ANNALES
UNIVERSITATIS MARIAE CURIE - SKLODOWSKA
LUBLIN - POLONIA
VOL. L, 1 SECTIO A 1996

AISHAH S. ABDULLAH, ROSIHAN M. ALI (Penang)
and VIKRAMADITYA SINGH† (Patiala)

On Functions Starlike with Respect to a Boundary Point

ABSTRACT. Let U be the unit disc $|z| < 1$ and $\mathcal{G}$ be the class of functions $f(z) = 1 + \sum_{n=1}^{\infty} d_n z^n$ analytic and non-vanishing in U, and satisfying $\mathrm{Re}\left\{2\frac{zf'(z)}{f(z)} + \frac{1+z}{1-z}\right\} > 0$ in U. We examine the importance of the Koebe function $z/(1-z)^2$ to the class $\mathcal{G}$ and obtain sharp inequalities involving the coefficients d_1, d_2 and d_3.

1. Introduction.

Let $U = \{z : |z| < 1\}$ be the unit disc and $S^*(\alpha)$, $0 \le \alpha \le 1$, denote the class of analytic functions f in U normalized so that $f(0) = f'(0) - 1 = 0$, and such that

$$\mathrm{Re}\,\frac{zf'(z)}{f(z)} > \alpha, \quad z \in U.$$

Thus $f \in S^*(\alpha)$ maps U univalently onto a domain starlike with respect to the origin. We shall denote the class $S^*(0)$ simply by S^*. Further let S denote the familiar class of normalized analytic univalent functions in U.

The class $S^*(\alpha)$ has been extensively investigated during the last fifty years. However, not much seems to be known about the class of analytic functions that map U onto domains that are starlike with respect to a

1991 *Mathematics Subject Classification.* 30C45.

boundary point. Egerváry [1] seems among the early researchers to have come across such functions in his investigations on the Cesáro partial sums of the geometric series $\sum_{n=1}^{\infty} z^n$. However, Robertson [5] was the first to initiate a systematic study of this class, and we follow his terminology in this paper.

Definition. Let $\mathcal{G}$ denote the class of functions $f(z) = 1 + d_1 z + d_2 z^2 + \cdots + d_n z^n + \cdots$, analytic and non-vanishing in U and such that

$$\text{(1)} \qquad \operatorname{Re}\left\{2\frac{zf'(z)}{f(z)} + \frac{1+z}{1-z}\right\} > 0\,, \quad z \in U\,.$$

It was shown [5] that a function f belongs to $\mathcal{G}$ if and only if there exists a function $g \in S^*(1/2)$ such that

$$\text{(2)} \qquad f(z) = (1-z)\frac{g(z)}{z}\,.$$

This is equivalent to the condition $f \in \mathcal{G}$ if and only if there exists an $h \in S^*$ such that

$$\text{(3)} \qquad \phi(z) = \frac{h(z)(1-z)^2}{z} = f^2(z)\,.$$

Furthermore, either f is identically equal to the constant 1, or ϕ is close-to-convex with respect to h satisfying

$$\operatorname{Re}\left\{-\frac{z\phi'(z)}{h(z)}\right\} > 0\,, \quad z \in U\,.$$

Moreover, the coefficients d_n of $f \in \mathcal{G}$ satisfy

$$\text{(4)} \qquad |d_n| \le n|d_1|\,.$$

Inequality (4) is the general inequality for close-to-convex functions [4]. The equality is attained for any positive integer n and the function

$$f(z) = \frac{1-z}{\sqrt{1-2z\cos\theta + z^2}}\,, \quad 0 < \theta < 2\pi\,, \quad d_1 = \cos\theta - 1$$

which satisfies (2) and for which

$$\lim_{\theta\to 0}\left[\frac{f(z)-1}{\cos\theta - 1}\right] = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} nz^n\,.$$

From (3) we notice a peculiar role of the Koebe function or its rotations. These functions have several special properties [2]. In this paper we examine the special role of the Koebe and the generalized Koebe functions. In addition, we obtain sharp inequalities involving the coefficients of functions in $\mathcal{G}$.

2. Results.

Theorem 1. *The functions $g_x(z) = z(1-xz)^{-2}$, $|x| = 1$ are the only functions $g \in S$ so that for any $h \in S^*$, $h \neq g$, the function $f(z) = h(z)/g(z)$ is close-to-convex.*

Proof. Let $h(z) = z + \sum_{n=2}^{\infty} b_n z^n \in S^*$, and $g_x(z) = z + 2xz^2 + \cdots$. Then $|b_2| \leq |2x| = 2$ and since $g_x \neq h$, we have $b_2 \neq 2x$. Hence, for $0 \leq \rho \leq 1$, $\rho b_2 - 2x$ remains bounded away from zero.

Let $0 < \rho < 1$ and define

$$f_\rho(z) = \frac{h(\rho z)}{\rho g_x(z)}.$$

Then f_ρ is analytic in U and

$$\operatorname{Re}\left\{-\frac{1}{x}\,\frac{\rho z f_\rho'(z)}{h(\rho z)}\right\} > 0, \quad z \in U.$$

Thus f_ρ is univalent and close-to-convex in U, so $(f_\rho(z) - 1)/(\rho b_2 - 2x)$ is a normalized close-to-convex function in U. Since this class is compact and $\rho b_2 - 2x$ is bounded away from zero, we can take the limit $\rho \to 1$ and conclude that $f(z) = h(z)/g_x(z)$ is close-to-convex in U.

If $g(z) = z + a_2 z^2 + \cdots \in S$ but $g \neq g_x$, $|x| = 1$, then $|a_2| < 2$. There are many triples $(\lambda, \varepsilon, \delta) \in (0,1) \times \partial U \times \partial U$ satisfying

$$\lambda\varepsilon + (1-\lambda)\delta = \frac{a_2}{2},$$

so that we can find one such triple for which

$$h(z) = \frac{z}{(1-\varepsilon z)^{2\lambda}(1-\delta z)^{2-2\lambda}} \in S^*$$

is different from g. But

$$h(z) = z + 2(\lambda\varepsilon + \delta(1-\lambda))z^2 + \cdots = z + a_2 z^2 + \cdots,$$

and a simple calculation shows that $f'(0) = 0$ for $f(z) = h(z)/g(z)$ not identical to a constant. Thus such a function f is not even locally univalent at $z = 0$. Hence g does not have the property needed in the theorem

We are thankful to St. Ruscheweyh for suggesting this proof.

We observe that $g \in S$ in some sense cannot be relaxed. Indeed, the functions

$$g_\nu(z) = \frac{z}{\nu(1+z^2)+2z}, \quad -1 \le \nu \le 1, \quad \nu \ne 0$$

are univalent in U and belong to S only for $\nu = 1$. For $-1 < \nu < 1$ these functions are meromorphic univalent in U. If $h \in S^*$ and

$$f(z) = h(z)\left(2z + \nu(1+z^2)\right)/z, \tag{5}$$

then

$$\frac{zf'(z)}{h(z)} = -\frac{(1-z^2)}{z}\nu + \left\{2 + \nu\left(\frac{1}{z} + z\right)\right\}\frac{zh'(z)}{h(z)}$$

and it is clear that $\operatorname{Re}\{zf'(z)/h(z)\} > 0$ for $-1 \le \nu \le 1$.

If $h(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then (5) along with (4) for $f(z)$ gives the following sharp inequality for coefficients of starlike functions:

$$|2a_n + \nu(a_{n+1} + a_{n-1}| \le n|2 + \nu a_2|.$$

Theorem 2. *Let $g \in S^*(\alpha)$, $0 \le \alpha \le 1$, and let*

$$\phi(z) = (1-z)^{2(1-\alpha)} g(z)/z. \tag{6}$$

Then either ϕ is the constant 1, or $\phi(z)$, $[\phi(z)]^{1/2(1-\alpha)}$, $[\phi(z)]^{1/(1-\alpha)}$, and $\log \phi(z)$ are close-to-convex in U.

Proof. We first observe that, if $g \in S^*(\alpha)$ then

$$g_1(z) = \frac{g(z)}{(1-z)^{2\alpha}} \quad \text{and} \quad g_2(z) = z\left[\frac{g(z)}{z}\right]^{1/(1-\alpha)}$$

are in S^*. Further, for ϕ defined by (6),

$$[\phi(z)]^{1/(1-\alpha)} = \frac{(1-z)^2}{z} g_2(z) \quad \text{and} \quad [\phi(z)]^{1/2(1-\alpha)} = \frac{(1-z)}{z} h(z),$$

where g_2 is defined above and $h \in S^*(1/2)$. From [5] we deduce that $[\phi(z)]^{1/(1-\alpha)}$ and $[\phi(z)]^{1/2(1-\alpha)}$ are close-to-convex if ϕ is not a constant. Notice that

$$\frac{z\phi'(z)}{\phi(z)} = \frac{z\phi'(z)}{g_1(z)} \frac{z}{(1-z)^2}, \quad g_1 \in S^* \tag{7}$$

and

$$\frac{z\phi'(z)}{\phi(z)} = \left(\frac{zg'(z)}{g(z)} - \alpha\right) - (1-\alpha)\,\frac{1+z}{1-z}\,.$$

Hence

$$-\frac{1}{1-\alpha}\,\frac{z\phi'(z)}{g_1(z)} = \frac{1-z^2}{z} - \frac{(1-z)^2}{z}\,\frac{1}{1-\alpha}\left(\frac{zg'(z)}{g(z)} - \alpha\right).$$

Therefore $\mathrm{Re}\left\{-\frac{1}{1-\alpha}\,\frac{z\phi'(z)}{g_1(z)}\right\} > 0$ and ϕ is close–to–convex. From (7) we also conclude that $\log\phi(z)$ is close–to–convex.

The following yields some interesting coefficient bounds.

Theorem 3. *If $f \in \mathcal{G}$ with*

(8) $$f(z) = \frac{(1-z)g(z)}{z} = 1 + \sum_{n=1}^{\infty} d_n z^n\,, \quad g \in S^*(1/2),$$

and

(9) $$S_n(z) = 1 + \sum_{k=1}^{n} d_k z^k\,, \quad S_0(z) = 1\,,$$

then the functions

(10) $$\phi_n(z) = \frac{1}{z^n}\left(1 - \frac{S_{n-1}(z)}{f(z)}\right) + \frac{S_{n-1}(1)}{f(z)}\,, \quad n \geq 1\,,$$

are analytic in U and

(11) $$|\phi_n(z)| \leq 1\,.$$

In particular

(12) $$|S_n(1)| \leq 1\,,$$

(13) $$|S_n(1)|^2 + \sum_{m=1}^{p} |d_{n+m}|^2 \leq 1 + \sum_{n=1}^{p} |d_n|^2\,, \quad p \geq 1\,,$$

and

(14) $$|d_{n+1} - d_1 S_n(1)| \leq 1 - |S_n(1)|^2\,.$$

Proof. For $z, \zeta \in U$, let

$$\phi(z,\zeta) = \frac{\zeta}{g(\zeta)}\frac{g(z)-g(\zeta)}{z-\zeta}, \quad g \in S^*(1/2).$$

Then by [6], $\operatorname{Re}\phi(z,\zeta) > 1/2$. Hence in view of (8)

$$\phi(z,\zeta) = \frac{1}{1-z/\zeta} - \frac{f(z)}{f(\zeta)}\left(\frac{\zeta}{\zeta-z} - \frac{1}{1-z}\right).$$

Expansion of $\phi(z,\zeta)$ in powers of z yields

$$\phi(z,\zeta) = 1 + \sum_{n=1}^{\infty}\phi_n(\zeta)z^n,$$

where $\phi_n(\zeta)$ is defined by (9) and (10). As $\operatorname{Re}\phi(z,\zeta) > 1/2$, (11) follows.

Notice that

$$\phi_n(\zeta) = \frac{S_n(1) + d_{n+1}\zeta + \cdots + d_{n+m}\zeta^m + \cdots}{1 + \sum_{n=1}^{\infty} d_n\zeta^n}.$$

The inequalities (12) and (13) now follow from the fact that $|\phi_n(\zeta)| \le 1$ and $\phi_n(\zeta)$ is analytic for $\zeta \in U$. The inequality (14) is a consequence of the fact that, if

$$\phi_n(\zeta) = a_0 + a_1\zeta + a_2\zeta^2 + \cdots, \quad |\phi_n(\zeta)| < 1, \quad \zeta \in U,$$

then $|a_1| \le 1 - |a_0|^2$.

If $f \in \mathcal{G}$, inequality (11) for $n = 1$ gives

$$\left| f(z) - \frac{1-z}{1-|z|^2} \right| \le \frac{|z||1-z|}{1-|z|^2}, \quad z \in U,$$

which yields a distortion theorem for functions of the class $\mathcal{G}$.

Theorem 4. *If $f(z) = 1 + \sum_{n=1}^{\infty} d_n z^n \in \mathcal{G}$, then the coefficients d_n satisfy the following sharp inequalities:*

(15) $$|2d_2 + 1 - d_1^2| \le 1,$$

(16) $$|2d_2 - (1+d_1)(1+3d_1)| \le 1,$$

(17) $$|2d_2 - 2d_1(1+d_1)| \le 1 - |1+d_1|^2,$$

(18) $$|3d_3 - 3d_1d_2 + 1 + d_1^3| \le 1,$$

(19) $$|3d_3 - d_2(4+7d_1) - (1+d_1)(1+d_1-3d_1^2)| \le 1,$$

and

(20) $$|3d_3 - d_2(8+11d_1) + (1+d_1)(1+7d_1+9d_1^2)| \le 1.$$

We need the following for the proof of Theorem 4:

Lemma 1 [3]. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *and* $g(z) = \sum_{k=0}^{\infty} b_k z^k$ *be analytic in* U. *Then* $|g(z)| \le |f(z)|$, $z \in U$, *if and only if*

$$\sum_{j=0}^{\infty} \left\{ \left| \sum_{k=0}^{\infty} a_k z_{k+j} \right|^2 - \left| \sum_{k=0}^{\infty} b_k z_{k+j} \right|^2 \right\}$$

is positive semidefinite on the family of all sequences $\{z_k\}$ *satisfying*

$$\limsup_{k\to\infty} |z_k|^{1/k} < 1 .$$

Proof of Theorem 4. Let $d_0 = 1$ and

$$\psi(z) = 2\frac{zf'(z)}{f(z)} + \frac{1+z}{1-z} = \frac{1 + \sum_{n=1}^{\infty} [(2n+1)d_n + (3-2n)d_{n-1}]z^n}{1 + \sum_{n=1}^{\infty} (d_n - d_{n-1})z^n}$$
$$= 1 + 2(1+d_1)z + 2(2d_2 + 1 - d_1^2)z^2 + 2(3d_3 - 3d_1d_2 + d_1^3 + 1)z^3 + \cdots .$$

Then $\psi \in P$. Hence

$$\phi(z) = \frac{\psi(z)-1}{\psi(z)+1} = \frac{(1+d_1)z + 2(d_2 z^2 + (3d_3 - d_2)z^3 + \cdots}{1 + 2d_1 z + (3d_2 - d_1)z^2 + \cdots}$$

is analytic in U and satisfies $|\phi(z)| < 1$.

Applying Lemma 1 to the function $\phi(z)/z$ with

$$z_0 = b(1-d_1) + \lambda(d_1 - d_2), \quad z_1 = b, \quad z_2 = \lambda, \quad z_k = 0, \; k \ge 3$$

gives

$$\left| b(1 - d_1^2 + 2d_2) + \lambda\{(d_1 - d_2)(1+d_1) + (3d_3 - d_2)\} \right|^2$$
$$\le |\lambda|^2 + |b + 2\lambda d_1|^2 - |\lambda(1+d_1)|^2 .$$

The choice $\lambda = 0$ gives (15) and the choice $b = \lambda(1-d_1)$ gives (18). Further, the choice $b = -\lambda(1+3d_1)$ gives (19).

If we choose

$$z_0 = -b(1 + 3d - 1) - \lambda(5d_2 - d_1), \quad z_1 = b, \quad z_2 = \lambda, \quad z_k = 0, \; k \ge 3$$

and apply Lemma 1 again to $\phi(z)/z$, we obtain

$$|b\{(2d_2 - (1+d_1)(1+3d_1)\} + \lambda\{3d_3 - d_2 - (5d_2 - d_1)(1+d_1)\}|^2$$
$$\le |\lambda|^2 + |b + 2\lambda d_1|^2 - |\lambda(1+d_1)|^2 .$$

For $\lambda = 0$ this gives (16) and the choice $b = -\lambda(1 + 3d_1)$ yields (20).

Similarly, for $z_0 = 1$, $z_1 = y$, $z_k = 0$, $k \geq 2$, Lemma 1 yields

$$\left(1 - |1 + d_1|^2\right) + 4\,\mathrm{Re}\left\{y(d_1 - d_2(1 + \overline{d}_1))\right\} + |y|^2\left(3|d_1|^2 - 4|d_2|^2 - 2\,\mathrm{Re}\,d_1\right) \geq 0\,.$$

Upon completing squares, we deduce that

$$\left(3|d_1|^2 - 4|d_2|^2 - 2\,\mathrm{Re}\,d_1\right)\left(1 - |1 + d_1|^2\right) \geq 4|d_1 - d_2(1 + \overline{d}_1)|^2\,,$$

that is

$$\left(1 - |1 + d_1|^2\right)\left(4|d_2|^2 - 4|d_1|^2\right) + 4\left|d_2(1 + \overline{d}_1) - d_1\right|^2 \leq \left(1 - |1 + d_1|^2\right)^2\,.$$

Since

$$4\left|d_2(1 + \overline{d}_1) - d_1\right|^2 - 4|d_2|^2|1 + d_1|^2 - 4|d_1|^2 = -8\,\mathrm{Re}\left\{d_2\overline{d}_1(1 + \overline{d}_1)\right\},$$

this establishes (17).

Sharpness of the above inequalities follows from the fact that the defining equation (1) yields

$$\psi(z) = 2\,\frac{zf'(z)}{f(z)} + \frac{1 + z}{1 - z} = 1 + \sum_{n=1}^{\infty} p_n z^n\,,$$

and it is readily seen that (15) and (18) are respectively equivalent to

$$|p_2| \leq 2 \qquad \text{and} \qquad |p_3| \leq 2\,.$$

The inequality (17) corresponds to the well-known inequality

$$|p_2 - p_1^2/2| \leq 2 - |p_1|^2/2\,.$$

If we take the relationship (3) for f and take

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n\,, \qquad h \in S^*$$

then inequalities (15) and (16) are easily seen to be equivalent to the inequalities $|a_3 - a_2^2/2| \leq 1$ and $|a_3 - a_2^2| \leq 1$.

It appears that the inequalities involving coefficients of functions $f \in \mathcal{G}$ not only give the familiar well-known inequalities for coefficients of functions with positive real part and S^*, but they also give rise to some less-known results. Thus (19) is seen equivalent to

$$|p_3 - p_1p_2| \leq 2$$

and (17) to

$$|a_3 - 3a_2^2/4| \leq 1 - |a_2|^2/4\,. \tag{21}$$

The inequality (21) is identical with an inequality proved by Trimble [7] for convex functions.

References

[1] Egerváry, E., *Abbildungseigenschaften der arithmetischen Mittel der geometrischen Reihe*, Math. Z. **42** (1937), 221–230.

[2] Hallenbeck, D. J. and T. H. MacGregor, *Linear problems and convexity techniques in geometric function theory*, Pitman, 1984.

[3] Leverenz, C. R., *Hermitian forms in function theory*, Trans. Amer. Math. Soc. **286** (1984), 675–688.

[4] Reade, M. O., *On close-to-convex univalent functions*, Michigan Math. J. **3** (1955), 59–62.

[5] Robertson, M. S., *Univalent functions starlike with respect to a boundary point*, J. Math. Anal. Appl. **81** (1981), 327–345.

[6] Ruscheweyh, St. and T. Sheil-Small, *Hadamard products of schlicht functions and the Pólya-Schoenberg conjecture*, Comment. Math. Helv. **48** (1974), 119–135.

[7] Trimble, S. Y., *A coefficient inequality for convex univalent functions*, Proc. Amer. Math. Soc. **48** (1975), 266–267.

School of Mathematical Sciences
Universiti Sains Malaysia
Penang, Malaysia

received October 9, 1995

Punjabi University
Patiala, India