ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN-POLONIA
VOL. LI.2, 25 SECTIO A 1997

HONG-KUN XU*

Banach Space Properties of Opial's Type and Fixed Point Theorems of Nonlinear Mappings

ABSTRACT. Banach space properties of Opial's type are discussed; in particular, examples are given to separate the nonstrict Opial property, the Opial property, the locally uniform Opial property and the uniform Opial property from each other. It is also proved that if the Opial modulus $r_X(\cdot)$ of a Banach space X has a positive value at some $c \in (0,1)$, then X has the weak normal structure and every asymptotically regular and asypmtotically nonexpansive semigroup of self-mappings of a weakly compact convex subset C of X has a common fixed point.

1. Introduction and preliminaries. Let X be a Banach space and C a nonempty weakly compact convex subset of X. Recall that a mapping $T : C \to C$ is nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. In order to study convergence of the Picard iterates $\{T^n x\}$ of T, Z. Opial [9] introduced the following property of Banach spaces which is now called Opial's property.

*Supported in part by FRD (South Africa).
1991 *Mathematics Subject Classification.* Primary 46B20, 47H10; Secondary 47H09, 47H15.
Key words and phrases. Opial's property, locally uniform Opial property, uniform Opial property, Opial's modulus, renorming, fixed point, asymptotically regular, asymptotically nonexpansive semigroup.

Definition 1.1. A Banach space X is said to satisfy Opial's property if for a given sequence $\{x_n\}$ in X with $x_n \rightharpoonup x$, we have

$$\overline{\lim_{n\to\infty}} \|x_n - x\| < \overline{\lim_{n\to\infty}} \|x_n - y\| \quad \forall y \in X \setminus \{x\}.$$

(Throughout the paper, we use '$\rightharpoonup$' to stand for weak convergence and '$\to$' for strong convergence.)

If in the above definition the strict inequality '$<$' is replaced by the nonstrict inequality '$\le$', then the space X is said to satisfy the nonstrict (or weak) Opial property. It is well known that all Hilbert spaces and l^p $(1 < p < \infty)$ satisfy Opial's property, while L^p fails unless $p = 2$. The following is Opial's example [9] (with a correction of the values that the function φ takes).

Example 1.2. Let φ be a periodic function with period 2π such that

$$\varphi(t) = \begin{cases} 1, & \text{if } 0 \le t \le \frac{3}{4}\pi, \\ -\frac{3}{5}, & \text{if } \frac{3}{4}\pi < t \le 2\pi. \end{cases}$$

Let $\Phi_n(t) = \varphi(nt)$, $n \ge 1$. Then $\Phi_n \rightharpoonup 0$ in $L^p[0, 2\pi]$ for all $1 < p < \infty$. However, since

$$\lambda_p'(0) = -\frac{5\pi p}{4}\left[\frac{3}{5} - \left(\frac{3}{5}\right)^{p-1}\right],$$

where

$$\lambda_p(c) = \lim_{n\to\infty} \|\Phi_n(t) - c\|^p,$$

it follows that $c = 0$ is a minimizer of $\lambda_p(c)$ if and only if $p = 2$. So $L^p[0, 2\pi]$ does not satisfy Opial's property for any $p \ne 2$.

On the other hand, Hilbert spaces H and l^p for $1 < p < \infty$ do satisfy a property stronger than Opial's property as shown in the proposition below.

Proposition 1.3 (cf. [6],[7],[9]). *Let H be a Hilbert space and $1 < p < \infty$.*

(1) *If $x_n \rightharpoonup x$ in H, then*

$$\overline{\lim_{n\to\infty}} \|x_n - y\|^2 = \overline{\lim_{n\to\infty}} \|x_n - x\|^2 + \|y - x\|^2 \quad \forall y \in H.$$

(2) *If $x_n \rightharpoonup x$ in l^p, then*

$$\overline{\lim_{n\to\infty}} \|x_n - y\|^p = \overline{\lim_{n\to\infty}} \|x_n - x\|^p + \|y - x\|^p, \quad \forall y \in l^p.$$

This leads to the notion of the uniform Opial property proposed first by S. Prus [12]. Here we adopt another alternative.

Definition 1.4 [10]. The Opial modulus of a Banach space X is defined as the function $r_X : [0, \infty) \to \mathbf{R}$ given by

$$r_X(c) = \inf\left\{ \varliminf_{n\to\infty} \|x_n + x\| - 1 : x_n \rightharpoonup 0,\ \varliminf_{n\to\infty} \|x_n\| \geq 1,\ \|x\| \geq c \right\},\ c \geq 0.$$

If $r_X(c) > 0$ for all $c > 0$, then X is said to satisfy the uniform Opial property. The following proposition lists some basic properties of Opial's modulus.

Proposition 1.5 [10], [13]. *Let X be a Banach space. Then*

(i) $c - 1 \leq r_X(c) \leq c \quad \forall c \geq 0$; *hence* $r_X(c) > 0$ *for all* $c > 1$.
(ii) r_X *is continuous on* $[0, \infty)$ *and* $\frac{1+r_X(c)}{c}$ *is nondecreasing on* $(0, \infty)$.
(iii) *If X is uniformly convex and satisfies the nonstrict Opial property, then X satisfies the uniform Opial property.*

A few Banach spaces have their Opial modulus calculated.

Proposition 1.6 [8], [11]. *Let* $1 < p < \infty$ *and* $1 \leq q < \infty$. *Then*

(a) $r_{l^p}(c) = (1 + c^p)^{\frac{1}{p}} - 1, \quad c \geq 0.$
(b) $r_{l_{p,q}}(c) = \min\left\{ (1 + c^p)^{\frac{1}{p}} - 1, (1 + c^q)^{\frac{1}{q}} - 1 \right\}, \quad \forall c \geq 0$, *where* $l_{p,q}$ *is the usual* l^p *space but renormed by*

$$|x| = \left(\|x^+\|_p^q + \|x^-\|_p^q \right)^{\frac{1}{q}},$$

x^+ *and* x^- *being the positive and negative part of* x, *respectively, and* $\|\cdot\|_p$ *being the usual* l^p *norm. (More properties of* $l_{p,q}$ *may be found in* [2].*)*

Sometimes the sequence under consideration is fixed. This leads to the notion of the locally uniform Opial property introduced in [8].

Definition 1.7. A Banach space X is said to satisfy the locally uniform Opial property if for each weakly null sequence $\{x_n\}$ with $\varliminf_{n\to\infty} \|x_n\| \geq 1$, we have

$$\inf\left\{ \varliminf_{n\to\infty} \|x_n + x\| - 1 : \|x\| \geq c \right\} > 0 \quad \forall c > 0.$$

A useful equivalence is given below.

Proposition 1.8 [8]. *A Banach space X satisfies the locally uniform Opial property if and only if given any sequences $\{x_n\}$ and $\{y_m\}$ in X, the condition $x_n \rightharpoonup x$ and $\overline{\lim}_{m\to\infty} \overline{\lim}_{n\to\infty} \|x_n - y_m\| \le \overline{\lim}_{n\to\infty} \|x_n - x\|$ imply $y_m \to x$.*

2. Renorming. D. van Dulst [3] proved the following result.

Theorem 2.1. *Every separable Banach space $(X, \|\cdot\|)$ can be equivalently renormed to satisfy Opial's property.*

We include briefly van Dulst's proof which helps to construct several examples in the next section.

Sketch of Proof of Theorem 2.1. Because of separability, X is isometrically embedded into another Banach space Y with a basis $\{e_n\}_{n=1}^{\infty}$. We may assume that $\|e_n\| = 1$ for all $n \ge 1$ so that the associated conjugate basis $\{e_n^*\}_{n=1}^{\infty}$ is bounded. Let $\{P_n\}_{n=1}^{\infty}$ be the associated sequence of projections, i.e.,

$$P_n y = \sum_{i=1}^{n} \langle y, e_i^* \rangle e_i, \quad \forall y = \sum_{i=1}^{\infty} \langle y, e_i^* \rangle e_i \in Y.$$

Let $\|x\|_1 := \|x\|_2 + \|x\|_3, \quad x \in X$, where

$$\|x\|_2 := \sup_{n \ge 0} \|x - P_n x\| \quad (P_0 := 0) \quad \text{and} \quad \|x\|_3 := \sum_{n=1}^{\infty} \frac{1}{2^n} |\langle x, e_n^* \rangle|.$$

Then $\|\cdot\|_1$ is a norm equivalent to the original norm $\|\cdot\|$ of X and it satisfies the nonstrict Opial property. The contribution of the seminorm $\|\cdot\|_3$ is to force the nonstrict inequality '$\le$' in the nonstrict Opial property of the norm $\|\cdot\|_2$ to become the strict inequality '$<$' in Opial property of the norm $\|\cdot\|_1$. Also we observe that if $\|\cdot\|_3$ is replaced by $\varepsilon\|\cdot\|_3$ with $\varepsilon > 0$ small enough, then the norm $\|\cdot\|_1$ defined above still satisfies Opial's property. But in the latter case, the norm $\|\cdot\|_1$ is close to the norm $\|\cdot\|_2$ in the sense of the Banach-Mazur distance so that some geometric properties of $(X, \|\cdot\|_2)$ can be carried over to $(X, \|\cdot\|_1)$. □

3. Examples. This section is devoted to several examples of Banach spaces which separate various properties of Opial's type from each other. Obviously, the uniform Opial property implies the locally uniform Opial property which implies the Opial property which in turn implies the nonstrict Opial property. The examples constructed below show that none of the inverse implications is valid.

Example 3.1. Banach spaces that have the nonstrict Opial property but fail to have the Opial property. There are plenty of such spaces, for example, c_0 and $(l^2 \oplus \mathbf{R})$ with the norm $\|(x,r)\| = \max\{\|x\|_2, |r|\}$.

Example 3.2. Banach spaces that have the Opial property but lack the locally uniform Opial property. Apply van Dulst's norm in Theorem 3.1 to c_0 to get an equivalent norm $|x|$ on c_0 which satisfies Opial's property, given by

$$|x| = \|x\|_\infty + \sum_{n=1}^{\infty} \frac{|a_n|}{2^n}, \quad x = \{a_n\} \in c_0,$$

where $\|\cdot\|_\infty$ is the usual norm of c_0. To see that c_0 with this norm fails to satisfy the locally uniform Opial property, we take $x_n = e_n \rightharpoonup 0$ and $y_m = e_m$. Then it is straightforward that

$$\lim_m \lim_n |x_n + y_m| = 1 = \lim_n |x_n|, \quad \text{but} \quad y_m \not\to 0.$$

It follows from Proposition 1.8 that $(X, |\cdot|)$ does not satisfy the locally uniform Opial property.

Example 3.3. Let $X = l^2 \oplus l^3$ be normed by

$$|z| = \|z\|_\infty + \varepsilon \sum_{i=1}^{\infty} \frac{|x_i| + |y_i|}{2^i}, \quad z = (x,y),\ x = \{x_i\} \in l^2,\ y = \{y_i\} \in l^3,$$

where $\|z\|_\infty = \max\{\|x\|_2, \|y\|_3\}$, with $\|\cdot\|_2$ and $\|\cdot\|_3$ the usual l^2 and l^3 norms, repectively. As the norm $\|\cdot\|_\infty$ satisfies the nonstrict Opial property, $|\cdot|$ does satisfy the Opial property. Next we show that $|\cdot|$ fails to satisfy the locally uniform Opial property. Take a sequence $z^n = (x^n, y^n) \in X$ such that $z^n \rightharpoonup 0$ and $|z^n| \to 1$ with $\|x^n\|_2 \to 1$ and $\|y^n\|_3 \to b < 1$. Choose another sequence $w^m = (u^m, v^m) \in X$ such that $\|u^m\|_2 \to 0$, $\|v^m\|_3 \to c$ with $b^3 + c^3 \leq 1$, and $v_i^m \to 0$ as $m \to \infty$ for each $i \geq 1$. It follows from Proposition 1.3 that

$$\varlimsup_m \varlimsup_n |z^n - w^m| = \varlimsup_m \left(\max\left\{ \left(1 + \|u^m\|_2^2\right)^{\frac{1}{2}}, \left(b^3 + \|v^m\|_3^3\right)^{\frac{1}{3}} \right\} + \varepsilon \sum_{i=1}^{\infty} \frac{|u_i^m| + |v_i^m|}{2^i} \right) = \max\{1, (b^3 + c^3)^{\frac{1}{3}}\} = 1 = \varlimsup_n |z^n|.$$

But $w^m \not\to 0$. So again by Proposition 1.8, $(X, |\cdot|)$ lacks the locally uniform Opial property.

Remark. Recall that the normal structure coefficient of a Banach space X is defined by

$$N(X) = \inf\left\{\frac{\operatorname{diam}(A)}{r_A(A)}\right\},$$

where the infimum is taken over all bounded convex subsets A of X with more than one point, $\operatorname{diam}(A) = \sup\{\|x-y\| : x, y \in A\}$ is the diameter of A and $r_A(A) = \inf_{x\in A}\sup\{\|x-y\| : y \in A\}$ is the Chebyshev radius of A with respect to itself. X is said to have the uniform normal structure if $N(X) > 1$. As is well-known, uniform normal structure plays an important role in both Banach space theory and fixed point theory. It is known that $N((l^2 \oplus l^3)_\infty)) = \min\{N(l^2), N(l^3)\} = \min\{\sqrt{2}, \sqrt[3]{2}\} = \sqrt[3]{2} > 1$. It is easy to see that the Banach-Mazur distance $d((X, |\cdot|), (X, \|\cdot\|)) \le 1 + 2\varepsilon$. From Theorem 5 of Bynum [1] where it is proved that if E and F are two isomorphic Banach spaces, then $N(E) \le d(E, F)N(F)$, it follows that the space $(X, |\cdot|)$ in example 3.3 has the uniform normal structure provided $\varepsilon < (\sqrt[3]{2} - 1)/2$.

Example 3.4. A Banach space which has the locally uniform Opial property but lacks the uniform Opial property. Let

$$X = \left(\sum_{i=2}^{\infty} \oplus l^i\right)_2.$$

Then X satisfies the locally uniform Opial property; see [12] for details. However, it lacks the uniform Opial property, because for each $c > 0$, $r_X(c) \le r_{l^i}(c) = (1 + c^i)^{1/i} - 1 \to 0$ as $i \to \infty$.

4. Fixed Point Theorems. Let X be a Banach space, C a nonempty weakly compact convex subset of X, and $T : C \to C$ a mapping. Opial property plays an important role in the metric fixed point theory. The following result is now well-known (cf. [3]); see [5] for a comprehensive theory of nonexpansive mappings.

Theorem 4.1. *If X satisfies Opial property and T is nonexpansive, then T has a fixed point. Moreover, $I - T$ is demiclosed on C; namely, for $\{x_n\} \subset C$, $x_n \rightharpoonup x$ and $(I - T)x_n \to y \Rightarrow (I - T)x = y$.*

The following is a more recent result [8].

Theorem 4.2. *Suppose X satisfies the uniform Opial property and T is of asymptotically nonexpansive type; that is, for every $x \in C$,*

$$\overline{\lim_{n\to\infty}} \sup\{\|T^n x - T^n y\| - \|x - y\| : y \in C\} \leq 0.$$

If T^N is continuous for some integer $N \geq 1$, then T has a fixed point.

Let now G be an unbounded subset of $[0, \infty)$ such that for every $h, t \in G$ we have $h + t \in G$ and $h - t \in G$ if $h \geq t$. Let $\mathcal{S} = \{S_t : t \in G\}$ be a semigroup of self-mappings of C; i.e., $\mathcal{S}$ fulfils the conditions:

(i) $S_{t+s}(x) = S_t S_s(x)$, $x \in C, t, s \in G$;

(ii) for each $x \in C$, the mapping $t \to S_t(x)$ is continuous from G to C when G is equipped with the relative topology of $\mathbf{R}$.

Recall that $\mathcal{S}$ is said to be asymptotically regular on C if

$$\lim_{\substack{t\in G\\ t\to\infty}} \|S_{t+s}x - S_t x\| = 0 \quad \forall s \geq 0,\ x \in C.$$

For a Lipschitz mapping $T : C \to C$ we use $|T|$ to denote its Lipschitz constant, i.e.,

$$|T| = \sup\left\{\frac{\|Tx - Ty\|}{\|x - y\|} : x, y \in C,\ x \neq y\right\}.$$

For a semigroup $\mathcal{S}$ of Lipschitz self-mappings of C, we put

$$\sigma(\mathcal{S}) = \varliminf_{\substack{t\in G\\ t\to\infty}} |S_t|.$$

Theorem 4.3. *Let X be a Banach space such that $r_X(c) > 0$ for some $0 < c < 1$. (Here r_X is the Opial modulus of X.) Let C be a weakly compact convex subset of X and $\mathcal{S} = \{S_t\}_{t\in G}$ an asymptotically regular semigroup of Lipschitz mappings of C such that $\sigma(\mathcal{S}) \leq 1$. Then $\mathcal{S}$ has a common fixed point, i.e., a point $z \in C$ for which $S_t x = x$ for all $t \in G$.*

Proof. Select a sequence $\{t_n\} \subset G$, $t_n \to \infty$, such that $\lim_n |S_{t_n}| = \sigma(\mathcal{S}) \leq 1$. Without loss of generality, we may assume that C is separable, so that by passing to a subsequence if necessary, we may further assume that for every $x, y \in C$, $\{S_{t_n} x\}$ is weakly convergent and $\lim_n \|S_{t_n} x - y\|$ does exist. Define a function $q : C \to C$ by $q(x) = w - \lim_n S_{t_n} x$, $x \in C$ and a function $p : C \to \mathbf{R}$ by $p(x) = \lim_n \|S_{t_n}x - q(x)\|$, $x \in C$. The key point to the proof is the following assertion:

$$p(q(x)) \le c\, p(x), \quad \forall x \in C. \tag{4.1}$$

Write $y = q(x)$. If $p(x) = 0$, then $S_{t_n} \to y$ and hence by continuity and the asymptotic regularity of $\mathcal{S}$, we have for every $m \ge 1$,

$$S_{t_m} y = \lim_n S_{t_m} S_{t_n} x = \lim_n S_{t_m + t_n} = y.$$

It follows that $p(y) = 0$ and (4.1) is proved. Assume now $p(x) > 0$. If (4.1) were not valid, then we would have some $x \in C$ such that $p(y) > c\, p(x)$, where $y = q(x)$. By definition of the Opial modulus of X, we get

$$\begin{aligned} \overline{\lim_m} \lim_n \|S_{t_n}x - S_{t_m}y\| &= \overline{\lim_m} \lim_n \left\| \frac{(S_{t_n}x - y) + (y - S_{t_m}y)}{p(x)} \right\| p(x) \\ &\ge p(x) \left[1 + r_X \left(\frac{\lim_m \|S_{t_m}y - y\|}{p(x)} \right) \right]. \end{aligned} \tag{4.2}$$

On the other hand, using the asymptotic regularity of $\mathcal{S}$, we have

$$\begin{aligned} \overline{\lim_m} \lim_n \|S_{t_n}x - S_{t_m}y\| &= \overline{\lim_m} \lim_n \|S_{t_m}S_{t_n}x - S_{t_m}y\| \\ &\le \overline{\lim_m} \lim_n |S_{t_m}|\, \|S_{t_n}x - y\| \\ &\le \lim_n \|S_{t_n}x - y\| = p(x) \quad \text{as } \sigma(\mathcal{S}) \le 1. \end{aligned} \tag{4.3}$$

Also, by the weak lower semicontinuity of the norm, we obtain

$$\begin{aligned} p(y) = \lim_m \|S_{t_m}y - q(y)\| &\le \lim_m \overline{\lim_n} \|S_{t_m}y - S_{t_n}y\| \\ &\le \lim_m \overline{\lim_n} |S_{t_n}|\, \|S_{t_m - t_n}y - y\| = \lim_m \|S_{t_m}y - y\|. \end{aligned} \tag{4.4}$$

Combining (4.2) - (4.4) and noting $p(y) > c\, p(x)$ we get

$$p(x) \ge p(x) \left[1 + r_X \left(\frac{\lim_m \|S_{t_m}y - y\|}{p(x)} \right) \right] \ge p(x)[1 + r_X(c)].$$

This is a contradiction as $r_X(c) > 0$ and $p(x) > 0$. Hence (4.1) is proved. Now we can construct a sequence $\{x_n\}_{n=0}^{\infty}$ in C as follows.

$$x_0 \in C \text{ arbitrary}, \quad x_m = w - \lim_n S_{t_n} x_{m-1}, \quad m = 1, 2, \cdots.$$

By (4.1) we see that $p(x_m) \le c\,p(x_{m-1}),\quad m \ge 1$. Since

$$\begin{aligned}\|x_{m+1} - x_m\| &\le \lim_n \|x_{m+1} - S_{t_n} x_m\| + \lim_n \|S_{t_n} x_m - x_m\| \\ &\le p(x_m) + \overline{\lim_n} \lim_k \|S_{t_n} x_m - S_{t_k} x_{m-1}\| \\ &\le p(x_m) + \overline{\lim_n} \lim_k |S_{t_n}|\, \|S_{t_k - t_n} x_{m-1} - x_m\| \\ &\le p(x_m) + \lim_k \|S_{t_k} x_{m-1} - x_m\| \\ &= p(x_m) + p(x_{m-1}) \\ &\le (1+c)p(x_{m-1}) \le (1+c)c^{m-1} p(x_0),\end{aligned}$$

it follows that $\{x_m\}$ is a Cauchy sequence. Let $z = \lim x_m$. Since $p(x_m) = \lim_{n\to\infty} \|S_{t_n} x_m - x_{m+1}\| \to 0$ as $m \to \infty$, we see that $S_{t_n} z \to z$ as $n \to \infty$. By continuity and asymptotic regularity of $\mathcal{S}$, we get for each $s \in G$ that $S_s z = \lim_n S_s S_{t_n} z = \lim_n S_{s+t_n} z = \lim_n S_{t_n} z = z$. This completes the proof.

□

Remark. If, in addition, the space X satisfies the nonstrict Opial property, then Kuczumow [8] was able to relax the Lipschitz condition imposed upon $\mathcal{S}$ by showing that $\sigma(\mathcal{S}) < 1 + r_X(c)$ is sufficient for the existence of a common fixed point of $\mathcal{S}$.

Corollary 4.4. *Let X be a Banach space such that $r_X(c) > 0$ for some $0 < c < 1$, C a weakly compact convex subset of X, and $\mathcal{S} = \{S(t) : t \ge 0\}$ a semigroup of self-mappings of C which is asymptotically regular on C. Assume there exists a function $k : [0,\infty) \to [0,\infty)$, with $\overline{\lim}_{t\to\infty} k(t) \le 1$, such that for each $t > 0$,*

$$\|S(t)x - S(t)y\| \le k(t)\|x - y\| \quad \forall x, y \in C.$$

Then there is $z \in C$ for which $S(t)z = z$ for all $t \ge 0$.

Corollary 4.5. *Let X and C be as in* Corollary 5.4 *and $T : C \to C$ an asymptotically nonexpansive mapping (i.e., there exists a sequence $\{k_n\}$ with $\overline{\lim}_{n\to\infty} k_n \le 1$ such that $\|T^n x - T^n y\| \le k_n \|x - y\|$ for all $x, y \in C$ and $n \ge 1$). Assume T is asymptotically regular on C (i.e., $\lim_{n\to\infty} \|T^{n+1}x - T^n x\| = 0$ for all $x \in C$). Then T has a fixed point.*

Recall that a Banach space X is said to have the weak fixed point property (FPP) for nonexpansive mappings if for every nonempty weakly compact convex subset C, every nonexpansive mapping $T : C \to C$ has a fixed point; see the book [5] by Goebel and Kirk for much more.

Corollary 4.6. *If $r_X(c) > 0$ for some $0 < c < 1$, then X has the weak fixed point property for nonexpansive mappings.*

Proof. Let C be a weakly compact convex subset of X and $T : C \to C$ a nonexpansive mapping. The averaged mapping $S := (I + T)/2$ is asymptotically regular on C (see [4]). Therefore, Theorem 5.3 assures the existence of a fixed point for S (and for T).

□

In fact, we can prove more than the weak FPP.

Proposition 4.7. *If X is a Banach space such that $r_X(c) > 0$ for some $0 < c < 1$, then X has the weak normal structure; i.e., for every weakly compact convex subset K of X consisting of more than one point, there exists a point $z \in K$ for which* diam $K > r_K(K)$.

Proof. Suppose X fails to have the weak normal structure. Then (cf. [5]) we have a sequence $\{x_n\}$ in X satisfying

$$x_n \rightharpoonup 0, \ \lim_n \|x_n\| = 1, \text{ and } \operatorname{diam}\{x_n\} = 1.$$

Let $K = \overline{\text{co}}\{x_n\}$. Then K is diametral; i.e., $\sup_{y \in K} \|x - y\| = 1$ for every $x \in K$. Since $0 \in K$ and $0 < c < 1$, $cx_m \in K$ for all $m \geq 1$. It follows that

$$1 \geq \overline{\lim_{n \to \infty}} \|x_n - cx_m\| \geq 1 + r_X(c\|x_m\|) \stackrel{m \to \infty}{\longrightarrow} 1 + r_X(c).$$

This is a contradiction for $r_X(c) > 0$. □

References

[1] Bynum, W. L., *Normal structure coefficients for Banach spaces*, Pacific J. Math. **86** (1980), 427–436.

[2] Dominguez Benavides, T., G. Lopez Acedo and H.K. Xu, *Qualitative and quantitative properties for the space $l_{p,q}$*, Houston J. Math. **22** (1996), 89–100.

[3] van Dulst, D., *Equivalent norms and the fixed point property for nonexpansive mappings*, J. London Math. Soc. **25** (1982), 139–144.

[4] Edelstein, M. and R.C. O'Brien, *Nonexpansive mappings, asymptotic regularity and successive approximations*, J. London Math. Soc. **17** (1978), 547–554.

[5] Goebel, K. and W.A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge University Press, 1990.

[6] Kuczumow, T., *Opial's modulus and fixed points of semigroups of mappings*, Proc. Amer. Math. Soc. (to appear).

[7] Lim, T. C., *Asymptotic centers and nonexpansive mappings in some conjugate spaces*, Pacific J. Math. **90** (1980), 135–143.

[8] Lin, P. K.,K. K. Tan and H. K. Xu, *Demiclosedness principle and asymptotic behavior for asymptotically nonexpansive mappings*, Nonlinear Anal. **24** (1995), 929–946.

[9] Opial, Z., *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc. **73** (1967), 591–597.

[10] Prus, S., *Banach spaces with the uniform Opial property*, Nonlinear Anal. **19** (1992), 697–704.

[11] Xu, H. K., *Geometrical coefficients of Banach spaces and nonlinear mappings*, Recent Advances on Metric Fixed Point Theory (T. Dominguez Benavides, Ed.), Proceedings of the International Workshop on Metric Fixed Point Theory, Seville, Spain, 1995, Universidad de Seville Press, 1996, pp. 161–178.

[12] Xu, H. K. and G. Marino, *Uniform property (K) and its related properties*, Bull. Austral. Math. Soc. (1998) (to appear).

Department of Mathematics
University of Durban-Westville
Private Bag X54001, Durban 4000, South Africa
e-mail: hkxu@pixie.udw.ac.za

received October 1, 1997