by $\tilde{\psi}$. Hence $\bar{\partial}(\tilde{\psi}\circ\check{\psi}) = 0$ a.e. on Δ, and so $\tilde{\psi}\circ\check{\psi} \in \mathbb{Q}_\Delta(1)$. Since $\hat{\partial}\psi = \hat{\partial}\tilde{\psi} = \gamma$, we see that $\psi = \tilde{\psi}$ on Δ. □

Corollary 1.4. *If $K \geq 1$ and if $g \in H$ satisfies $\|g\|_H = 1$, then there exists $\gamma \in \mathbb{Q}_\mathbf{T}$ such that*

$$\|B_\gamma\|^2 = \|B_\gamma(g)\|_H^2 = K(\gamma) = K \,. \tag{1.29}$$

Moreover, γ admits a unique regular $K(\gamma)$-qc. Teichmüller extension ψ to Δ with the complex dilatation given by (1.15), *where $k := (K-1)/(K+1)$ and $f := B_\gamma(g)$.*

Proof. Given $K \geq 1$ and $g \in H$, let $\|g\|_H = 1$ and let k and f be as above. By the Mapping Theorem [LV, p. 194] (also cf. [B] and [ŁK, p. 45]), there exists a solution ψ of the Beltrami equation (1.15) being a K-qc. self-mapping of Δ. Hence $\gamma := \hat{\partial}\psi \in \mathbb{Q}_\mathbf{T}(K)$. Theorem 1.3 now shows that (1.29) holds and γ admits a unique regular $K(\gamma)$-qc. Teichmüller extension ψ to Δ with the complex dilatation given by (1.15). □

2. The smallest eigenvalue of a quasisymmetric automorphism of the unit circle. If $f \in \operatorname{Re} L^1(\mathbf{T})$ then, by (0.3), $\operatorname{Im} f_\Delta$ is a real-valued harmonic function on Δ. A classical result states that the function $\operatorname{Im} f_\Delta$ has a finite non-tangential limit a.e. on $\mathbf{T}$ and

$$\hat{\partial}_r \operatorname{Im} f_\Delta(z) = \lim_{r\to 1^-} \operatorname{Im} f_\Delta(rz) = \frac{1}{\pi}\operatorname{Re}\operatorname{PV}\int_\mathbf{T}\frac{f(u)}{z-u}du$$

for a.e. $z \in \mathbf{T}$; cf. e.g. [G, p. 103]. For every $f := [f/\doteq] \in H$, define

$$A(f) := [\hat{\partial}_r \operatorname{Im} f_\Delta/\doteq] \,. \tag{2.1}$$

Since $\mathcal{D}[\operatorname{Re} F] = \mathcal{D}[\operatorname{Im} F]$ for $F \in \dot{A}^2(\Delta)$, we conclude from (2.1) and the definition of the space H that

$$A(H) = H \quad , \quad A^2 = -I \quad \text{and} \quad \|A\| = 1 \,, \tag{2.2}$$

and so the operator A maps isometrically H onto itself. Therefore, the operator

$$A_\gamma := B_\gamma A B_\gamma^{-1} \,, \tag{2.3}$$

called the generalized harmonic conjugation operator, is a linear homeomorphism of H onto itself; cf. [P2]. We recall that a real number λ is said to be an eigenvalue of $\gamma \in \mathbb{Q}_{\mathbf{T}}$ if there exists $f \in H$ with $\|f\|_H = 1$ such that

$$(\lambda + 1)A(f) = (\lambda - 1)A_\gamma(f) \,; \tag{2.4}$$

cf. [P3, Definition 1.1]. For every $\gamma \in \mathbb{Q}_{\mathbf{T}}$ write Λ_γ^* for the set of all eigenvalues of γ and define

$$\lambda_*(\gamma) = \min\{\lambda > 0 : \lambda \in \Lambda_\gamma^*\}$$

whenever $\Lambda_\gamma^* \neq \emptyset$ and the minimum exists, while $\lambda_*(\gamma) = \infty$ otherwise. From [P3, Thm. 1.4] it follows that $\lambda_*(\gamma) = \infty$ for $\gamma \in \mathbb{Q}_{\mathbf{T}}(1)$, and

$$\lambda_*(\gamma) \geq (K(\gamma) + 1)/(K(\gamma) - 1) \quad \text{for } \gamma \in \mathbb{Q}_{\mathbf{T}} \setminus \mathbb{Q}_{\mathbf{T}}(1) \,. \tag{2.5}$$

A sufficient condition on γ for the equality in (2.5) to hold, was obtained in [P4, Thm. 2.2]. We use this result to show the following

Theorem 2.1. *Let $\gamma \in \mathbb{Q}_{\mathbf{T}} \setminus \mathbb{Q}_{\mathbf{T}}(1)$. Then*

$$\lambda_*(\gamma) = (K(\gamma) + 1)/(K(\gamma) - 1) \tag{2.6}$$

iff there exists $g \in H$ such that $\|g\|_H = 1$ and that

$$\|B_\gamma(g)\|_H^2 = K(\gamma) \,. \tag{2.7}$$

Proof. Assume first that (2.7) holds. Then Theorem 1.3 shows that γ admits a regular qc. Teichmüller extension ψ to Δ with the complex dilatation (1.15), where $f := B_\gamma(g)$ and $k := (K(\gamma) - 1)/(K(\gamma) + 1)$. Therefore (2.6) follows from [P4, Thm. 2.2].

Conversely, assume that (2.6) holds. Then there exists $f \in H$ such that $\|f\|_H = 1$ and that (2.4) holds with λ replaced by $\lambda_*(\gamma)$. Hence, by (2.2) and (2.5) we have

$$(\lambda_*(\gamma) + 1)\|f\|_H = (\lambda_*(\gamma) - 1)\|A_\gamma(f)\|_H$$

and consequently, by (2.3) and (2.6),

$$K(\gamma) = \|B_\gamma A B_\gamma^{-1}(f)\|_H .$$

Set $g := (\sqrt{K(\gamma)})^{-1} AB_\gamma^{-1}(f)$. Theorem 1.2, (0.15) and (2.2) now imply that

$$\|g\|_H \leq (\sqrt{K(\gamma)})^{-1}\|A\|\|B_\gamma^{-1}\|\|f\|_H = (\sqrt{K(\gamma)})^{-1}\|B_{\check{\gamma}}\|$$
$$\leq (\sqrt{K(\gamma)})^{-1}\sqrt{K(\check{\gamma})} = 1;$$

moreover,

$$K(\gamma) = \|B_\gamma A B_\gamma^{-1}(f)\|_H = \sqrt{K(\gamma)}\|B_\gamma(g)\|_H$$
$$\leq \sqrt{K(\gamma)}\|B_\gamma\|\|g\|_H \leq K(\gamma).$$

Combining the above inequalities we see, by (0.13), that $\|g\|_H = 1$ and the equality (2.7) holds. □

Corollary 2.2. *If $\gamma \in \mathbb{Q}_{\mathbf{T}}$ and if (2.7) holds for some $g \in H$ such that $\|g\|_H = 1$, then there exists $f \in H$ satisfying*

$$\text{(2.8)} \qquad \|f\|_H = 1 \quad , \quad \|B_\gamma^{-1}(f)\|_H^2 = K(\gamma) .$$

In particular, $\|B_\gamma\| = \|B_\gamma^{-1}\| = \sqrt{K(\gamma)}$.

Proof. Given $\gamma \in \mathbb{Q}_{\mathbf{T}} \setminus \mathbb{Q}_{\mathbf{T}}(1)$ assume (2.7) holds for some $g \in H$ such that $\|g\|_H = 1$. Therefore (2.6) holds by Theorem 2.1. From [P3, Thm. 1.4 (v)] it follows that $\Lambda_\gamma^* = \Lambda_{\check{\gamma}}^*$, and hence

$$\lambda_*(\gamma) = \min\{\lambda > 0 : \lambda \in \Lambda_\gamma^*\} = \min\{\lambda > 0 : \lambda \in \Lambda_{\check{\gamma}}^*\} = \lambda_*(\check{\gamma}) .$$

Combining this with (2.6) we obtain

$$\lambda_*(\check{\gamma}) = \lambda_*(\gamma) = (K(\gamma)+1)/(K(\gamma)-1) = (K(\check{\gamma})+1)/(K(\check{\gamma})-1) .$$

Applying Theorem 2.1 again, with γ replaced by $\check{\gamma}$, we see, by (0.15), that there exists $f \in H$ satisfying (2.8). Moreover, combining (2.7) and (2.8) with (0.13) we have

$$\sqrt{K(\gamma)} = \|B_\gamma(g)\|_H \leq \|B_\gamma\| \leq \sqrt{K(\gamma)},$$
$$\sqrt{K(\check{\gamma})} = \|B_{\check{\gamma}}(f)\|_H \leq \|B_{\check{\gamma}}\| \leq \sqrt{K(\check{\gamma})},$$

and hence $\|B_\gamma\| = \sqrt{K(\gamma)} = \sqrt{K(\check{\gamma})} = \|B_{\check{\gamma}}\| = \|B_\gamma^{-1}\|$ as claimed.

If $\gamma \in \mathbb{Q}_{\mathbf{T}}(1)$ then, by (0.13) and (0.15), we obtain $\|B_\gamma\| \leq \sqrt{K(\gamma)} = 1$ and $\|B_\gamma^{-1}\| = \|B_{\check{\gamma}}\| \leq \sqrt{K(\check{\gamma})} = 1$. Hence the operators B_γ and B_γ^{-1} are isometries of H onto itself, and the corollary follows. □

3. The Schober constant $\lambda_s(\Gamma)$. Given a Jordan curve $\Gamma \subset \mathbb{C}$ write $H(\Gamma)$ for the family of all real-valued functions F continuous on $\hat{\mathbb{C}}$ and harmonic on $\Omega \cup \Omega_* = \hat{\mathbb{C}} \setminus \Gamma$ which satisfy $0 < \mathcal{D}_\Omega[F] + \mathcal{D}_{\Omega_*}[F] < \infty$, where Ω and $\Omega_* \ni \infty$ are complementary domains to Γ. Define

$$\frac{1}{\lambda_s(\Gamma)} := \sup\left\{ \frac{|\mathcal{D}_\Omega[F] - \mathcal{D}_{\Omega_*}[F]|}{\mathcal{D}_\Omega[F] + \mathcal{D}_{\Omega_*}[F]} : F \in H(\Gamma) \right\}$$

provided the supremum is a positive number, while $\lambda_s(\Gamma) = \infty$ otherwise. For a short survey of basic properties of the curve functional $\lambda_s(\Gamma)$ we refer the reader to the Schober article [S]; also see the references given there. Let Φ and Φ_* denote conformal mappings of Δ and $\mathbb{C} \setminus \overline{\Delta}$ onto the domains Ω and Ω_*, respectively. Such mappings exist by the Riemann mapping theorem; cf. for instance [R, Thm. 14.8]. Moreover, by the Taylor-Osgood-Carathéodory theorem both the mappings Φ and Φ_* have homeomorphic extensions to the closures $\overline{\Omega}$ and $\overline{\Omega}_*$, respectively; cf. for instance [R, Thm. 14.19]. Then $\gamma := \hat{\partial}\check{\Phi}_* \circ \hat{\partial}\Phi$ is a sense-preserving homeomorphic self-mapping of $\mathbf{T}$. We recall that every homeomorphism γ assigned to Γ in this way is said to be a welding homeomorphism of $\Gamma \subset \mathbb{C}$. The class of all welding homeomorphisms of Γ will be denoted by $\mathrm{Weld}(\Gamma)$. For $z \in \mathbb{C} \setminus \{0\}$ set $\hbar(z) := 1/z$, and let $\hbar(0) := \infty$, $\hbar(\infty) := 0$. If a Jordan curve $\Gamma \subset \mathbb{C}$ admits a K-qc. reflection Ψ then $\psi := \overline{\hbar} \circ \check{\Phi}_* \circ \Psi \circ \Phi$ is a K-qc. extension of $\gamma := \hat{\partial}\check{\Phi}_* \circ \hat{\partial}\Phi$ to Δ. Conversely, if ψ is a K-qc. extension of γ to Δ then Ψ,

$$\Psi(z) := \begin{cases} \Phi_* \circ \overline{\hbar} \circ \psi \circ \check{\Phi}(z) & , z \in \overline{\Omega}, \\ \check{\Psi}(z) & , z \in \Omega_*, \end{cases} \tag{3.1}$$

is a K-qc. reflection in Γ. Thus for every $K \geq 1$,

(3.2)
a Jordan curve $\Gamma \subset \mathbb{C}$ *admits a* K*-qc. reflection iff* $\mathrm{Weld}(\Gamma) \subset \mathbb{Q}_{\mathbf{T}}(K)$.

Lemma 3.1. *For every quasicircle* $\Gamma \subset \mathbb{C}$ *the following equality holds*

$$\frac{1}{\lambda_s(\Gamma)} = \frac{(\max\{\|B_\gamma\|, \|B_\gamma^{-1}\|\})^2 - 1}{(\max\{\|B_\gamma\|, \|B_\gamma^{-1}\|\})^2 + 1}, \tag{3.3}$$

where $\gamma \in \mathrm{Weld}(\Gamma)$.

Proof. Given a quasicircle $\Gamma \subset \mathbb{C}$ let $F \in H(\Gamma)$. Define $G := F \circ \Phi$ and $G_* := F \circ \Phi_* \circ \overline{\hbar}$. By the conformal invariance of the Dirichlet integral we have

$$\mathcal{D}[G] = \mathcal{D}_\Omega[F] \quad \text{and} \quad \mathcal{D}[G_*] = \mathcal{D}_{\Omega_*}[F] , \tag{3.4}$$

and consequently, by (0.10), $\mathbf{g} := [g/\doteq], \mathbf{g}_* := [g_*/\doteq] \in \boldsymbol{H}$, where $g := \hat{\partial}G$ and $g_* := \hat{\partial}G_*$. Since both the functions g and g_* are continuous on $\mathbf{T}$ and $g = g_* \circ \gamma$, we conclude from (0.12) that

$$\mathbf{g} = B_\gamma(\mathbf{g}_*) \quad \text{and} \quad \mathbf{g}_* = B_\gamma^{-1}(\mathbf{g})\,. \tag{3.5}$$

From (3.4), (3.5) and (0.6) it follows that

$$\begin{aligned}\frac{\mathcal{D}_\Omega[F] - \mathcal{D}_{\Omega_*}[F]}{\mathcal{D}_\Omega[F] + \mathcal{D}_{\Omega_*}[F]} = \frac{\mathcal{D}[G] - \mathcal{D}[G_*]}{\mathcal{D}[G] + \mathcal{D}[G_*]} &= \frac{\|\mathbf{g}\|_{\boldsymbol{H}}^2 - \|\mathbf{g}_*\|_{\boldsymbol{H}}^2}{\|\mathbf{g}\|_{\boldsymbol{H}}^2 + \|\mathbf{g}_*\|_{\boldsymbol{H}}^2} \\ &= \frac{\|B_\gamma(\mathbf{g}_*)\|_{\boldsymbol{H}}^2 - \|\mathbf{g}_*\|_{\boldsymbol{H}}^2}{\|B_\gamma(\mathbf{g}_*)\|_{\boldsymbol{H}}^2 + \|\mathbf{g}_*\|_{\boldsymbol{H}}^2} \le \frac{\|B_\gamma\|^2 - 1}{\|B_\gamma\|^2 + 1}\,,\end{aligned}$$

and similarly

$$\frac{\mathcal{D}_{\Omega_*}[F] - \mathcal{D}_\Omega[F]}{\mathcal{D}_\Omega[F] + \mathcal{D}_{\Omega_*}[F]} = \frac{\|B_\gamma^{-1}(\mathbf{g})\|_{\boldsymbol{H}}^2 - \|\mathbf{g}\|_{\boldsymbol{H}}^2}{\|B_\gamma^{-1}(\mathbf{g})\|_{\boldsymbol{H}}^2 + \|\mathbf{g}\|_{\boldsymbol{H}}^2} \le \frac{\|B_\gamma^{-1}\|^2 - 1}{\|B_\gamma^{-1}\|^2 + 1}\,.$$

Combining the above inequalities we obtain

$$\frac{|\mathcal{D}_\Omega[F] - \mathcal{D}_{\Omega_*}[F]|}{\mathcal{D}_\Omega[F] + \mathcal{D}_{\Omega_*}[F]} \le \frac{(\max\{\|B_\gamma\|, \|B_\gamma^{-1}\|\})^2 - 1}{(\max\{\|B_\gamma\|, \|B_\gamma^{-1}\|\})^2 + 1}\,,$$

and consequently

$$\frac{1}{\lambda_s(\Gamma)} \le \frac{(\max\{\|B_\gamma\|, \|B_\gamma^{-1}\|\})^2 - 1}{(\max\{\|B_\gamma\|, \|B_\gamma^{-1}\|\})^2 + 1}\,. \tag{3.6}$$

It remains to show the inverse inequality of (3.6). Fix $g_* \in \operatorname{Re}\mathbb{P}(\mathbf{T})$ and let $g := g_* \circ \gamma$. For $z \in \hat{\mathbb{C}}$ define $F(z) := \mathcal{P}[g] \circ \check{\Phi}(z)$ if $z \in \Omega$, $F(z) := \mathcal{P}[g_*] \circ \overline{h} \circ \check{\Phi}_*(z)$ if $z \in \Omega_*$ and $F(z) := \hat{\partial}(\mathcal{P}[g] \circ \check{\Phi}(z))$ if $z \in \Gamma$. Since both the functions g and g_* are continuous on $\mathbf{T}$, we see that for every $z \in \Gamma$

$$\hat{\partial}(\mathcal{P}[g] \circ \check{\Phi})(z) = g \circ \hat{\partial}\check{\Phi}(z) = g_* \circ \gamma \circ \hat{\partial}\check{\Phi}(z) = g_* \circ \hat{\partial}\check{\Phi}_*(z) = \hat{\partial}(\mathcal{P}[g_*] \circ \overline{h} \circ \check{\Phi}_*)(z)\,.$$

Therefore the function F is continuous on $\hat{\mathbb{C}}$.

We can assume that $\|\mathbf{g}_*\|_{\boldsymbol{H}} = 1$, where $\mathbf{g}_* := [g_*/\doteq] \in \boldsymbol{H}$. By (0.12), $g \doteq \mathcal{S}_0 B_\gamma(\mathbf{g}_*)$. The conformal invariance of the Dirichlet integral now shows, by (0.6), that

$$\mathcal{D}_\Omega[F] = \mathcal{D}[\mathcal{P}[g]] = 2\|B_\gamma(\mathbf{g}_*)\|_{\boldsymbol{H}}^2 < \infty$$

and

$$\mathcal{D}_{\Omega_*}[F] = \mathcal{D}[\mathcal{P}[g_*]] = 2\|\mathbf{g}_*\|_{\boldsymbol{H}}^2 = 2\,.$$

Hence $F \in H(\Gamma)$ and

$$\frac{\|B_\gamma(g_*)\|_H^2 - 1}{\|B_\gamma(g_*)\|_H^2 + 1} \le \frac{|\mathcal{D}_\Omega[F] - \mathcal{D}_{\Omega_*}[F]|}{\mathcal{D}_\Omega[F] + \mathcal{D}_{\Omega_*}[F]} \le \frac{1}{\lambda_s(\Gamma)} . \tag{3.7}$$

By (0.9), $\|B_\gamma\| = \sup\{\|B_\gamma(f)\|_H : \mathcal{S}_0(f) \in \operatorname{Re}\mathbb{P}(\mathbf{T}) \text{ and } \|f\|_H = 1\}$. Then (3.7) leads to

$$\frac{\|B_\gamma\|^2 - 1}{\|B_\gamma\|^2 + 1} \le \frac{1}{\lambda_s(\Gamma)} .$$

The same conclusion can be drawn for the inverse operator B_γ^{-1}, and so

$$\frac{(\max\{\|B_\gamma\|, \|B_\gamma^{-1}\|\})^2 - 1}{(\max\{\|B_\gamma\|, \|B_\gamma^{-1}\|\})^2 + 1} \le \frac{1}{\lambda_s(\Gamma)} .$$

Combining this with (3.6) we obtain (3.3). □

Assume $\Gamma \subset \mathbb{C}$ is a quasicircle. By (3.2) we see that Ψ is an extremal K-qc. reflection in Γ iff $\psi := \bar{h} \circ \bar{\Phi}_* \circ \Psi \circ \Phi$ is an extremal K-qc. extension of γ to $\mathbf{\Delta}$, $K \ge 1$. Moreover, the complex dilatations of ψ and Ψ are related by the equality

$$\frac{\bar{\partial}\psi}{\partial\psi} = \overline{\left[\frac{(\partial\Psi \circ \Phi)\Phi'}{(\bar{\partial}\Psi \circ \Phi)\overline{\Phi'}}\right]} \quad \text{a.e. on } \mathbf{\Delta} . \tag{3.8}$$

This observation is the key for the proof of the following

Theorem 3.2. *Given $K > 1$ suppose that $\Gamma \subset \mathbb{C}$ admits a regular K-qc. reflection Ψ with the complex dilatation*

$$\frac{\bar{\partial}\Psi}{\partial\Psi} = \frac{K+1}{K-1}\frac{\overline{G'}}{G'} \quad \text{a.e. on } \Omega , \tag{3.9}$$

where $G \in \dot{A}^2(\Omega)$ is a non-constant function. Then for each $\gamma \in \mathrm{Weld}(\Gamma)$

$$\lambda_s(\Gamma) = \frac{K+1}{K-1} = \frac{\|B_\gamma\|^2 + 1}{\|B_\gamma\|^2 - 1} , \tag{3.10}$$

and Ψ is a unique extremal K-qc. reflection in Γ.

Proof. Given a non-constant function $G \in \dot{A}^2(\Omega)$, let $F := G \circ \Phi - G \circ \Phi(0)$. Then $F \in \dot{A}(\mathbf{\Delta})$, $\mathcal{D}[F] > 0$ and $F' = (G' \circ \Phi)\Phi'$. By (3.8) and (3.9) we have

$$\frac{\bar{\partial}\psi}{\partial\psi} = \frac{\overline{(\partial\Psi \circ \Phi)}\,\overline{\Phi'}}{\overline{(\bar{\partial}\Psi \circ \Phi)}\,\Phi'} = \frac{K-1}{K+1}\frac{\overline{(G' \circ \Phi)}\,\overline{\Phi'}}{G' \circ \Phi\,\Phi'} = \frac{K-1}{K+1}\frac{\overline{F'}}{F'} \quad \text{a.e. on } \mathbf{\Delta} . \tag{3.11}$$

By (0.10), $f := [\hat{\partial}_r \operatorname{Re} F / \div] \in H$ and $F = \mathcal{S}_0(f)_\Delta$.
By (0.11), $4\|f\|_H^2 = \mathcal{D}[F] > 0$. Theorem 1.3 now yields (1.29) for some $g \in H$ satisfying $\|g\|_H = 1$. Therefore $\|B_\gamma\| = \|B_\gamma^{-1}\|$ by Corollary 2.2. Hence (3.10) follows from (1.29) and Lemma 3.1. Moreover, from Theorem 1.3 and (3.11) we conclude that ψ is a unique extremal K-qc. extension of γ to Δ, and hence that Ψ is a unique extremal K-qc. reflection in Γ. □

Corollary 3.3. *For every $K > 1$ and every non-constant function $G \in \dot{A}^2(\Omega)$ there exists a quasicircle $\Gamma \subset \mathbb{C}$ which admits a unique extremal K-qc. reflection Ψ with the complex dilatation given by* (3.9), *and for each $\gamma \in \operatorname{Weld}(\Gamma)$ the equality* (3.10) *holds.*

Proof. Fix a non-constant function $G \in \dot{A}^2(\Omega)$. Then the function $F := G\circ\Phi - G\circ\Phi(0) \in \dot{A}(\Delta)$ is also non-constant. Following the proof of Corollary 1.4 we see that there exists $\gamma \in \mathbb{Q}_{\mathbf{T}}$ which admits a unique regular $K(\gamma)$-qc. Teichmüller extension ψ to Δ with the complex dilatation given by (3.11), where $K := K(\gamma)$. It can be shown that γ is a welding homeomorphism of some quasicircle $\Gamma \subset \mathbb{C}$; cf. e.g. [P1] or [V]. Then the mapping Ψ, given by (3.1), is a unique extremal K-qc. reflection and, by (3.8), its complex dilatation satisfies the equation (3.9). Then Theorem 3.2 shows that (3.10) holds for each $\gamma \in \operatorname{Weld}(\Gamma)$. □

Theorem 3.4. *Given $K > 1$ suppose that $\Gamma \subset \mathbb{C}$ admits a regular K-qc. reflection Ψ. If*

(3.12) $$\lambda_s(\Gamma) = \frac{K+1}{K-1}$$

and if there exists a sequence $G_n \in H(\Gamma)$, $\mathcal{D}_\Omega[G_n] = 1$, $n \in \mathrm{N}$, such that

(3.13) $$\frac{1}{\lambda_s(\Gamma)} = \lim_{n\to\infty} \frac{|\mathcal{D}_\Omega[G_n] - \mathcal{D}_{\Omega_*}[G_n]|}{\mathcal{D}_\Omega[G_n] + \mathcal{D}_{\Omega_*}[G_n]},$$

and that

(3.14) $$\mathcal{D}_\Omega[G_n - G_m] \to 0 \quad \text{as } n, m \to \infty,$$

then the equation (3.9) *holds for some non-constant function $G \in \dot{A}^2(\Omega)$ and the equation* (3.10) *holds. In particular, Ψ is a unique extremal K-qc. reflection in Γ.*

Proof. Let $\gamma := \hat{\partial}\Phi_* \circ \hat{\partial}\Phi \in \operatorname{Weld}(\Gamma)$. For every $n \in \mathrm{N}$, set $F_n := G_n \circ \Phi$, $F_{n,*} := G_n \circ \Phi_* \circ \bar{h}$ and $f_n := \hat{\partial} F_n$. By the conformal invariance of the Dirichlet integral we have

(3.15) $$\mathcal{D}[F_n] = \mathcal{D}_\Omega[G_n] = 1 < \infty \text{ and } \mathcal{D}[F_{n,*}] = \mathcal{D}_{\Omega_*}[G_n] < \infty, \quad n \in \mathrm{N}.$$

Then for each $n \in \mathbb{N}$, $F_n \in \dot{H}^2(\Delta)$, and consequently, by (0.10) and (0.11), we see that $f_n := [f_n / \doteqdot] \in H$ and $2\|f_n\|_H^2 = \mathcal{D}[F_n] = 1$. Since each $f_n \in C(\mathbf{T})$ and $\hat{\partial} F_n = \hat{\partial} F_{n,*} \circ \gamma$, we conclude from (0.12) and (0.6) that

$$2\|B_\gamma^{-1}(f_n)\|_H^2 = \mathcal{D}[F_{n,*}], \quad n \in \mathbb{N}. \tag{3.16}$$

Applying the conformal invariance of the Dirichlet integral once again we see, by (3.14) and (0.6), that

$$2\|f_n - f_m\|_H^2 = \mathcal{D}[F_n - F_m] = \mathcal{D}_\Omega[G_n - G_m] \to 0 \quad \text{as } n, m \to \infty .$$

Therefore there exists $f \in H$ such that $\|f - f_n\|_H \to 0$ as $n \to \infty$, and hence $\|f\|_H > 0$. By this, (3.13), (3.15), (3.16) and by the continuity of the operator B_γ^{-1} we obtain

$$\begin{aligned} \frac{1}{\lambda_s(\Gamma)} &= \lim_{n\to\infty} \frac{|\mathcal{D}_\Omega[G_n] - \mathcal{D}_{\Omega_-}[G_n]|}{\mathcal{D}_\Omega[G_n] + \mathcal{D}_{\Omega_-}[G_n]} = \lim_{n\to\infty} \frac{|\mathcal{D}[F_n] - \mathcal{D}[F_{n,*}]|}{\mathcal{D}[F_n] + \mathcal{D}[F_{n,*}]} \\ &= \lim_{n\to\infty} \frac{|\|B_\gamma^{-1}(f_n)\|_H^2 - \|f_n\|_H^2|}{\|B_\gamma^{-1}(f_n)\|_H^2 + \|f_n\|_H^2} = \frac{|\|B_\gamma^{-1}(f)\|_H^2 - \|f\|_H^2|}{\|B_\gamma^{-1}(f)\|_H^2 + \|f\|_H^2} . \end{aligned}$$

Hence by (3.12) we have

$$\|B_\gamma^{-1}(f)\|_H^2 = K\|f\|_H^2 \quad \text{or} \quad \|B_\gamma^{-1}(f)\|_H^2 = K^{-1}\|f\|_H^2,$$

and consequently

$$\|B_\gamma^{-1}(f)\|_H^2 = K\|f\|_H^2 \quad \text{or} \quad \|B_\gamma(g)\|_H^2 = K\|g\|_H^2, \tag{3.17}$$

where $g := B_\gamma^{-1}(f)$. Set $k := (K-1)(K+1)^{-1}$. Suppose the second equality in (3.17) holds. Then $K(\gamma) = K$ by Theorem 1.2 and (3.2). Theorem 1.3 now shows that γ admits a regular qc. Teichmüller extension ψ to Δ with the complex dilatation (1.15) and ψ is uniquely extremal. Then the mapping Ψ, as given by (3.1), is a unique extremal K-qc. reflection and, by (3.8), its complex dilatation satisfies (3.9) with a non-constant function $G := \mathcal{S}_0(f)_\Delta \circ \Phi \in A^2(\Omega)$. If the first equality in (3.17) holds, then by Corollary 2.2, the second equality in (3.17) holds for some $g \in H$, $\|g\| > 0$, and the rest of the proof runs as before. □

4. Complementary remarks.

Remark 4.1. Theorem 1.3 states additionally that if $\gamma \in \mathrm{Q}_{\mathbf{T}}$ admits a regular qc. Teichmüller extension ψ of to Δ with the complex dilatation (1.14), then ψ is uniquely extremal, provided F is a square of an analytic function which is square integrable on Δ. In this way we have proved, by the way, a special case of Strebel's theorem; cf. [St1], [St2] and [ŁK, p. 153-154].

From Theorems 1.3 and 2.1 we obtain

Remark 4.2. Under the assumptions in the first part of Theorem 1.3 the following equality holds

$$\lambda_*(\gamma) = \frac{\|B_\gamma\|^2 + 1}{\|B_\gamma\|^2 - 1}. \tag{4.1}$$

From Remark 4.2 we get

Remark 4.3. Theorem 3.2 and Corollary 3.3 hold with the equality (3.10) replaced by

$$\lambda_s(\Gamma) = \lambda_*(\gamma) = \frac{K+1}{K-1}. \tag{4.2}$$

Every analytic Jordan curve $\Gamma \subset \mathbb{C}$ is a quasicircle, which is clear e.g. from [LV, p. 97].

This can be also deduced from the inclusion $\text{Weld}(\Gamma) \subset \mathbb{A}_{\mathbf{T}} \subset \mathbb{Q}_{\mathbf{T}}$ and (3.2).

Combining Kühnau's result [Kü1, Satz 5] with Theorem 3.2 yields

Remark 4.4. In case Γ is an analytic Jordan curve Theorem 3.2 is reduced to Kühnau's result [Kü2, p. 302].

The idea of using welding homeomorphisms in the study of topics covered by Section 3 appears in [Kü3], too.

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Wydział Filozofii KUL
Katedra Funkcji Analitycznych
Al. Racławickie 14
20–950 Lublin, Poland

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