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Moments of Certain Inflated Probability Distributions

ABSTRACT. We consider properties of three classes of discrete probability distributions, namely the so-called Inflated Factorial Series Distributions (IFSD), Inflated Modified Factorial Series Distributions (IMFSD) and Inflated Modified Power Series Distributions (IMPSD). The formulas for moments and recurrence relations for the moments of those inflated distributions are derived. The obtained results generalize or extend some theorems established by Janardan [7], Sibuza and Shimizu [12], Gupta [5], Gerstenkorn [6] and Grzegórska [4].

1. Introduction. The Factorial Series Distributions (FSD) introduced by Berg [1] and the Modified Power Series Distributions (MPSD) defined by Gupta [5] were discussed, among other things, by Janardan [7]. This note deals with the mixtures of those distributions and the degenerate distributions. They are called inflated probability distributions (cf. [10]). It appears that the inflated probability distributions sometimes better describe random phenomena than the classic probability distributions alone. They describe mixed populations consisting of two groups of individuals, the individuals of the first group follow the simple distribution, while those of the second group always contribute to rth cell. Those distributions are applicable in the cases where simple distributions describe the situation well except for the rth cell which is inflated, that is, there are more observations with r than could be expected on the basis of a simple distribution. Models of random phenomena described by inflated distributions were presented for instance by Cohen [3], Panday [9] and Singh [11].

We are interested in moments of different classes of inflated probability distributions. In Section 2 there are given definitions of Inflated Factorial Series Distributions (IFSD), Inflated Modified Factorial Series Distributions (IMFSD) and Inflated Modified Power Series Distributions (IMPSD). Ordinary and factorial moments of IFSD are established by difference operators in Section 3. Formulas for ordinary and factorial moments of IMFSD are contained in Section 4. Ordinary and factorial moments of IMPSD are given in Section 5. Recurrence relations for central moments and similar relations for factorial moments are given in Section 6. The obtained results generalize formulas from [7], [12], [5], [6] and [4].

2. Definitions and notations.

Definition 2.1. A discrete random variable X is said to have an inflated probability function (p.f.) if its p.f. is a mixture of p.f. degenerate at the point s and a p.f. of discrete random variable Y, i.e. if

$$(2.1)\quad p(x) = \begin{cases} 1-\alpha+\alpha P[Y=x], & x=s, \\ \alpha P[Y=x], & x\neq s, \quad x\in N\cup\{0\};\ 0<\alpha\leq 1. \end{cases}$$

Definition 2.2. A discrete random variable X is said to have an IFSD if its p.f. is given by

$$(2.2)\qquad p_\theta(x) = \begin{cases} 1-\alpha+\alpha\frac{\theta^{(x)}a(x)}{f(\theta)}, & x=s, \\ \alpha\frac{\theta^{(x)}a(x)}{f(\theta)}, & x\neq s,\ x\in N\cup\{0\}; \\ & 0<\theta<\infty,\ 0<\alpha\leq 1, \end{cases}$$

where $f(\theta)$ admits a factorial series expansion $\sum a(x)\theta^{(x)}$ in θ, with coefficients $a(x)\geq 0$ independent of θ and simply related to the xth forward difference of $f(\theta)$ at $\theta=0$, namely

$$(2.3)\qquad a(x)=\frac{\Delta^x f(0)}{x!}.$$

Here $\Delta f(x)=f(x+1)-f(x)$ and $\theta^{(x)}=\theta(\theta-1)\dots(\theta-x+1)$.

Definition 2.3. A discrete random variable X is said to have an IMPSD if its p.f. is given by

$$(2.4)\quad p_\theta(x) = \begin{cases} 1-\alpha+\alpha\frac{[g(\theta)]^x a(x)}{f(\theta)}, & x=s, \\ \alpha\frac{[g(\theta)]^x a(x)}{f(\theta)}, & x\neq s, \quad x\in N\cup\{0\};\ 0<\alpha\leq 1, \end{cases}$$

where $f(\theta) = \sum a(x)[g(\theta)]^x$, $g(\theta)$ is positive, finite and differentiable, while the coefficients $a(x)$ are nonnegative and independent of θ.

Now we introduce a more general class of distributions.

Definition 2.4. A discrete random variable X is said to have an Inflated Modified Factorial Series Distribution (IMFSD) if its p.f. is given by

$$(2.5)\quad p_\theta(x) = \begin{cases} 1-\alpha+\alpha\frac{[g(\theta)]^{(x)}a(x)}{f(\theta)}, & x = s, \\ \alpha\frac{[g(\theta)]^{(x)}a(x)}{f(\theta)}, & x \neq s, \quad x \in N\cup\{0\};\ 0<\alpha\leq 1, \end{cases}$$

where $f(\theta) = \sum a(x)[g(\theta)]^{(x)}$, $g(\theta)$ is finite and differentiable and $a(x)$ is given by

$$(2.6)\qquad a(x) = \frac{\Delta_g{}^x f(0)}{x!}.$$

where $\Delta_g f(x) = f(g(x)+1) - f(g(x))$.

We will use the following notation for the moments:
m'_r-rth ordinary moment of discrete distribution,
m_r-rth ordinary moment of inflated distribution,
$m'_{(r)}$-rth factorial moment of discrete distribution,
$m_{(r)}$-rth factorial moment of inflated distribution,
μ'_r-rth central moment of discrete distribution,
μ_r-rth central moment of inflated distribution.

We are going to use the following operators:

$$\begin{aligned} Ef(x) = f(x+1),\ & \Delta f(x) = f(x+1) - f(x), \\ & \nabla f(x) = f(x) - f(x-1), \\ E_g f(x) = f(g(x)+1),\ & \Delta_g f(x) = f(g(x)+1) - f(g(x)), \\ & \nabla_g f(x) = f(g(x)) - f(g(x)-1). \end{aligned}$$

We note that

$$E \equiv I + \Delta,\ E^{-1} \equiv I - \nabla,\ E^n f(x) = f(x+n)$$

and

$$E_g \equiv I + \Delta_g,\ E_g{}^{-1} \equiv I - \nabla_g,\ E_g{}^n f(x) = f(g(x)+n).$$

Moreover, we have

$$(2.7)\qquad j^r = E^j 0^r = (I+\Delta)^j 0^r = \sum_{k=1}^{j} \binom{j}{k} \Delta^k 0^r,$$

where $\Delta^k 0^r = \Delta^k x^r \mid x = 0$ and $\Delta^k 0^r = 0$ for $k = 0$ and $k > r$.

We shall use the Stirling numbers of the second kind defined by

$$S(i,j) = \begin{cases} \frac{\Delta^j 0^i}{j!}, & i \geq j, \\ 0, & i < j. \end{cases} \tag{2.8}$$

3. Moments of IFSD. By the definition (2.1) we have the following obvious

Lemma 3.1. *If a discrete random variable X has an inflated p.f. (2.1) then the ordinary moments m_r, the factorial moments $m_{(r)}$ and the central moments μ_r of r.v. X have the form*

$$m_r = (1-\alpha)s^r + \alpha m'_r, \tag{3.1}$$

$$m_{(r)} = (1-\alpha)s^{(r)} + \alpha m'_{(r)}, \tag{3.2}$$

$$\mu_r = \alpha\beta(s - m'_1)^r(\alpha^{r-1} - (-\beta)^{r-1}) + \alpha\sum_{j=2}^{r}\binom{r}{j}(\beta(m'_1 - s))^{r-j}\mu'_j, \tag{3.3}$$

respectively, where m'_r denote the ordinary moments, $m'_{(r)}$ denote the factorial moments and μ'_r the central moments of r.v. Y (cf. (2.1)) and $\beta = 1-\alpha$, $0 < \alpha \leq 1$.

Proof. Formulas (3.1) and (3.2) follow from the definition of ordinary and factorial moments and from (2.1). To prove (3.3) observe that

$$\mu_r = \beta(s - m_1)^r + \alpha\sum_x (x - m_1)^r P[Y = x].$$

Using the equality

$$\sum_x (x - m_1)^r P[Y = x] = \sum_x \sum_{j=0}^{r}\binom{r}{j}(x - m'_1)^j(m'_1 - m_1)^{r-j}P[Y = x]$$

$$= \sum_{j=0}^{r}\binom{r}{j}[\beta(m'_1 - s)]^{r-j}\mu'_j$$

we have

$$\mu_r = \beta\alpha^r(s-m_1')^r + \alpha\sum_{j=0}^{r}\binom{r}{j}[\beta(m_1'-s)]^{r-j}\mu_j'.$$

From the equalities $\mu_0 = 1$ and $\mu_1 = 0$ we obtain (3.3). For the factorial moments of an inflated p.f. we have the following

Lemma 3.2. *If a r.v. Y has the ordinary moments m_r' of the form*

$$m_r' = \sum_{j=1}^{r} C_j \Delta^j 0^r \tag{3.4}$$

then the factorial moments $m_{(r)}$ of an inflated p.f. (2.1) *are given by the formula*

$$m_{(r)} = \begin{cases} [(1-\alpha)\binom{s}{r} + \alpha C_r]r!, & s \geq r, \\ \alpha C_r r!, & s < r. \end{cases} \tag{3.5}$$

Proof. Let $\mathbf{S} = [S(i,j)]$ be an $r \times r$ matrix of the Stirling numbers of the second kind. Obviously $\mathbf{S}$ is a nondegenerate matrix (cf. [7]). From $\binom{s}{j} = 0$ for $s < j$ and from (2.7) we get

$$m_r = (1-\alpha)\sum_{j=1}^{r}\binom{s}{j}\Delta^j 0^r + \alpha\sum_{j=1}^{r} C_j\Delta^j 0^r = \sum_{j=1}^{r}\left[(1-\alpha)\binom{s}{j} + \alpha C_j\right]\Delta^j 0^r.$$

Using the equality

$$m_r = \sum_{j=1}^{r} m_{(j)} S(r,j)$$

we obtain

$$\sum_{j=1}^{r}\left\{\left[(1-\alpha)\binom{s}{j} + \alpha C_j\right] j! - m_{(j)}\right\} S(r,j) = 0. \tag{3.6}$$

Since $S(i,j) > 0$ for $i \geq j$, we have (3.5).

The relations (3.2) - (3.4) and the formulas for the ordinary moments of FSD given in [7] imply the following

Theorem 3.3. *The rth ordinary moment m_r and rth factorial moment $m_{(r)}$ of* IFSD *are given by the formulas*

$$m_r = (1-\alpha)s^r + \alpha\sum_{k=1}^{r}\frac{\theta^{(k)}\Delta^k f(\theta-k)}{f(\theta)k!}\Delta^k 0^r, \quad r \geq 1, \tag{3.7}$$

and

$$(3.8) \qquad m_{(r)} = \begin{cases} (1-\alpha)s^{(r)} + \alpha\frac{\theta^{(r)}}{f(\theta)}\Delta^r f(\theta - r), & s \geq r, \\ \alpha\frac{\theta^{(r)}}{f(\theta)}\Delta^r f(\theta - r), & s < r, \end{cases}$$

respectively.

Example 3.4. Inflated binomial distribution.
Suppose X is a r.v. with p.f.

$$P[X = x] = \begin{cases} 1 - \alpha + \alpha\binom{\theta}{x}(\frac{a}{1+a})^x(\frac{1}{1+a})^{\theta - x}, & x = s, \\ \alpha\binom{\theta}{x}(\frac{a}{1+a})^x(\frac{1}{1+a})^{\theta - x}, & x \neq s, \quad x = 0, 1, ..., \theta. \end{cases}$$

Observe that this is an IFSD with the series function $f(\theta) = (1 + a)^\theta$. From (3.7) and (3.8) we get

$$m_r = (1-\alpha)s^r + \alpha\sum_{j=1}^{r}\frac{\theta^{(j)}}{j!}(\frac{a}{1+a})^j\Delta^j 0^r, \quad r \geq 1,$$

and

$$m_{(r)} = \begin{cases} (1-\alpha)s^{(r)} + \alpha\theta^{(r)}a^r(1+a)^{-r}, & s \geq r, \\ \alpha\theta^{(r)}a^r(1+a)^{-r}, & s < r. \end{cases}$$

Example 3.5. Inflated hypergeometric distribution.
Let X have p.f. given by

$$P[X = x] = \begin{cases} 1 - \alpha + \alpha\binom{\theta}{x}\binom{m}{n-x}\Big/\binom{\theta+m}{n}, & x = s, \\ \alpha\binom{\theta}{x}\binom{m}{n-x}\Big/\binom{\theta+m}{n}, & x \neq s, \end{cases}$$

$x = max(0, n - m), ..., min(n, \theta)$.
In this case $f(\theta) = \binom{\theta+m}{n}$. Using (3.7) and (3.8) we have

$$m_r = (1-\alpha)s^r + \alpha\sum_{k=1}^{r}\frac{\theta^{(k)}n^{(k)}}{k!(\theta+m)^{(k)}}\Delta^k 0^r, r \geq 1,$$

and

$$m_{(r)} = \begin{cases} (1-\alpha)s^{(r)} + \alpha\frac{\theta^{(r)}n^{(r)}}{(\theta+m)^{(r)}}, & s \geq r, \\ \alpha\frac{\theta^{(r)}n^{(r)}}{(\theta+m)^{(r)}}, & s < r. \end{cases}$$

Example 3.6. Inflated Stevens-Craig distribution.
Now we consider a r.v. with the p.f. which is a mixture of the Stevens-Craig distribution considered in [7] and inflated distribution at the point s, given by the formula

$$P[X=x]=\begin{cases} 1-\alpha+\alpha N^{(x)}S(n,x)N^{-n}, & x=s,\\ \alpha N^{(x)}S(n,x)N^{-n}, & x\neq s,\quad x=1,2,\dots,\end{cases}$$

where $S(n,x)$ are Stirling numbers of the second kind defined by (2.9). This distribution has the series function $f(N)=N^n$. Thus from (3.7) and (3.8) we obtain

$$m_r=(1-\alpha)s^r+\alpha\sum_{k=1}^{r}\frac{N^{(k)}\nabla^k N^n}{N^n k!}\Delta^k 0^r, r\geq 1,$$

and

$$m_{(r)}=\begin{cases} (1-\alpha)s^{(r)}+\alpha\frac{N^{(r)}\nabla^r N^n}{N^n}, & s\geq r,\\ \alpha\frac{N^{(r)}\nabla^r N^n}{N^n}, & s<r.\end{cases}$$

4. Moments of IMFSD. Now we give the moment formulas for the class of Inflated Modified Factorial Distributions.

Theorem 4.1. *The rth ordinary moment m_r and rth factorial moment $m_{(r)}$ of IMFSD are given by the formulas*

$$m_r=(1-\alpha)s^r+\alpha\sum_{k=1}^{r}\frac{[g(\theta)]^{(k)}\Delta_g^k f(\theta-k)}{f(\theta)}S(k,r), r\geq 1, \tag{4.2}$$

and

$$m_{(r)}=\begin{cases} (1-\alpha)s^{(r)}+\alpha\frac{[g(\theta)]^{(r)}}{f(\theta)}\Delta_g^r f(\theta-r), & s\geq r,\\ \alpha\frac{[g(\theta)]^{(r)}}{f(\theta)}\Delta_g^r f(\theta-r), & s<r,\end{cases} \tag{4.3}$$

respectively.

Proof. We can observe that

$$m_r=(1-\alpha)s^r+\alpha\sum_{x=0}^{\infty}\frac{x^r[g(\theta)]^{(x)}\Delta_g^x f(0)}{f(\theta)x!}$$

$$= (1-\alpha)s^r + \alpha\sum_{x=0}^{\infty}\left(\sum_{k=0}^{x}\binom{x}{k}\Delta^x 0^r\right)\frac{[g(\theta)]^{(x)}\Delta_g^x f(0)}{f(\theta)x!}$$

$$= (1-\alpha)s^r + \alpha\sum_{k=1}^{r}\sum_{x=k}^{\infty}\frac{[g(\theta)]^{(x)}\Delta_g^x f(0)\Delta^k 0^r}{(x-k)!f(\theta)x!}$$

$$= (1-\alpha)s^r + \alpha\sum_{k=1}^{r}\sum_{y=0}^{\infty}\frac{[g(\theta)]^{(y+k)}\Delta_g^{y+k} f(0)}{f(\theta)y!}S(r,k)$$

$$= (1-\alpha)s^r + \alpha\sum_{k=1}^{r}\frac{[g(\theta)]^{(k)}}{f(\theta)}S(r,k)\Delta_g^k\left[\sum_{y=0}^{\infty}\frac{[g(\theta)-k]^{(y)}\Delta_g^y f(0)}{y!}\right]$$

$$= (1-\alpha)s^r + \alpha\sum_{k=1}^{r}\frac{[g(\theta)]^{(k)}\Delta_g^k f(g(\theta)-k)}{f(\theta)}S(r,k).$$

Using Lemma 3.2. we get (4.3).

Corollary 4.2. *The mean* m_1 *of* IMFSD *satisfies*

$$m_1 = (1-\alpha)s + \alpha\frac{g(\theta)}{f(\theta)}[f(g(\theta)) - f(g(\theta)-1)] \tag{4.4}$$

Remark 4.3. *Observe that for* $g(\theta)=\theta$ *we obtain the formulas for the moments of* IFSD.

Now we consider a few types of generalized hypergeometric distributions which are mixtures of degenerate distributions and distributions classified in [8] and [12].

Example 4.4. Let a r.v. X have the following p.f.

$$P[X=x] = \begin{cases} 1-\alpha+\alpha\binom{n-n\theta}{x}\binom{-n-1}{-m-x}/\binom{-n\theta-1}{-m}, & x=s,\\ \alpha\binom{n-n\theta}{x}\binom{-n-1}{-m-x}/\binom{-n\theta-1}{-m}, & x\neq s,\end{cases}$$

where $x = max(0, 1-m-n), \ldots, min(n-n\theta, -m)$ is the number of failures. For $\alpha = 1$ this is a p.f. of type IIIA from [8]. In this case we have

$$g(\theta) = n - n\theta, \quad a(x) = \binom{-n-1}{-m-x}\frac{1}{x!},$$
$$f(\theta) = \binom{-n\theta-1}{-m} = \binom{g(\theta)-n-1}{-m}.$$

From (4.2) and (4.3) we obtain the following

$$m_r = (1-\alpha)s^r + \alpha\sum_{k=1}^{r}(n-n\theta)^{(k)}\left[\binom{-m}{k}\Big/\binom{-n\theta-1}{k}\right]S(r,k),\ r\geq 1,$$

and

$$m_{(r)} = \begin{cases} (1-\alpha)s^{(r)} + \alpha(n-n\theta)^{(r)}\binom{-m}{r}\Big/\binom{-n\theta-1}{r}, & s\geq r,\\ \alpha(n-n\theta)^{(r)}\binom{-m}{r}\Big/\binom{-n\theta-1}{r}, & s<r.\end{cases}$$

Example 4.5. Suppose that a r.v. X has the following p.f.

$$P[X=x] = \begin{cases} 1-\alpha+\alpha\binom{n\theta-n}{x}\binom{n-1}{-m-x}\Big/\binom{n\theta-1}{-m}, & x=s,\\ \alpha\binom{n\theta-n}{x}\binom{n-1}{-m-x}\Big/\binom{n\theta-1}{-m}, & x\neq s,\end{cases}$$

where $x = max(0, 1-m-n), ..., min(n\theta-n, -m)$ is the number of failures. In this case we have

$$g(\theta) = n\theta - n,\ a(x) = \binom{n-1}{-m-x}\frac{1}{x!},$$
$$f(\theta) = \binom{n\theta-1}{-m} = \binom{g(\theta)+n-1}{-m}.$$

For $\alpha = 1$ this is a p.f. of type IV from [8]. Using (4.2) and (4.3) we get

$$m_r = (1-\alpha)s^r + \alpha\sum_{k=1}^{r}(n\theta-n)^{(k)}\left[\binom{-m}{k}\Big/\binom{n\theta-1}{k}\right]S(r,k),\ r\geq 1,$$

and

$$m_{(r)} = \begin{cases} (1-\alpha)s^{(r)} + \alpha(n\theta-n)^{(r)}\binom{-m}{r}\Big/\binom{n\theta-1}{r}, & s\geq r,\\ \alpha(n\theta-n)^{(r)}\binom{-m}{r}\Big/\binom{n\theta-1}{r}, & s<r.\end{cases}$$

Example 4.6. Let a r.v. X have the p.f. given by the formula

$$P[X=x] = \begin{cases} 1-\alpha+\alpha\binom{-\theta}{x}\binom{-\gamma+n-1}{n-x}\Big/\binom{-\theta-\gamma+n-1}{n}, & x=s,\\ \alpha\binom{-\theta}{x}\binom{-\gamma+n-1}{n-x}\Big/\binom{-\theta-\gamma+n-1}{n}, & x\neq s,\end{cases}$$

where $x = 0, 1, ..., n$ and $n < \gamma + 1$.
For $\alpha = 1$ this is a p.f. of type A2 from [12]. We have

$$g(\theta) = -\theta,\ a(x) = \binom{-\gamma + n - 1}{n - x}\frac{1}{x!},$$
$$f(\theta) = \binom{-\theta - \gamma + n - 1}{n}.$$

From(4.2) and (4.3) we get in this case

$$m_r = (1-\alpha)s^r + \alpha\sum_{k=1}^{r}(-\theta)^{(k)}\left[\binom{n}{k}\Big/\binom{-\theta-\gamma+n-1}{k}\right]S(r,k),\ r \geq 1,$$

and

$$m_{(r)} = \begin{cases} (1-\alpha)s^{(r)} + \alpha(-\theta)^{(r)}\binom{n}{r}\Big/\binom{-\theta-\gamma+n-1}{r}, & s \geq r, \\ \alpha(-\theta)^{(r)}\binom{n}{r}\Big/\binom{-\theta-\gamma+n-1}{r}, & s < r. \end{cases}$$

Example 4.7. Let a r.v. X have a p.f.

$$P[X = x] = \begin{cases} 1 - \alpha + \alpha\binom{n-\theta}{x}\binom{\zeta-\delta+n-1}{n-\delta-x}\Big/\binom{-\theta+\zeta-\delta+2n-1}{n-\delta}, & x = s, \\ \alpha\binom{n-\theta}{x}\binom{\zeta-\delta+n-1}{n-\delta-x}\Big/\binom{-\theta+\zeta-\delta+2n-1}{n-\delta}, & x \neq s, \end{cases}$$

where $x = 0, 1, ...$
If $\alpha = 1$ then this p.f. is a p.f. of type B1 from [12]. Here

$$g(\theta) = n - \theta,\ a(x) = \binom{\zeta - \delta + n - 1}{n - \delta - x}\frac{1}{x!},$$
$$f(\theta) = \binom{n - \theta + \zeta - \delta + n - 1}{n - \delta}.$$

If we use (4.2) and (4.3) we will get

$$m_r = (1-\alpha)s^r + \alpha\sum_{k=1}^{r}(n-\theta)^{(k)}\left[\binom{n-\delta}{k}\Big/\binom{-\theta+\zeta-\delta+2n-1}{k}\right]S(r,k),\ r \geq 1,$$

and

$$m_{(r)} = \begin{cases} (1-\alpha)s^{(r)} + \alpha(n-\theta)^{(r)}\binom{n-\delta}{r}\Big/\binom{-\theta+\zeta-\delta+2n-1}{r}, & s \geq r, \\ \alpha(n-\theta)^{(r)}\binom{n-\delta}{r}\Big/\binom{-\theta+\zeta-\delta+2n-1}{r}, & s < r. \end{cases}$$

5. Moments of IMPSD. Now we consider the mixtures of MPSD introduced in [5] and the inflated probability distribution. Using (3.3), (3.4) and a formula given in [7] we have the following

Theorem 5.1. *The ordinary moments* m_r *and the factorial moments* $m_{(r)}$ *of* IMPSD *are given by the formulas*

$$m_r = (1-\alpha)s^r + \alpha\sum_{k=1}^{r}\frac{[g(\theta)]^k f_g^{(k)}(\theta)}{f(\theta)k!}\Delta^k 0^r, r \geq 1, \tag{5.1}$$

and

$$m_{(r)} = \begin{cases} (1-\alpha)s^{(r)} + \alpha\frac{[g(\theta)]^r}{f(\theta)} f_g^{(r)}(\theta), & s \geq r, \\ \alpha\frac{[g(\theta)]^r}{f(\theta)} f_g^{(r)}(\theta), & s < r, \end{cases} \tag{5.2}$$

respectively, where $f_g^{(r)}(\theta)$ *is the rth derivative of* $f(\theta)$ *with respect to* $g(\theta)$, *given in the form*

$$f_g^{(r)}(\theta) = \sum_{k=0}^{\infty}\frac{(k+r)!}{k!}a(k+r)[g(\theta)]^k. \tag{5.3}$$

Now we consider some examples.

Example 5.2. Inflated generalized Poisson distribution.
Suppose X is a discrete r.v. whose p.f. is given by

$$P[X = x] = \begin{cases} 1-\alpha+\alpha\theta^x(1+ax)^{x-1}e^{-\theta(1+ax)}/x!, & x = s, \\ \alpha\theta^x(1+ax)^{x-1}e^{-\theta(1+ax)}/x!, & x \neq s, \end{cases} \tag{5.4}$$

for $x = 0,1,2,\ldots; \theta > 0, \mid \theta a \mid < 1$.
In this case we have

$$a(x) = \frac{(1+ax)^{x-1}}{x!},\ f(\theta) = e^\theta,\ g(\theta) = \theta e^{-a\theta}.$$

From (5.1) and (5.2) we get

$$m_r = (1-\alpha)s^r + \alpha\sum_{k=1}^{r}\frac{\theta^k}{k!}e^{-\theta(1+ak)} f_g^{(k)}(\theta)\Delta^k 0^r, r \geq 1,$$

and

$$m_{(r)} = \begin{cases} (1-\alpha)s^{(r)} + \alpha\frac{\theta^r}{k!}e^{-\theta(1+ar)} f_g^{(r)}(\theta), & s \geq r, \\ \alpha\frac{\theta^r}{k!}e^{-\theta(1+ar)} f_g^{(r)}(\theta), & s < r. \end{cases}$$

Example 5.3. Inflated generalized negative binomial distribution.
Let a r.v. X have the following p.f.

$$(5.5)\qquad P[X=x]=\begin{cases} 1-\alpha+\alpha\frac{n\Gamma(n+bx)[\theta(1-\theta)^{b-1}]^x}{x!\Gamma(n+bx-x+1)(1-\theta)^{-n}}, & x=s,\\ \alpha\frac{n\Gamma(n+bx)[\theta(1-\theta)^{b-1}]^x}{x!\Gamma(n+bx-x+1)(1-\theta)^{-n}}, & x\neq s,\end{cases}$$

for $x=0,1,2,...;0<\theta<1,|\theta b|<1$.
Here

$$a(x)=\frac{n\Gamma(n+bx)}{x!\Gamma(n+bx-x+1)},\ f(\theta)=(1-\theta)^{-n},\ g(\theta)=\theta(1-\theta)^{b-1}.$$

From (5.1) and (5.2) we get

$$m_r=(1-\alpha)s^r+\alpha\sum_{k=1}^{r}\frac{\theta^k}{k!}(1-\theta)^{(k(b-1)+n)}f_g^{(k)}(\theta)\Delta^k 0^r, r\geq 1,$$

and

$$m_{(r)}=\begin{cases}(1-\alpha)s^{(r)}+\alpha\theta^r(1-\theta)^{(r(b-1)+n)}f_g^{(r)}(\theta), & s\geq r,\\ \alpha\theta^r(1-\theta)^{(r(b-1)+n)}f_g^{(r)}(\theta), & s<r.\end{cases}$$

We have some special cases:
(a) If $b=0$ then X has inflated binomial distribution with the p.f.

$$P[X=x]=\begin{cases}1-\alpha+\alpha\binom{n}{x}\theta^x(1-\theta)^{n-x}, & x=s,\\ \alpha\binom{n}{x}\theta^x(1-\theta)^{n-x}, & x\neq s.\end{cases}$$

In this case

$$a(x)=\binom{n}{x},\ f(\theta)=(1-\theta)^{-n},\ g(\theta)=\frac{\theta}{1-\theta}.$$

Using (5.1) and (5.2) we get

$$m_r=(1-\alpha)s^r+\alpha\sum_{k=1}^{r}\theta^k\binom{n}{k}\Delta^k 0^r, r\geq 1,$$

and

$$m_{(r)}=\begin{cases}(1-\alpha)s^{(r)}+\alpha\theta^r n^{(r)}, & s\geq r,\\ \alpha\theta^r n^{(r)}, & s<r.\end{cases}$$

(b) If $b = 1$ then X has an inflated negative binomial distribution with

$$a(x) = \binom{n+x-1}{x}, \; f(\theta) = (1-\theta)^{-n}, \; g(\theta) = \theta.$$

In this cases we obtain

$$m_r = (1-\alpha)s^r + \alpha \sum_{k=1}^{r} \left(\frac{\theta}{1-\theta}\right)^k \binom{n+k-1}{k} \Delta^k 0^r, r \geq 1,$$

and

$$m_{(r)} = \begin{cases} (1-\alpha)s^{(r)} + \alpha(\frac{\theta}{1-\theta})^r (n+r-1)^{(r)}, & s \geq r, \\ \alpha(\frac{\theta}{1-\theta})^r (n+r-1)^{(r)}, & s < r. \end{cases}$$

Example 5.4. Inflated generalized logarithmic series distribution. Suppose X has the p.f.

$$(5.6) \qquad P[X = x] = \begin{cases} 1 - \alpha + \alpha \frac{n\Gamma(bx)[\theta(1-\theta)^{b-1}]^x}{x\Gamma(x)\Gamma(bx-x+1)[-ln(1-\theta)]}, & x = s, \\ \alpha \frac{n\Gamma(bx)[\theta(1-\theta)^{b-1}]^x}{x\Gamma(x)\Gamma(bx-x+1)[-ln(1-\theta)]}, & x \neq s. \end{cases}$$

One can see that

$$a(x) = \frac{\Gamma(bx)}{x\Gamma(x)\Gamma(bx-x+1)}, \; f(\theta) = -\ln(1-\theta), \; g(\theta) = \theta(1-\theta)^{b-1}.$$

From (5.1) and (5.2) we get

$$m_r = (1-\alpha)s^r + \alpha \sum_{k=1}^{r} \frac{[\theta(1-\theta)^{b-1}]^k}{-ln(1-\theta)} f_g^{(k)}(\theta) \Delta^k 0^r, r \geq 1,$$

and

$$m_{(r)} = \begin{cases} (1-\alpha)s^{(r)} + \alpha \frac{[\theta(1-\theta)]^r}{-ln(1-\theta)} f_g^{(r)}(\theta), & s \geq r, \\ \alpha \frac{[\theta(1-\theta)]^r}{-ln(1-\theta)} f_g^{(r)}(\theta), & s < r, \end{cases}$$

where $f_g^{(r)}(\theta) = \sum_{k=0}^{\infty} \frac{\Gamma((k+r)b)[\theta(1-\theta)^{b-1}]^k}{\Gamma((k+r)b+1)k!}$.

If $b = 1$ then X has an inflated Fisher's logarithmic series distribution. Thus we obtain

$$m_r = (1-\alpha)s^r + \alpha \sum_{k=1}^{r} \frac{\theta^k}{k[-ln(1-\theta)](1-\theta)^k} \Delta^k 0^r, r \geq 1,$$

and

$$m_{(r)} = \begin{cases} (1-\alpha)s^{(r)} + \alpha\frac{(r-1)!\theta^r}{-ln(1-\theta)(1-\theta)^r} f_g^{(r)}(\theta), & s \geq r, \\ \alpha\frac{(r-1)!\theta^r}{-ln(1-\theta)(1-\theta)^r} f_g^{(r)}(\theta), & s < r. \end{cases}$$

Example 5.5. Inflated lost games distribution.
Suppose X has the p.f.

$$\text{(5.7)} \qquad P[X = x] = \begin{cases} 1-\alpha+\alpha\frac{a}{2x-a}\binom{2x-a}{x}\frac{[\theta(1-\theta)]^x}{\theta^a}, & x = s, \\ \alpha\frac{a}{2x-a}\binom{2x-a}{x}\frac{[\theta(1-\theta)]^x}{\theta^a}, & x \neq s, \end{cases}$$

for $x = a, a+1, ...;\ a \geq 1,\ 0 < \theta < \frac{1}{2}$.
Then

$$a(x) = \frac{a}{2x-a}\binom{2x-a}{x},\ f(\theta) = \theta^a,\ g(\theta) = 1-\theta.$$

In this case we get the following formulas

$$m_r = (1-\alpha)s^r + \alpha\sum_{k=0}^{r}\frac{[\theta(1-\theta)]^k}{\theta^a}\frac{\Delta^k 0^r}{k!} f_g^{(k)}(\theta), r \geq 1,$$

and

$$m_{(r)} = \begin{cases} (1-\alpha)s^{(r)} + \alpha\frac{[\theta(1-\theta)]^r}{\theta^a} f_g^{(r)}(\theta), & s \geq r, \\ \alpha\frac{[\theta(1-\theta)]^r}{\theta^a} f_g^{(r)}(\theta), & s < r, \end{cases}$$

where $f_g^{(r)}(\theta) = \sum_{k=0}^{\infty}\frac{(k+r)!a^2}{[2(k+r)-a](2x-a)}\binom{2(k+r)-a}{k+r}\frac{[\theta(1-\theta)]^k}{k!}$.

Now we consider the relations between μ_r and μ_r' for IMPSD. Using results obtained in [5] and Lemma 3.1. we obtain

Theorem 5.6. *If X is a r.v. with the p.f.* (2.4) *then*

$$\text{(5.8)} \qquad \begin{aligned} \mu_r = \frac{g(\theta)}{g'(\theta)}\alpha\sum_{j=2}^{r}\binom{r}{j}[\beta(m_1'-s)]^{r-j}\left[\frac{d\mu_{r-1}'}{d\theta} + (r-1)\frac{dm_1}{d\theta}\mu_{r-2}'\right] \\ + \alpha\beta(s-m_1)^r\left(\alpha^{r-1} + (-\beta)^{r-1}\right) \end{aligned}$$

for $r = 2, 3, \ldots$.

Example 5.7. Inflated generalized Poisson distribution.
Suppose X has the p.f. given by (5.4). From [5] we have $m_1' = \frac{\theta}{1-a\theta}$. Hence

$$m_1 = \beta s + \alpha\frac{\theta}{1-a\theta} \text{ and } \frac{dm_1}{d\theta} = \frac{\alpha}{(1-a\theta)^2}.$$

From (5.8) we have

$$\mu_r = \alpha^{r+1}\beta\left(s - \frac{\theta}{1-a\theta}\right)^r\left(\alpha^{r-1} + (-\beta)^{r-1}\right)$$

$$+\frac{\alpha\theta}{1-a\theta}\sum_{j=2}^{r}\binom{r}{j}\left[-\beta(s+\frac{\theta}{1-a\theta})\right]^{r-j}\left[\frac{d\mu'_{r-1}}{d\theta} + \frac{\alpha(r-1)}{(1-a\theta)^2}\mu'_{r-2}\right].$$

Example 5.8. Inflated generalized negative binomial distribution. Let X has the p.f. (5.5). In this case we have

$$m_1 = \beta s + \alpha\frac{n\theta}{1-b\theta} \quad \text{and} \quad \frac{dm_1}{d\theta} = \frac{\alpha n}{(1-b\theta)^2}.$$

From (5.8) we have

$$\mu_r = \alpha^{r+1}\beta\left(s + \frac{n\theta}{1-b\theta}\right)^r\left(\alpha^{r-1} + (-\beta)^{r-1}\right)$$

$$+\frac{\alpha\theta(1-\theta)}{1-b\theta}\sum_{j=2}^{r}\binom{r}{j}\left[\beta(\frac{n\theta}{1-b\theta} - s)\right]^{r-j}\left[\frac{d\mu'_{r-1}}{d\theta} + \frac{(r-1)n}{(1-b\theta)^2}\mu'_{r-2}\right].$$

Example 5.9. Inflated generalized logarithmic series distribution. Suppose X has the p.f. given by (5.6). In this case we have

$$m_1 = \beta s - \frac{\alpha\theta}{(1-b\theta)ln(1-\theta)} \text{ and } \frac{dm_1}{d\theta} = \frac{\alpha[\theta(b\theta-1) - (1-\theta)ln(1-\theta)]}{(b\theta-1)^2(1-\theta)ln^2(1-\theta)}.$$

From (5.8) we have

$$\mu_r = \alpha^{r+1}\beta\left[s - \frac{\theta}{(b\theta-1)ln(1-\theta)}\right]^r\left(\alpha^{r-1} + (-\beta)^{r-1}\right)$$

$$+\frac{\alpha\theta(1-\theta)}{1-b\theta}\sum_{j=2}^{r}\binom{r}{j}\left[-\beta(s - \frac{\theta}{(b\theta-1)ln(1-\theta)})\right]^{r-j}$$

$$\times\left[\frac{d\mu'_{r-1}}{d\theta} + \frac{\alpha(r-1)[\theta(b\theta-1) - (1-\theta)ln(1-\theta)]}{(b\theta-1)^2(1-\theta)ln^2(1-\theta)}\mu'_{r-2}\right].$$

Example 5.10. Inflated lost games distribution.

Supose X has the p.f. (5.7). Then we have

$$m_1 = \beta s + \alpha\frac{a(1-\theta)}{1-2\theta} \quad\text{and}\quad \frac{dm_1}{d\theta} = \frac{\alpha a}{(1-2\theta)}.$$

From (5.8) we have

$$\mu_r = \alpha^{r+1}\beta\left(s - \frac{a(1-\theta)}{1-2\theta}\right)^r\left(\alpha^{r-1} + (-\beta)^{r-1}\right)$$

$$+\frac{\alpha\theta(1-\theta)}{1-2\theta}\sum_{j=2}^{r}\binom{r}{j}\left[-\beta(s - \frac{a(1-\theta)}{1-2\theta})\right]^{r-j}\left[\frac{d\mu'_{r-1}}{d\theta} + \frac{(r-1)\alpha a}{(1-2\theta)^2}\mu'_{r-2}\right].$$

6. The recurrence relations for the central and factorial moments of IMPSD. In this section we give recurrence relations for the central and factorial moments of some inflated probability distributions. They contain as particular cases those from [6] and furnish recurrence relations for the factorial moments of MPSD as given in [7]. Moreover, we give relations between cumulants and ordinary moments of IMPSD.

Theorem 6.1. *The $(r+1)$th ordinary moment m_{r+1}, $(r+1)$th central moment $\mu_{(r+1)}$ and the $(r+1)$th factorial moment $m_{(r+1)}$ of* IMPSD *are given by*

$$m_{r+1} = \frac{g(\theta)}{g'(\theta)}\frac{dm_r}{d\theta} + \frac{m_1 m_r}{\alpha} - \frac{\beta}{\alpha}s[m_r + s^{r-1}(m_1 - s)], \tag{6.1}$$

$$\mu_{r+1} = \frac{g(\theta)}{g'(\theta)}\left[\frac{d\mu_r}{d\theta} + r\frac{dm_1}{d\theta}\mu_{r-1}\right]$$

$$-\frac{\beta}{\alpha}(s-m_1)\mu_r + \frac{\beta}{\alpha}(s-m_1)^{r-1}, \tag{6.2}$$

$$m_{(r+1)} = \frac{g(\theta)}{g'(\theta)}\frac{dm_{(r)}}{d\theta} - [r - \frac{1}{\alpha}m_1]m_{(r)} + \frac{\beta}{\alpha}[s^{(r)}(s-m_1) + sm_{(r)}], \tag{6.3}$$

where $\beta = 1-\alpha$, respecrively.

Proof. Observe that $f'(\theta) = \sum_x xa(x)[g(\theta)]^{x-1}g'(\theta)$ and

$$m_1 = \beta\alpha + \alpha\frac{g(\theta)}{g'(\theta)}\frac{f'(\theta)}{f(\theta)}. \tag{6.4}$$

Moreover,

$$m_r = (1-\alpha)s^r + \alpha \sum_x x^r a(x)[g(\theta)]^x f(\theta)^{-1}.$$

Differentiating this formula with respect to θ, we obtain

$$\frac{dm_r}{d\theta} = \alpha \sum_x x^r a(x) x [g(\theta)]^{x-1} g'(\theta) f(\theta)^{-1}$$

$$-\alpha \sum_x x^r a(x)[g(\theta)]^x f'(\theta) f(\theta)^{-2}.$$

Hence we get

$$\frac{g(\theta)}{g'(\theta)} \frac{dm_r}{d\theta} = m_{r+1} - \frac{f'(\theta)}{f(\theta)} \frac{g(\theta)}{g'(\theta)} m_r + (1-\alpha)s^r \left(\frac{f'(\theta)}{f(\theta)} \frac{g(\theta)}{g'(\theta)} - s \right).$$

Using (6.4) we obtain

$$\frac{g(\theta)}{g'(\theta)} \frac{dm_r}{d\theta} = m_{r+1} - \frac{m_1 m_r}{\alpha} + \frac{1-\alpha}{\alpha} s^r \left[m_r + s^{r-1}(m_1 - s) \right]$$

which implies (6.1).

To prove (6.2) we observe that

$$\mu_r = \beta(s-m_1)^r + \alpha \sum_x (x-m_1)^r \frac{a(x)[g(\theta)]^x}{f(\theta)}.$$

Differentiating the last formula with respect to θ, we get

$$\frac{d\mu_r}{d\theta} = -r\frac{dm_1}{d\theta} \beta (s-m_1)^{r-1} - r\frac{dm_1}{d\theta} \alpha \sum_x (x-m_1)^{r-1} \frac{a(x)[g(\theta)]^x}{f(\theta)}$$

$$+\alpha \sum_x (x-m_1)^r \frac{x a(x)[g(\theta)]^{x-1} g'(\theta)}{f(\theta)} - \frac{f'(\theta)}{f(\theta)} \alpha \sum_x (x-m_1)^r \frac{a(x)[g(\theta)]^x}{f(\theta)}.$$

Hence we have

$$g(\theta)\frac{d\mu_r}{d\theta} = -rg(\theta)\frac{dm_1}{d\theta}\mu_{r-1} + g'(\theta)\mu_{r+1} - g'(\theta)\beta(s-m_1)^{r+1}$$

$$+ \left[g'(\theta) m_1 - \frac{f'(\theta)}{f(\theta)} g(\theta) \right] \mu_r + \left[g'(\theta) m_1 - \frac{f'(\theta)}{f(\theta)} g(\theta) \right] \beta (s-m_1)^{r-1}.$$

In view of (6.4) we get

$$g'(\theta)m_1 - \frac{f'(\theta)}{f(\theta)}g(\theta) = \frac{\beta}{\alpha}(s-m_1)^{r-1}g'(\theta).$$

Hence we have

$$\frac{g(\theta)}{g'(\theta)}\frac{d\mu_r}{d\theta} = -r\frac{g(\theta)}{g'(\theta)}\frac{dm_1}{d\theta}\mu_{r-1} + \mu_{r+1} - \frac{\beta}{\alpha}(s-m_1)^{r+1}\mu_r - \frac{\beta}{\alpha}(s-m_1)^{r+1}.$$

This gives the formula (6.2). To prove (6.3) we note that

$$m_{(r)} = \beta s^{(r)} + \alpha\sum_x x^{(r)}\frac{a(x)[g(\theta)]^x}{f(\theta)}. \tag{6.5}$$

Differentiating with respect to θ the formula (6.5) we obtain

$$\begin{aligned}\frac{dm_{(r)}}{d\theta} &= \alpha\sum_x x^{(r+1)}\frac{a(x)[g(\theta)]^{x-1}g'(\theta)}{f(\theta)}\\ &+ \left(r - \frac{f'(\theta)}{f(\theta)}g(\theta)\right)\alpha\sum_x x^{(r)}\frac{a(x)[g(\theta)]^{x-1}}{f(\theta)}\end{aligned}$$

which can be written as follows

$$\begin{aligned}g(\theta)\frac{dm_{(r)}}{d\theta} &= g'(\theta)m_{(r+1)} + \left(g'(\theta)r - \frac{f'(\theta)}{f(\theta)}g(\theta)\right)m_{(r)}\\ &- \beta s^{(r)}\left[(s-r)g'(\theta) + g'(\theta)r - \frac{f'(\theta)}{f(\theta)}g(\theta)\right].\end{aligned}$$

Now observe that (6.4) implies

$$\frac{f'(\theta)}{f(\theta)}g(\theta) = \frac{1}{\alpha}g'(\theta)m_1 - \frac{\beta s}{\alpha}g'(\theta).$$

In view of the last formula we get

$$\begin{aligned}g(\theta)\frac{dm_{(r)}}{d\theta} &= g'(\theta)m_{(r+1)} + g'(\theta)(r - \frac{1}{\alpha}m_1)m_{(r)}\\ &- g'(\theta)\frac{\beta}{\alpha}\left[s^{(r)}(s-m_1) + sm_{(r)}\right].\end{aligned}$$

Hence (6.3) follows.

Example 6.2. Inflated generalized Poisson distribution.
Suppose X has the p.f. given by (5.4). From (6.1) we get

$$m_{r+1} = \frac{\theta}{1-a\theta}\left(\frac{dm_r}{d\theta} + m_r - \beta s^r\right) + \beta s^{r+1}.$$

From (6.2) we have

$$\mu_{r+1} = \frac{\theta}{1-a\theta}\left(\frac{d\mu_r}{d\theta} + \frac{r\alpha}{(1-a\theta)^2}\mu_{r-1}\right) - \beta\left(s - \frac{\theta}{1-a\theta}\right)\mu_r$$
$$+ \beta\alpha^{r+1}\left(s - \frac{\theta}{1-a\theta}\right)^{r+1}.$$

From (6.3) we obtain

$$m_{(r+1)} = \frac{\theta}{1-a\theta}\frac{dm_{(r)}}{d\theta} + m_{(r)}\left(2s\frac{\beta}{\alpha} - r + \frac{\theta}{1-a\theta}\right) + \frac{\beta}{\alpha}s^{(r)}\left(2s + \frac{\theta}{1-a\theta}\right).$$

Example 6.3. Inflated generalized negative binomial distribution.
Let X has the p.f. (5.5). From (6.1)-(6.3) we get

$$m_{r+1} = \frac{\theta}{1-b\theta}\left[(1-\theta)\frac{dm_r}{d\theta} + n(m_r - \beta s^r)\right] + \beta s^{r+1},$$

$$\mu_{r+1} = \frac{\theta(1-\theta)}{1-b\theta}\left[\frac{d\mu_r}{d\theta} + \frac{rn}{(1-b\theta)^2}\mu_{r-1}\right] - \beta\left(s + \frac{n\theta}{1-b\theta}\right)\mu_r$$
$$-\beta\left(s + \frac{n\theta}{1-b\theta}\right)^{r+1},$$

$$m_{(r+1)} = \frac{\theta(1-\theta)}{1-b\theta}\frac{dm_{(r)}}{d\theta} + m_{(r)}\left(2s\frac{\beta}{\alpha} - r + \frac{n\theta}{1-b\theta}\right)$$
$$-\beta s^{(r)}\left(s + \frac{\alpha n\theta}{1-b\theta}\right).$$

Example 6.4. Inflated generalized logarithmic series distribution.
Suppose X has a p.f. given by (5.6). Using (6.1)-(6.3) we get

$$m_{r+1} = \frac{\theta(1-\theta)}{1-b\theta}\left[(1-\theta)\frac{dm_r}{d\theta} + \frac{m_r - \beta s^r}{(\theta-1)ln(1-\theta)}\right] + \beta s^{r+1},$$

$$
\begin{aligned}
\mu_{r+1} = {} & \frac{\theta(1-\theta)}{1-b\theta}\left[\frac{d\mu_r}{d\theta} + \frac{r\alpha}{(1-b\theta)ln(1-\theta)}\right. \\
& \left. \times\left(\frac{\theta}{(1-\theta)ln(1-\theta)} - \frac{1}{1-b\theta}\right)\mu_{r-1}\right] \\
& + \beta\left(s + \frac{\theta}{(1-b\theta)ln(1-\theta)}\right)\mu_r + \beta\alpha^r\left(s + \frac{\theta}{(1-\theta)ln(1-\theta)}\right)^{r+1}, \\
m_{(r+1)} = {} & \frac{\theta(1-\theta)}{1-b\theta}\frac{dm_{(r)}}{d\theta} + m_{(r)}\left(2s\frac{\beta}{\alpha} - r + \frac{\theta}{(1-b\theta)ln(1-\theta)}\right) \\
& - \beta s^{(r)}\left(s + \frac{\theta}{(1-b\theta)ln(1-\theta)}\right).
\end{aligned}
$$

Example 6.5. Inflated lost games distribution.
Let X has a p.f. (5.7). In this case using (6.1)-(6.3) we obtain the following recurrence relations

$$
m_{r+1} = (\theta-1)\left[\frac{dm_r}{d\theta} + \frac{a}{\theta}(m_r - \beta s^r)\right] + \beta s^{r+1},
$$

$$
\begin{aligned}
\mu_{r+1} = {} & \frac{\theta(1-\theta)}{1-2\theta}\left[\frac{d\mu_r}{d\theta} + \frac{ar\alpha}{(1-2\theta)^2}\mu_{r-1})\right] - \beta\left(s - \frac{a(1-\theta)}{(1-2\theta)}\right)\mu_r \\
& + \beta\alpha^r\left(\frac{a(1-\theta)}{1-2\theta} - s\right)^{r+1},
\end{aligned}
$$

$$
\begin{aligned}
m_{(r+1)} = {} & \frac{\theta(1-\theta)}{1-2\theta}\frac{dm_{(r)}}{d\theta} + m_{(r)}\left(2s\frac{\beta}{\alpha} - r + \frac{a(1-\theta)}{1-2\theta}\right) \\
& - \beta s^{(r)}\left(s - \frac{a(1-\theta)}{1-2\theta}\right).
\end{aligned}
$$

The following theorem establishes a relation between cumulants κ_r and ordinary moments m_r.

Theorem 6.6. *The* $(r+1)$*th cumulant* κ_r *of* IMPSD *is given by*

$$
\begin{aligned}
\kappa_r = {} & \frac{g(\theta)}{g'(\theta)}\sum_{j=1}^{r}\binom{r-1}{j-1}m_{r-j}\frac{d\kappa_j}{d\theta} - \sum_{j=2}^{r}\binom{r-1}{j-2}m_{r+1-j}\kappa_j \\
& + \frac{\beta}{\alpha}s^r(m_1 - s)\left[\sum_{j=1}^{r}\binom{r-1}{j-1}\kappa_j - 1\right],
\end{aligned}
\tag{6.6}
$$

where $\beta = 1-\alpha$, $0 < \alpha \le 1$ *and* m_r *denotes* r*th ordinary moment of* IMPSD.

Proof. Using the following results obtained in [5]:

$$m_r = \sum_{j=1}^{r} \binom{r-1}{j-1} m_{r-j}\kappa_j,$$

$$\frac{dm_r}{d\theta} = \sum_{j=1}^{r} \binom{r-1}{j-1} \left[\frac{dm_{r-j}}{d\theta}\kappa_j + m_{r-j}\frac{d\kappa_j}{d\theta}\right]$$

and (6.1) we get

$$\sum_{j=1}^{r} \binom{r-1}{j-1} m_{r+1-j}\kappa_j = \frac{g(\theta)}{g'(\theta)} \sum_{j=1}^{r} \binom{r-1}{j-1} \left[\frac{dm_{r-j}}{d\theta}\kappa_j + m_{r-j}\frac{d\kappa_j}{d\theta}\right]$$

$$+\frac{1}{\alpha}(m_1 - \beta s)\sum_{j=1}^{r} \binom{r-1}{j-1} m_{r-j}\kappa_j - \frac{\beta}{\alpha}s^r(m_1 - s).$$

This gives

$$\kappa_{r+1} = \frac{g(\theta)}{g'(\theta)} \sum_{j=1}^{r} \binom{r-1}{j-1} m_{r-j}\frac{d\kappa_j}{d\theta}$$

$$+\sum_{j=1}^{r} \binom{r-1}{j-1} \left[\frac{g(\theta)}{g'(\theta)}\frac{dm_{r-j}}{d\theta} + \frac{1}{\alpha}(m_1 - \beta s)m_{r-j} - \frac{\beta}{\alpha}s^r(m_1 - s)\right]\kappa_j$$

$$+\frac{\beta}{\alpha}s^r(m_1 - s)\left[\sum_{j=1}^{r} \binom{r-1}{j-1}\kappa_j - 1\right] - \sum_{j=1}^{r} \binom{r}{j-1} m_{r+1-j}\kappa_j.$$

Making use of (6.1) again we obtain (6.6) after some obvious simplifications.

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