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Nonlinear Volterra Integral Equation with Discontinuous Right-Hand Side

Abstract. The existence of a maximal continuous solution to a nonlinear discontinuous Volterra integral equation is established.

1. Introduction. Our aim is to present a theorem concerning the existence of a maximal continuous solution to a nonlinear integral equation of the Volterra type:

$$x(t) = u(t) + \int_0^t f(t, \tau, x(\tau))\, d\tau \,. \tag{1}$$

Usually techniques for deriving existence criteria are based on Schauder fixed point theorem and need f to be continuous in x, see e.g. [1], [3], [4], [7] or [2, Chapter 12]. We assume f to be right continuous and non-decreasing in that variable which allows us to use the inequalities method (called sometimes Perron method [7]). Conditions we assume for f are similar to those of [6] (conditions (C1),...,(C5)). It is worth of mention however that only ordinary differential equations are considered there.

2. Main result. Given the functions $u : [0,1] \to \mathbb{R}$ and $f : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$. Assume that:

$(C1)$ u is continuous.

($C2$) For any $(t,x) \in [0,1] \times \mathbb{R}$, $f(t,\cdot,x)$ is measurable.
($C3$) For any $t \in [0,1]$ and almost all $\tau \in [0,1]$, $f(t,\tau,\cdot)$ is non-decreasing and

$$\lim_{y \downarrow x} f(t,s,y) = f(t,s,x) .$$

($C4$) For any $(t,x) \in [0,1] \times \mathbb{R}$, and almost all $\tau \in [0,1]$,

$$|f(t,\tau,x)| \leq M(t) ,$$

where $M : [0,1] \to [0,\infty]$ is an integrable function.
($C5$) Denote by $\mathcal{M}$ the family of all continuous functions $x : [0,1] \to \mathbb{R}$ satisfying the inequality

$$(2) \qquad |x(t)| \leq |u(t)| + \int_0^t M(\tau)\, d\tau .$$

We assume that all functions

$$[0,1] \ni t \longmapsto \int_0^t f(t,\tau,x(\tau))\, d\tau ,$$

where $x \in \mathcal{M}$, are equicontinuous.

Theorem. *There exists a maximal continuous solution x^* to the equation*

$$x(t) = u(t) + \int_0^t f(t,\tau,x(\tau))\, d\tau, \; t \in [0,1] .$$

Proof. Let ω be a common modulus of continuity of all functions

$$[0,1] \ni t \longmapsto \int_0^t f(t,\tau,x(\tau))\, d\tau ,$$

where $x \in \mathcal{M}$. Denote by $\mathcal{X}$ the set of all $x \in \mathcal{M}$ such that

$$(3) \qquad |x(t) - x(s)| \leq |u(t) - u(s)| + \omega(|t-s|) + \left| \int_s^t M(\tau)\, d\tau \right|$$

and

$$(4) \qquad x(t) \leq u(t) + \int_0^t f(t,\tau,x(\tau))\, d\tau ,$$

for all $s,t \in [0,1]$, and let us define x^* by the formula

$$x^*(t) = \sup_{x \in \mathcal{X}} x(t) .$$

Let the operator L be defined for any continuous $x:[0,1]\to\mathbb{R}$ by the formula

$$(Lx)(t)=u(t)+\int_0^t f(t,\tau,x(\tau))\,d\tau.$$

By $(C3)$, we have

$$\lim_{y\uparrow x} f(t,\tau,y)\le f(t,\tau,x)\le \lim_{y\downarrow x} f(t,\tau,x),$$

for all $(t,x)\in[0,1]\times\mathbb{R}$ and almost all $\tau\in[0,1]$. Thus the composition $f(t,\cdot,x(\cdot))$ is a measurable function for any continuous x, see [5]. This, together with $(C4)$, implies that the operator L is well defined. Moreover, for any continuous $x:[0,1]\to\mathbb{R}$, the function Lx satisfies (2), and, for any $x\in\mathcal{M}$, Lx satisfies (3) because of the following estimate:

$$\begin{aligned}
|(Lx)(t)-(Lx)(s)| &= \Big|u(t)+\int_0^t f(t,\tau,x(\tau))d\tau \\
&\quad -\Big(u(s)+\int_0^s f(s,\tau,x(\tau))d\tau\Big)\Big| \\
&\le |u(t)-u(s)|+\Big|\int_0^t f(t,\tau,x(\tau))d\tau-\int_0^s f(s,\tau,x(\tau))d\tau\Big| \\
&\le |u(t)-u(s)|+\omega(|t-s|).
\end{aligned}$$

We are going to prove that $x^*\in\mathcal{X}$ and $Lx^*=x^*$, which is just we have to do. Let us observe first that $\mathcal{X}\ne\emptyset$ since the function

$$t\longmapsto u(t)-\int_0^t M(\tau)\,d\tau$$

belongs to $\mathcal{X}$. Hence, x^* is also well defined and it is easy to see that x^* satisfies (2) and (3). By $(C3)$, for any $x\in\mathcal{X}$ and any $t\in[0,1]$, we have

$$x(t)\le u(t)+\int_0^t f(t,\tau,x(\tau))\,d\tau\le u(t)+\int_0^t f(t,\tau,x^*(\tau))\,d\tau.$$

Thus, $x^*\le Lx^*$, which means that x^* satisfies (4), so that $x^*\in\mathcal{X}$. Let us set $y=Lx^*$. Clearly, y belongs to $\mathcal{M}$ and satisfies (3). By the inequality $x^*\le Lx^*=y$ and by $(C3)$, we get $Lx^*\le Ly=x^*$ and, finally, $Lx^*=x^*$. The proof is complete.

3. Remarks.

1. We assumed f to be right continuous in order to guarantee the measurability of the composition $f(t, \cdot, x(\cdot))$, for any continuous $x : [0,1] \to \mathbb{R}$. If the assumption were omitted, the function $f(t, \cdot, x(\cdot))$ would not be measurable in general, as the following example shows.

Let $E \subset [0,1]$ be the non-measurable Vitali's set and let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be defined by the formula

$$f(\tau, x) = \begin{cases} 0, \text{ if } x < \tau \text{ or } x = \tau \notin E, \\ 1, \text{ if } x = \tau \in E, \\ 2, \text{ if } x > \tau. \end{cases}$$

The function f is measurable in τ and non-decreasing in x. However, for $x(\tau) = \tau$, the composition $f(t, \cdot, x(\cdot))$ is non-measurable.

2. Our condition $(C5)$ is not satisfactory. It is clear that we need it in the proof. Similar conditions occur in many papers on the subject, see e.g. [1], [3] or condition (iv) of Theorem 5.1, p. 372 in [2]. In particular, condition $(G3)$ of [1] is the same as our $(C5)$. Some functions f discontinuous in t, can satisfy condition $(C5)$, see e.g. example 6.1 of [2, p. 375]. However, the ("almost continuous") function

$$f(t) = \begin{cases} 0, \text{ if } t \in [0,1] \setminus \{\frac{1}{2}\}, \\ 1, \text{ if } t = \frac{1}{2}, \end{cases}$$

does not satisfy that condition.

3. Under the same assumptions on u and f the existence of a minimal continuous solution to the equation (1) can be obtained.

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