ANNALES
UNIVERSITATIS MARIAE CURIE-SKLODOWSKA
LUBLIN-POLONIA
VOL. L, 9 SECTIO A 1996

WIESLAWA KACZOR,
MALGORZATA KOTER-MÓRGOWSKA (Lublin)

Firmly Lipschitzian Mappings

ABSTRACT. S. Reich and I. Shafrir in their paper [6] obtained some interesting results concerning asymptotic behaviour of firmly nonexpansive mappings. Our purpose is to extend some of their results to firmly k-Lipschitzian mappings with $k \geq 1$ and to give some other properties of such mappings.

1. Introduction. Let D be a subset of Banach space X. A mapping $T : D \to X$ is said to be *firmly Lipschitzian* with constant $k \geq 1$ (or *firmly k-Lipschitzian*), if for all $t \in [0,1]$ and for all $x, y \in D$

$$\| Tx - Ty \| \leq \| k(1-t)(x-y) + t(Tx - Ty) \| .$$

This is a generalization of the notion of firmly nonexpansive mappings indroduced by R. E. Bruck in [1]. Such mappings have some remarkable properties not shared by all nonexpansive mappings (see [1], [2], [3], [6]). The notion of firmly k-Lipschitzian mappings with $k > 1$ was introduced in [5], where firmly Lipschitzian and 2-rotative mappings were investigated.

It is easy to see that each firmly Lipschitzian mapping is Lipschitzian with the same constant.

In this paper we present some results concerning the general properties of firmly Lipschitzian mappings and their asymptotic behaviour.

2. Basic properties. Let now $J : X \to 2^{X^*}$ be the duality mapping. This means that for every $w \in X$

$$J(w) = \{w^* \in X^* : (w, w^*) = \| w \|^2 = \| w^* \|^2\}.$$

We have a well known (see e.g. [3])

Lemma. *Suppose $z, w \in X$. The following statements are equivalent*
(i) *For each $t \in [0,1]$, $\| w \| \leq \| (1-t)z + tw \|$,*
(ii) *There exists $j \in J(w)$ such that $\| w \|^2 \leq (z, j)$.*

We will need this lemma in proving

Theorem 1. *Let D be a subset of Banach space X. A mapping $T : D \to X$ is firmly k–Lipschitzian if and only if one of the following conditions holds*
(a) *For any $x, y \in D$ there is $j \in J(Tx - Ty)$, such that*

$$\| Tx - Ty \|^2 \leq k(x - y, j),$$

(b) *For any $x, y \in D$ and $t \in [0,1]$*

$$\| Tx - Ty \| \leq \frac{k}{1 + t(k-1)} \| (1-t)(x-y) + t(Tx - Ty) \|.$$

Proof. Assertion (a) is an immediate consequence of the lemma. To see this it is enough to put $z = k(x-y)$, $w = Tx - Ty$.

Suppose now that (a) holds. Then

$$\begin{aligned} k \| (1-t)(x-y) + t(Tx-Ty) \| \cdot \| Tx - Ty \| & \\ \geq k((1-t)(x-y) + t(Tx-Ty), j) & \\ = k(1-t)(x-y, j) + kt(Tx - Ty, j) & \\ \geq (1 - t + kt) \| Tx - Ty \|^2 & \end{aligned}$$

and (b) follows.

Suppose now that (b) holds. Then

$$\begin{aligned} \| Tx - Ty \| &\leq k \| (1-t)(x-y) + t(Tx-Ty) \| - t(k-1) \| Tx - Ty \| \\ &\leq \| k(1-t)(x-y) + kt(Tx - Ty) - t(k-1)(Tx - Ty) \| \\ &= \| k(1-t)(x-y) + t(Tx - Ty) \| \end{aligned}$$

and so T is firmly k–Lipschitzian.

Theorem 2. *Let $T : D \to X$ be firmly k–Lipschitzian mapping satisfying $\| Tx - Ty \| = k \| x - y \|$. Assume that T can be iterated at $x \in D$. Then for any positive integer n*

$$\| x - T^n x \| = \sum_{i=0}^{n-1} k^i \| x - Tx \| .$$

Proof. Let $\| T^i x - T^{i+1} x \| = k^i \| x - Tx \| = d_i$, $i = 0, 1, 2, \ldots$. Obviously $\| x - T^n x \| \leq \sum_{i=0}^{n-1} d_i$. We will prove the equality by induction on n. For $n = 1$ it is obvious. Suppose that $\| x - T^k x \| = \sum_{i=0}^{k-1} d_i$ for $k = 1, 2, \ldots, n$. Then

$$\begin{aligned} \sum_{i=0}^{n-1} d_i &= \| x - T^n x \| = \frac{1}{k} \| Tx - T^{n+1} x \| \\ &= \frac{1}{k} \cdot \frac{k}{k+1} \| x - T^n x + Tx - T^{n+1} x \| \\ &\leq \frac{1}{k+1} \| x - T^{n+1} x \| + \frac{1}{k+1} \| Tx - T^n x \| \\ &= \frac{1}{k+1} \| x - T^{n+1} x \| + \frac{k}{k+1} \sum_{i=0}^{n-2} d_i . \end{aligned}$$

Thus

$$\| x - T^{n+1} x \| \geq (k+1) \sum_{i=0}^{n-1} d_i - k \sum_{i=0}^{n-2} d_i = k d_{n-1} + \sum_{i=0}^{n-1} d_i = \sum_{i=0}^{n} d_i$$

and the proof is complete.

Suppose now that D is closed and convex. With each Lipschitzian mapping $T : D \to D$ a family of firmly Lipschitzian mappings is associated. Namely we have the following

Theorem 3. *Let $T : D \to D$, where D is closed and convex subset of X, be k–Lipschitzian. Then each member of the family of mappings $\{F_\alpha\}$ defined by $F_\alpha = (1 - \alpha T)^{-1}(1 - \alpha)I$, where I is the identity mapping and $\alpha \in (0, 1/k)$, is firmly Lipschitzian with the constant $(1 - \alpha)/(1 - \alpha k)$.*

Correctness of the definition of such F_α was proved in [4]. Moreover, it is easy to see that $\mathrm{Fix}\, F_\alpha = \mathrm{Fix}\, T$.

Proof of Theorem 3. First observe that

$$F_\alpha x = (1 - \alpha)x + \alpha T F_\alpha x .$$

Set $p = (1-t)x + tF_\alpha x$, $t \in [0,1]$. Then

$$F_\alpha x = (1-\alpha)\left(\frac{-t}{1-t}F_\alpha x + \frac{1}{1-t}p\right) + \alpha T F_\alpha x$$

and consequently

$$F_\alpha x = \left(1 - \frac{\alpha - t\alpha}{1-t\alpha}\right)p + \frac{\alpha - t\alpha}{1-t\alpha}TF_\alpha x .$$

Thus $F_\alpha x = F_\beta p$, where $\beta = (\alpha - t\alpha)/(1-t\alpha)$. (If $\alpha < 1/k$ then $\beta < 1/k$ and $\| F_\alpha x - F_\beta p \| = \beta \| TF_\alpha x - TF_\beta p \| \le \beta k \| F_\alpha x - F_\beta p \|$. So $\| F_\alpha x - F_\beta p \| = 0$.)

Similarly, taking $q = (1-t)y + tF_\alpha y$ we have $F_\alpha y = F_\beta q$. Therefore

$$\begin{aligned} \| F_\alpha x - F_\alpha y \| &= \| F_\beta p - F_\beta q \| \le \frac{1-\beta}{1-\beta k} \| p - q \| \\ &= \frac{1 - \frac{\alpha(1-t)}{1-\alpha t}}{1 - \frac{\alpha k(1-t)}{1-\alpha t}} \| (1-t)(x-y) + t(F_\alpha x - F_\alpha y) \| \\ &= \frac{\frac{1-\alpha}{1-\alpha k}}{1 + t\left(\frac{1-\alpha}{1-\alpha k} - 1\right)} \|(1-t)(x-y) + t(F_\alpha x - F_\alpha y)\| . \end{aligned}$$

This shows that F_α has property (b) from Theorem 1 with the constant $(1-\alpha)/(1-\alpha k)$. Hence it is firmly Lipschitzian with the same constant.

3. Asymptotic behaviour of firmly lipschitzian mappings. In this section we will try to say something about the convergence of the sequence of iterates of the firmly k-Lipschitzian mapping. Some interesting results concerning this problem in the case $k = 1$ were given in [6].

When $k > 1$ it is rather difficult to establish the limit of $\{T^n x\}$ or $\{\| T^{n+1}x - T^n x \|\}$. However, Theorem 2 has the following immediate corollary.

Corollary 1. *If $T : D \to D$ satisfies $\| Tx - Ty \| = k \| x - y \|$ for all $x, y \in D$ and is firmly k-Lipschitzian, then either $x \in Fix\,T$, or for any $m \ge 1$*

$$\lim_{n\to\infty} \| T^{n+m}x - T^n x \| = \lim_{n\to\infty} \| T^n x \| = \infty .$$

Assume now that $\| Tx - Ty \| \le k \| x - y \|$. The following theorem is a generalization of Theorem 1 in [6].

Theorem 4. *Let D be a subset of Banach space X and $T : D \to X$ a firmly $k-$Lipschitzian mapping with $k > 1$. If T can be iterated at the point $x \in D$ then for all $m \geq 1$*

$$\lim_{n\to\infty} \frac{\| T^{n+1}x - T^n x \|}{k^n} = \frac{k-1}{k^m - 1} \lim_{n\to\infty} \frac{\| T^{n+m}x - T^n x \|}{k^n}$$
$$= (k-1) \lim_{n\to\infty} \frac{\| T^n x \|}{k^n - 1}.$$

Proof. Since T is $k-$Lipschitzian, the sequences $\left\{ \frac{\|T^{n+m}x - T^n x\|}{k^n} \right\}$ for all $m \geq 1$ are nonincreasing and the limits

$$L = \lim_{n\to\infty} \frac{\| T^{n+1}x - T^n x \|}{k^n} \quad \text{and} \quad P_m = \lim_{n\to\infty} \frac{\| T^{n+m}x - T^n x \|}{k^n}$$

exist.
Moreover,

$$\|T^{n+m}x - T^n x\| \leq \| T^{n+m}x - T^{n+m-1}x \| + \dots + \| T^{n+1}x - T^n x \|$$
$$\leq \left(k^{m-1} + \dots + 1\right) \| T^{n+1}x - T^n x \| = \frac{k^m - 1}{k-1} \| T^{n+1}x - T^n x \|.$$

Thus

$$(*) \qquad P_m \leq \frac{k^m - 1}{k-1} L \quad \text{for all positive integers } m.$$

We will show, by induction on m, that

$$P_m = \frac{k^m - 1}{k-1} L.$$

For $m = 1$ the equality is obvious.

Assume that $P_j = [(k^j - 1)/(k-1)]L$ for $j = 1, 2, \dots, m$. Since T is firmly $k-$Lipschitzian, for all $t \in (0,1)$ we have

$$\| T^{n+m+1}x - T^{n+1}x \| \leq \| k(1-t)\left(T^{n+m}x - T^n x\right) + t\left(T^{n+m+1}x - T^{n+1}x\right) \|.$$

Chosing $t = k/(k+1)$ we get

$$\| T^{n+m+1}x - T^{n+1}x \| \leq \frac{k}{k+1} \| T^{n+m}x - T^n x + T^{n+m+1}x - T^{n+1}x \|$$
$$\leq \frac{k}{k+1} \| T^{n+m+1}x - T^{n+1}x \| + \frac{k}{k+1} \| T^{n+m}x - T^{n+1}x \|.$$

Hence

$$\frac{\| T^{n+m+1}x - T^n x \|}{k^{n+1}} \geq \frac{k+1}{k}\frac{\| T^{n+m+1}x - T^{n+1}x \|}{k^{n+1}} - \frac{\| T^{n+m}x - T^{n+1}x \|}{k^{n+1}}$$

and

$$\frac{1}{k}P_{m+1} \geq \frac{k+1}{k}P_m - P_{m-1} = \frac{k+1}{k}\cdot\frac{k^m-1}{k-1}L - \frac{k^{m-1}-1}{k-1}L\,.$$

Consequently

$$P_{m+1} \geq \frac{k^{m+1}-1}{k-1}L\,.$$

Therefore, using $(*)$ we complete the proof of equality

$$L = \frac{k-1}{k^m-1}P_m \quad \text{for all} \quad m \geq 1\,.$$

Now we will show that for every positive integer m

$$(**) \qquad \frac{\sum_{i=0}^{m-1} \| T^{i+1}x - T^i x \|}{\sum_{i=0}^{m-1} k^i} \leq \frac{1}{m}\sum_{i=0}^{m-1}\frac{\| T^{i+1}x - T^i x \|}{k^i}\,.$$

To see this, set $s = \sum_{i=0}^{m-1} k^i$. Let us also observe that for all $1 \leq i < m$

$$\frac{k^{i-1}+k^{i-2}+\ldots+1}{s} - \frac{i}{mk^{m-i}} > 0\,.$$

In fact, the monotonicity of the function $f(x) = (1-k^{-x})/x$, $x \geq 1$, implies $(1-k^{-i})/i > (1-k^{-m})/m$ which proves the above inequalities. In view of those, we obtain

$$\begin{aligned}
\sum_{i=0}^{m-1}\left(\frac{1}{s} - \frac{1}{mk^i}\right) \| T^{i+1}x - T^i x \| &= \sum_{i=0}^{m-2}\left(\frac{1}{s} - \frac{1}{mk^i}\right) \| T^{i+1}x - T^i x \| \\
&\quad + \left(\frac{1}{s} - \frac{1}{mk^{m-1}}\right) \| T^m x - T^{m-1}x \| \\
&\leq \sum_{i=0}^{m-3}\left(\frac{1}{s} - \frac{1}{mk^i}\right) \| T^{i+1}x - T^i x \| \\
&\quad + \left(\frac{1}{s} - \frac{1}{mk^{m-2}} + \frac{k}{s} - \frac{k}{mk^{m-1}}\right) \| T^{m-1}x - T^{m-2}x \| \\
&= \sum_{i=0}^{m-3}\left(\frac{1}{s} - \frac{1}{mk^i}\right) \| T^{i+1}x - T^i x \| \\
&\quad + \left(\frac{k+1}{s} - \frac{2}{mk^{m-2}}\right) \| T^{m-1}x - T^{m-2}x \|
\end{aligned}$$

$$\leq \sum_{i=0}^{m-4}\left(\frac{1}{s}-\frac{1}{mk^i}\right)\| T^{i+1}x - T^i x \|$$
$$+\left(\frac{1}{s}-\frac{1}{mk^{m-3}}+\frac{k^2+k}{s}-\frac{2}{mk^{m-3}}\right)\| T^{m-2}x - T^{m-3}x \|$$
$$=\sum_{i=0}^{m-4}\left(\frac{1}{s}-\frac{1}{mk^i}\right)\| T^{i+1}x - T^i x \|$$
$$+\left(\frac{k^2+k+1}{s}-\frac{3}{mk^{m-3}}\right)\| T^{m-2}x - T^{m-3}x \| \leq$$
$$\leq \ldots \leq \left(\frac{k^{m-1}+k^{m-2}+\ldots+1}{s}-\frac{m}{mk^{m-m}}\right)\| Tx - x \| = 0\,.$$

Now, from the equality $L = [(k-1)/(k^m-1)]P_m$ for all $m \geq 1$, we get

$$L \leq \frac{k-1}{k^m-1}\| T^m x - x \| \,.$$

With $(**)$ this yields

$$L \leq \varliminf_{m\to\infty}\frac{k-1}{k^m-1}\| T^m x - x \| \leq \varlimsup_{m\to\infty}\frac{k-1}{k^m-1}\| T^m x - x \|$$
$$\leq \varlimsup_{m\to\infty}\frac{k-1}{k^m-1}\sum_{i=0}^{m-1}\| T^{i+1}x - T^i x \|$$
$$\leq \varlimsup_{m\to\infty}\frac{1}{m}\sum_{i=0}^{m-1}\frac{\| T^{i+1}x - T^i x \|}{k^i}$$
$$= \lim_{m\to\infty}\frac{\| T^{m+1}x - T^m x \|}{k^m} = L\,,$$

and thus the limit $\lim_{n\to\infty}\frac{k-1}{k^n-1}\| T^n x - x \|$ exists and is equal to L. But $\lim_{n\to\infty}\frac{k-1}{k^n-1}\| T^n x - x \| = \lim_{n\to\infty}\frac{k-1}{k^n-1}\| T^n x \|$, which completes the proof of second equality.

Let us observe that the equalities of the above theorem can be written in the form:

$$\lim_{n\to\infty}\frac{\| T^{n+1}x - T^n x \|}{k^n} = \frac{1}{\sum_{i=0}^{m-i}k^i}\lim_{n\to\infty}\frac{\| T^{n+m}x - T^n x \|}{k^n}$$
$$= \lim_{n\to\infty}\frac{\| T^n x \|}{\sum_{i=0}^{n-1}k^i}\,.$$

Thus in case $k=1$ we get a result of S.Reich and I.Shafrir [6].

The following corollaries also generalize some results of [6].

Corollary 2. *If a firmly k-Lipschitzian mapping T has a nonempty fixed points set, then*

$$\lim_{n\to\infty} \frac{\| T^{n+1}x - T^n x \|}{k^n} \le (k-1) dist(x, Fix T)$$

for each point x at which T can be iterated.

Corollary 3. *Let $T: D \to D$ be firmly k-Lipschitzian and set* $d = \inf\{\| y - Ty \|: \ y \in D\}$.
Then for each $x \in D$ and for every $\varepsilon > 0$ there exists $x_\varepsilon \in D$ such that

$$\lim_{n\to\infty} \frac{\| T^{n+1}x - T^n x \|}{k^n} \le d + \varepsilon + (k-1) \| x - x_\varepsilon \| .$$

Proof. Let $\varepsilon > 0$. Then there is $x_\varepsilon \in D$ such that $\| Tx_\varepsilon - x_\varepsilon \| < d + \varepsilon$. We have

$$\begin{aligned}
\| T^n x_\varepsilon \| &\le \| T^n x_\varepsilon - T^{n-1} x_\varepsilon \| + \| T^{n-1}x_\varepsilon - T^{n-2}x_\varepsilon \| \\
&\quad + \ ... + \| Tx_\varepsilon - x_\varepsilon \| + \| x_\varepsilon \| \\
&\le (k^{n-1} + k^{n-2} + ... + 1) \| Tx_\varepsilon - x_\varepsilon \| + \| x_\varepsilon \| \\
&= \frac{k^n - 1}{k-1}(d+\varepsilon) + \| x_\varepsilon \|
\end{aligned}$$

and

$$\| T^n x \| \le \| T^n x_\varepsilon \| + k^n \| x - x_\varepsilon \| \le \frac{k^n - 1}{k-1}(d+\varepsilon) + \| x_\varepsilon \| + k^n \| x - x_\varepsilon \| .$$

Hence

$$\lim_{n\to\infty} \frac{k-1}{k^n - 1} \| T^n x \| \le d + \varepsilon + (k-1) \| x - x_\varepsilon \| .$$

Before stating next corollary let us recall that a mapping $T: D \to D$ is called *rotative* if there exist $m > 1$ and $0 \le a < m$ such that

$$\| T^m x - x \| \le a \| Tx - x \| \quad \text{for any } x \in D .$$

Corollary 4. *If T is firmly k-Lipschitzian with $k \ge 1$ and rotative then*

$$\lim_{n\to\infty} \frac{\| T^{n+1}x - T^n x \|}{k^n} = 0 .$$

Proof. Suppose that T is m-rotative. Then $\|T^{n+m}x - T^n x\| \leq a \|T^{n+1}x - T^n x\|$ and $\sum_{i=0}^{m-1} k^i L = P_m \leq aL$, where L and P_m are the limits defined in the proof of Theorem 4. But $\sum_{i=0}^{m-1} k^i \geq m > a$. Thus $L = 0$.

Unfortunately from $L = 0$, in the case $k > 1$, neither the existence of limit of the sequence $\{\| T^{n+1}x - T^n x \|\}$ nor its boundedness follow. However, if e.g. T is firmly k-Lipschitzian and 2-rotative with the constant $a \leq 1 + 1/k$ then the limit $\lim_{n\to\infty} \| T^{n+1}x - T^n x \|$ does exist. In fact,

$$\| T^{n+2}x - T^{n+1}x \| \leq \frac{k}{k+1} \| T^{n+2}x - T^n x + T^{n+2}x - T^{n+1}x \|$$

$$= \frac{k}{k+1} \| T^{n+2}x - T^n x \| \leq \frac{ka}{k+1} \| T^{n+1}x - T^n x \| \leq \| T^{n+1}x - T^n x \|,$$

meaning the sequence $\{\| T^{n+1}x - T^n x \|\}$ is nonincreasing.

Also if $T : D \to D$, where D is closed and convex, is rotative and k-Lipschitzian, then there exists $\gamma > 1$ such that the following holds. If $k \leq \gamma$ then we can find $\alpha_0 \in (0, 1/k)$ such that the sequence $\{\| F_{\alpha_0}^n x - F_{\alpha_0}^{n+1} x \|\}$ for associated mapping F_{α_0} is nonincreasing (see [4, proof of Theorem 2]). So the corresponding limits from Theorem 4 are for this mapping equal to zero.

References

[1] Bruck, R. E., *Nonexpansive projections on subsets of Banach spaces*, Pacific J. Math. **48** (1973), 341-357.

[2] Goebel, K. and W. A. Kirk, *Topics in metric fixed point theory*, Cambridge University Press, 1990.

[3] Goebel, K. and M. Koter, *Regularly nonexpansive mappings*, An. Ştiinţ. Univ."Al. I. Cuza" **24** (1978), 265-269.

[4] Goebel, K. and M. Koter, *Fixed points of rotative Lipschitzian mappings*, Rend. Sem. Mat. Fis. Milano **51** (1981), 145-156.

[5] Koter, M., *Fixed points of Lipschitzian 2-rotative mappings*, Boll. Un. Mat. Ital. Serie VI **V-C** (1986), no. N.1, 321- 339.

[6] Reich, S. and I. Shafrir, *The asymptotic behavior of firmly nonexpansive mappings*, Proc. Amer. Math. Soc. **101** (1987), 246-250.

Instytut Matematyki UMCS
Plac Marii Curie-Skłodowskiej 1
20-031 Lublin, Poland

received September 6, 1996