ANNALES
UNIVERSITATIS MARIAE CURIE – SKŁODOWSKA
LUBLIN – POLONIA
VOL. LI. 2, 24 SECTIO A 1997

WATARU TAKAHASHI

Weak and Strong Convergence Theorems for Families of Nonexpansive Mappings and Their Applications

ABSTRACT. In this article we discuss weak and strong convergence to common fixed points of nonexpansive mappings in a Hilbert space or a Banach space. We first deal with weak convergence theorems of Baillon's type in a Hilbert space or a Banach space. Furthermore, we discuss weak and strong convergence theorems of Mann's type in a Banach space. Finally, using these results, we discuss the problem of image recovery by convex combinations of nonexpansive retractions.

1. Introduction. Let C be a nonempty closed convex subset of a real Banach space E. Then a mapping $T : C \to C$ is called *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of T. Baillon [1] proved the first nonlinear ergodic theorem for nonexpansive mappings in the framework of a Hilbert space: Let C be a closed convex subset of a Hilbert space and let T be a nonexpansive mapping of C into itself. If the set $F(T)$ is nonempty then for each $x \in C$ the Cesàro means

$$S_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converge weakly to some $y \in F(T)$. In this case, putting $y = Px$ for each $x \in C$, P is a nonexpansive retraction of C onto $F(T)$ such that

$PT^n = T^nP = P$ for all $n = 0, 1, 2, \dots$ and $Px \in \overline{\text{co}}\{T^n x : n = 0, 1, 2, \dots\}$ for each $x \in C$, where $\overline{\text{co}}\, A$ is the closure of the convex hull of A. In [32, 33, 34] Takahashi proved the existence of such a retraction - "ergodic retraction" - for an amenable semigroup of nonexpansive mappings in a Hilbert space. Rodé [26] also found a sequence of means on the semigroup, generalizing the Cesàro means on positive integers, such that the corresponding sequence of mappings converges to an ergodic retraction onto the set of common fixed points.

These results were extended to a uniformly convex Banach space whose norm is Fréchet differentiable, in the case when the semigroup is commutative, by Hirano, Kido and Takahashi [8]. However, it has been an open problem whether Takahashi's result and Rodé's result can be fully extended to such a Banach space for an amenable semigroup; see [37]. On the other hand, Mann [20] introduced an iteration procedure for approximating fixed points of a mapping T in a Hilbert space as follows: $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n \quad \text{for} \ \ n \geq 1,$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$. Later, Reich [24] discussed this iteration procedure in a uniformly convex Banach space whose norm is Fréchet differentiable and obtained that the iterates $\{x_n\}$ converge weakly to a fixed point of T under $\Sigma_{n=1}^{\infty}\alpha_n(1 - \alpha_n) = \infty$. Recently, Tan and Xu [44] proved the following interesting result which generalizes the result of Reich [24]: Let C be a bounded closed convex subset of a uniformly convex Banach space E which satisfies Opial's condition or whose norm is Fréchet differentiable and let T be a nonexpansive mapping of C into itself. Then for any initial data x_1 in C, the iterates $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n T[\beta_n T x_n + (1 - \beta_n)x_n] + (1 - \alpha_n)x_n \quad \text{for } n \geq 1,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are chosen so that $\Sigma_{n=1}^{\infty}\alpha_n(1 - \alpha_n) = \infty$, $\Sigma_{n=1}^{\infty}\beta_n(1 - \alpha_n) < \infty$ and $\limsup_{n\to\infty} \beta_n < 1$, converge weakly to a fixed point of T.

Using a nonlinear ergodic theorem [23], Crombez [4] considered the problem of image recovery. Let H be a Hilbert space, let $C_1, C_2, \dots, C_r$ be nonempty closed convex subsets of H and let I be the identity operator on H. Then the problem of image recovery in a Hilbert space setting may be stated as follows: The original (unknown) image z is known a priori to belong to the intersection C_0 of r well-defined sets $C_1, C_2, \dots, C_r$ in a Hilbert space; given only the metric projections P_i of H onto $C_i (i = 1, 2, \dots, r)$, recover z by an iterative scheme. Crombez [4] proved the following: Let $T = \alpha_0 I + \Sigma_{i=1}^{r}\alpha_i T_i$ with $T_i = I + \lambda_i(P_i - I)$ for all i, $0 < \lambda_i < 2$, $\alpha_i \geq 0$

for $i = 0, 1, 2, \ldots, r$, $\Sigma_{i=0}^{r} \alpha_i = 1$, where each P_i is the metric projection of H onto C_i and $C_0 = \cap_{i=1}^{r} C_i$ is nonempty. Then, starting from an arbitrary element x of H, the sequence $\{T^n x\}$ converges weakly to an element of C_0. Lateron, Kitahara and Takahashi [14] dealt with the problem of image recovery by convex combinations of sunny nonexpansive retractions in uniformly convex Banach spaces.

In this article, we first state nonlinear ergodic theorems of Baillon's type for nonlinear semigroups of nonexpansive mappings. In particular, we state nonlinear ergodic theorems which answer affirmatively the problem posed during the Second World Congress on Nonlinear Analysis, Athens, Greece, 1996; see [37]. Next, we deal with weak and strong convergence theorems of Mann's type in a Banach space. Finally, using these results, we consider the problem of image recovery by convex combinations of nonexpansive retractions.

2. Preliminaries. Let E be a Banach space and let C be a nonempty closed convex subset of E. Then a mapping T of C into E is said to be *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for every $x, y \in C$. Let T be a mapping of C into E. Then we denote by $F(T)$ the set of fixed points of T and by $R(T)$ the range of T. A mapping T of C into E is said to be *asymptotically regular* if for every $x \in C$, $T^n x - T^{n+1} x$ converges to 0. Let D be a subset of C and let P be a mapping of C into D. Then P is said to be *sunny* if

$$P(Px + t(x - Px)) = Px$$

whenever $Px + t(x - Px) \in C$ for $x \in C$ and $t \ge 0$. A mapping P of C into C is said to be a *retraction* if $P^2 = P$. If a mapping P of C into C is a retraction, then $Pz = z$ for every $z \in R(P)$. A subset D of C is said to be a *sunny nonexpansive retract* of C if there exists a sunny nonexpansive retraction of C onto D.

Let E be a Banach space. Then, for every ε with $0 \le \varepsilon \le 2$, the *modulus $\delta(\varepsilon)$ of convexity* of E is defined by

$$\delta(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| \le 1, \|y\| \le 1, \|x - y\| \ge \varepsilon\right\}.$$

A Banach space E is said to be *uniformly convex* if $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$. E is also said to be *strictly convex* if $\|x + y\| < 2$ for $x, y \in E$ with $\|x\| \le 1$, $\|y\| \le 1$ and $x \ne y$. A uniformly convex Banach space is strictly convex. A closed convex subset C of a Banach space E is said to have *normal structure* if, for each closed convex bounded subset K of C which contains at least two points, there exists an element of K which is not a diametral point of K.

It is well known that a closed convex subset of a uniformly convex Banach space has normal structure and a compact convex subset of a Banach space has normal structure [35]. We also know the following

Theorem 2.1 [13]. *Let E be a reflexive Banach space and let C be a nonempty bounded closed convex subset of E which has normal structure. Let T be a nonexpansive mapping of C into itself. Then $F(T)$ is nonempty.*

Let E be a Banach space and let E^* be its dual, that is the space of all continuous linear functionals x^* on E. For every $x \in E$ and $x^* \in E^*$, (x, x^*) means the value of x^* at x. With each $x \in E$ we associate the set

$$J(x) = \{x^* \in E^* : (x, x^*) = \|x\|^2 = \|x^*\|^2\}.$$

From the Hahn-Banach theorem it follows immediately that $J(x) \neq \phi$ for any $x \in E$. Then the multivalued operator $J : E \to E^*$ is called the *duality mapping* of E. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the norm of E is said to be *Gâteaux differentiable* (and E is said to be *smooth*) if

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for all x and y in U. It is said to be *Fréchet differentiable* if for each x in U, this limit is attained uniformly for y in U. It is also said to be *uniformly Fréchet differentiable* (and E is said to be *uniformly smooth*) if the limit is attained uniformly for x, y in U. It is well known that if E is smooth, then the duality mapping J is single-valued. We also know that if E has a Fréchet differentiable norm, then J is norm to norm continuous. A Banach space E is said to satisfy *Opial's condition* [23] if $x_n \rightharpoonup x$ and $x \neq y$ imply $\lim_{n\to\infty} \inf \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$, where $\rightharpoonup$ denotes the weak convergence.

3. Convergence theorems of Baillon's type.

The first nonlinear ergodic theorem for nonexpansive mappings was established in 1975 by Baillon [1] in the framework of a Hilbert space.

Theorem 3.1 [1]. *Let C be a closed convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself. If the set $F(T)$ of fixed points of T is nonempty, then for each $x \in C$, the Cesàro means $S_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$ converge weakly to some $y \in F(T)$.*

This theorem was extended by Bruck [3] and Reich [24] to a uniformly convex Banach space whose norm is Fréchet differentiable.

Theorem 3.2 [3], [24]. *Let C be a closed convex subset of a uniformly convex Banach space E with a Fréchet differentiable norm. If $T : C \to C$ is a nonexpansive mapping with a fixed point, then the Cesàro means of $\{T^n x\}$ converge weakly to a fixed point of T.*

We first extend these theorems to a nonlinear semigroup of nonexpansive mappings in a Hilbert space or a Banach space. Let S be a semitopological semigroup, i.e., a semigroup with Hausdorff topology such that for each $s \in S$, the mappings $t \mapsto ts$ and $t \mapsto st$ of S into itself are continuous. Let $B(S)$ be the Banach space of all bounded real valued functions on S with supremum norm and let X be a subspace of $B(S)$ containing constants. Then, an element μ of X^* (the dual space of X) is called a *mean* on X if $\|\mu\| = \mu(1) = 1$. For each $s \in S$ and $f \in B(S)$, we define elements $\ell_s f$ and $r_s f$ of $B(S)$ given by

$$(\ell_s f)(t) = f(st) \quad \text{and} \quad (r_s f) = f(ts)$$

for all $t \in S$. Let $C(S)$ be the Banach space of all bounded continuous real valued functions on S and let $RUC(S)$ be the space of all bounded right uniformly continuous functions on S, i.e., all $f \in C(S)$ such that the mapping $s \mapsto r_s f$ is continuous. Then $RUC(S)$ is a closed subalgebra of $C(S)$ containing constants and invariant under ℓ_s and r_s, $s \in S$; see [21] for more details. Let $\{\mu_\alpha : \alpha \in A\}$ be a net of means on $RUC(S)$. Then $\{\mu_\alpha \in A\}$ is said to be *asymptotically invariant* if for each $f \in RUC(S)$ and $s \in S$,

$$\mu_\alpha(f) - \mu_\alpha(\ell_s f) \to 0 \quad \text{and} \quad \mu_\alpha(f) - \mu_\alpha(r_s f) \to 0.$$

Let us give an example of asymptotically invariant nets. Let $S = \{0, 1, 2, \dots\}$. Then for $f = (x_0, x_1, \dots) \in B(S)$ and $n \in N$, the real valued function μ_n defined by

$$\mu_n(f) = \frac{1}{n} \sum_{k=0}^{n-1} x_k$$

is a mean. Furthermore, since for $f = (x_0, x_1, \dots) \in B(S)$ and $m \in N$

$$|\mu_n(f) - \mu_n(r_m f)| = \left| \frac{1}{n} \sum_{k=0}^{n-1} x_k - \frac{1}{n} \sum_{k=0}^{n-1} x_{k+m} \right| \le \frac{1}{n} \cdot 2m\|f\| \to 0,$$

as $n \to \infty$, $\{\mu_n\}$ is an asymptotically invariant net of means.

Let S be a semitopological semigroup and let C be a nonempty subset of a Banach space E. Then a family $\mathcal{S} = \{T_s : s \in S\}$ of mappings of C into itself is called a *nonexpansive semigroup* on C if it satisfies the following:

(i) $T_{st}x = T_sT_tx$ for all $s, t \in S$ and $x \in C$;
(ii) for each $x \in C$, the mapping $s \mapsto T_sx$ is continuous;
(iii) for each $s \in S$, T_s is a nonexpansive mapping of C into itself.

For a nonexpansive semigroup $\mathcal{S} = \{T_s : s \in S\}$ on C, we denote by $F(\mathcal{S})$ the set of common fixed points of $T_s, s \in S$. If C is a nonempty subset of a Hilbert space H and $\mathcal{S} = \{T_s : s \in S\}$ is a nonexpansive semigroup on C such that $\{T_sx : s \in S\}$ is bounded for some $x \in C$, then we know that for each $u \in C$ and $v \in H$, the functions $f(t) = \|T_tu - v\|^2$ and $g(t) = (T_tu, v)$ are in $RUC(S)$. Let μ be a mean on $RUC(S)$. Then since for each $x \in C$ and $y \in H$, the real valued function $t \mapsto (T_tx, y)$ is in $RUC(S)$, we can define the value $\mu_t(T_tx, y)$ of μ at this function. By linearity of μ and of the inner product, this is linear in y; moreover, since

$$|\mu_t(T_tx, y)| \leq \|\mu\| \cdot \sup_t |(T_tx, y)| \leq (\sup_t \|T_tx\|) \cdot \|y\|,$$

it is continuous in y. So, by the Riesz theorem, there exists an $x_0 \in H$ such that

$$\mu_t(T_tx, y) = (x_0, y)$$

for every $y \in H$. We denote such an x_0 by $T_\mu x$; see [32, 35] for more details.

We can now state a nonlinear ergodic theorem for noncommutative semigroups of nonexpansive mappings in a Hilbert space.

Theorem 3.3 [36]. *Let C be a nonempty subset of a Hilbert space H and let S be a semitopological semigroup such that $RUC(S)$ has an invariant mean. Let $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive semigroup on C such that $\{T_tx : t \in S\}$ is bounded and $\cap_{s\in S}\overline{\mathrm{co}}\{T_{st}x : t \in S\} \subset C$ for some $x \in C$. Then, $F(\mathcal{S}) \neq \phi$. Further, for an asymptotically invariant net $\{\mu_\alpha : \alpha \in A\}$ of means on $RUC(S)$, the net $\{T_{\mu_\alpha}x : \alpha \in A\}$ converges weakly to an element $x_0 \in F(\mathcal{S})$.*

Using Theorem 3.3, we have Theorem 3.1. By the same method, we can prove the following nonlinear ergodic theorems:

Theorem 3.4. *Let C be a closed convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself. If $F(T)$ is nonempty, then for each $x \in C$,*

$$S_r(x) = (1 - r)\sum_{k=0}^{\infty} r^kT^kx,$$

converges weakly to an element $y \in F(T)$ as $r \uparrow 1$.

Theorem 3.5. *Let C be a closed convex subset of a Hilbert space H and let $\mathcal{S} = \{S(t) : t \in [0, \infty)\}$ be a nonexpansive semigroup on C. If $F(\mathcal{S})$ is nonempty, then for each $x \in C$,*

$$S_\lambda(x) = \frac{1}{\lambda}\int_0^\lambda S(t)x\,dt,$$

converges weakly to an element $y \in F(\mathcal{S})$ as $\lambda \to \infty$.

Next, we state a nonlinear ergodic theorem for nonexpansive semigroups in a Banach space. Before stating it, we give a definition. A net $\{\mu_\alpha\}$ of continuous linear functionals on $RUC(S)$ is called *strongly regular* if it satisfies the following conditions:

(i) $\sup_\alpha \|\mu_\alpha\| < +\infty$;
(ii) $\lim_\alpha \mu_\alpha(1) = 1$;
(iii) $\lim_\alpha \|\mu_\alpha - r_s^*\mu_\alpha\| = 0$ for every $s \in S$.

Theorem 3.6 [8]. *Let S be a commutative semitopological semigroup and let E be a uniformly convex Banach space with a Fréchet differentiable norm. Let C be a nonempty closed convex subset of E and let $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S})$ is nonempty. Then there exists a unique nonexpansive retraction P of C onto $F(\mathcal{S})$ such that $PT_t = T_tP = P$ for every $t \in S$ and $Px \in \overline{co}\{T_tx : t \in S\}$ for every $x \in C$. Further, if $\{\mu_\alpha\}$ is a strongly regular net of continuous linear functionals on $RUC(S)$, then for each $x \in C$, $T_{\mu_\alpha}T_tx$ converges weakly to Px uniformly in $t \in S$.*

We didn't know whether Theorem 3.6 would hold in the case when S is noncommutative (cf. [37]). Recently, Lau, Shioji and Takahashi [16] solved the problem as follows:

Theorem 3.7 [16]. *Let C be a closed convex subset of a uniformly convex Banach space E, let S be a semitopological semigroup which $RUC(S)$ has an invariant mean, and let $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive semigroup on C with $F(\mathcal{S}) \neq 0$. Then there exists a nonexpansive retraction P from C onto $F(\mathcal{S})$ such that $PT_t = T_tP = P$ for each $t \in \mathcal{S}$ and $Px \in \overline{co}\{T_tx : t \in S\}$ for each $x \in C$.*

This is a generalization of Takahashi's result [32] for an amenable semigroup of nonexpansive mappings on a uniformly convex Banach space. Furthermore, they extended Rodé's result [26] to an amenable semigroup of nonexpansive mappings on a uniformly convex Banach space whose norm is Fréchet differentiable.

Theorem 3.8 [16]. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let S be a semitopological semigroup. Let C be a closed convex subset of E and let $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive semigroup on C with $F(\mathcal{S}) \neq \phi$. Suppose that $RUC(S)$ has an invariant mean. Then there exists a unique nonexpansive retraction P from C onto $F(\mathcal{S})$ such that $PT_t = T_tP = P$ for each $t \in S$ and $Px \in \overline{co}\{T_tx : t \in S\}$ for each $x \in C$. Further, if $\{\mu_\alpha\}$ is an asymptotically invariant net of means on X, then for each $x \in C$, $\{T_{\mu_\alpha}x\}$ converges weakly to Px.*

To prove Theorem 3.8, they used Theorem 3.7 and the following lemma which has been proved in Lau, Nishiura and Takahashi [15].

Lemma 3.9 [15]. *Let E be a uniformly convex Banach space with a Fréchet differentiable mean and let S be a semitopological semigroup. Let C be a closed convex subset of E and let $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive semigroup on C with $F(\mathcal{S}) \neq \phi$. Then, for each $x \in C$, $F(\mathcal{S}) \cap \bigcap_{s \in S} \overline{co}\{T_{ts}x : t \in S\}$ consists of at most one point.*

The following theorem has been proved in Takahashi [32] and Lau, Nishiura and Takahashi [15] when E is a Hilbert space.

Theorem 3.10 [16]. *Let E be a uniformly convex Banach space with a Fréchet differentiable mean and let S be a semitopological semigroup. Let C be a closed convex subset of E and let $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive semigroup on C with $F(\mathcal{S}) \neq \phi$. Suppose that for each $x \in C$, $F(\mathcal{S}) \cap \bigcap_{s \in S} \overline{co}\{T_{ts}x : t \in S\}$ is nonempty. Then there exists a nonexpansive retraction P from C onto $F(\mathcal{S})$ such that $PT_t = T_tP = P$ for each $t \in S$ and $Px \in \overline{co}\{T_tx : t \in S\}$ for each $x \in C$.*

4. Convergence theorems of Mann's type. Reich [24] discussed the iteration procedure introduced by Mann [20] in a uniformly convex Banach space whose norm is Fréchet differentiable and obtained the following

Theorem 4.1 [24]. *Let C be a closed convex subset of a uniformly convex Banach space E with a Fréchet differentiable norm, let $T : C \to C$ be a nonexpansive mapping with a fixed point, and let $\{c_n\}$ be a real sequence such that $0 \leq c_n \leq 1$ and $\Sigma_{n=1}^{\infty} c_n(1 - c_n) = \infty$. If $x_1 \in C$ and*

$$x_{n+1} = c_nTx_n + (1 - c_n)x_n \quad \text{for } n \geq 1,$$

then $\{x_n\}$ converges weakly to a fixed point of T.

This theorem has been known for those uniformly convex Banach spaces that satisfy Opial's condition (cf. [7]). Tan and Xu [44] proved the following interesting result which generalizes the result of Reich [24].

Theorem 4.2 [44]. *Let C be a closed convex subset of a uniformly convex Banach space E which satisfies Opial's condition, or whose norm is Fréchet differentiable and let $T : C \to C$ be a nonexpansive mapping with a fixed point. Then for any initial data x_1 in C, the iterates $\{x_n\}$ defined by*

$$x_{n+1} = \alpha_n T[\beta_n T x_n + (1 - \beta_n)x_n] + (1 - \alpha_n)x_n \quad \text{for } n \geq 1,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are chosen so that $\Sigma_{n=1}^{\infty}\alpha_n(1 - \alpha_n) = \infty$, $\Sigma_{n=1}^{\infty}\beta_n(1 - \alpha_n) < \infty$, $\limsup_{n\to\infty}\beta_n < 1$, converge weakly to a fixed point of T.

To prove Theorem 4.2, Tan and Xu [44] used the following two lemmas.

Lemma 4.3 [24], [40]. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E with a Fréchet differentiable norm and let $\{T_1, T_2, T_3, \dots\}$ be a sequence of nonexpansive mappings of C into C such that $\cap_{n=1}^{\infty} F(T_n)$ is nonempty. Let $x \in C$ and put $S_n = T_n T_{n-1} \dots T_1$ for $n \geq 1$. Then, the set $U \cap \cap_{n=1}^{\infty} \overline{co}\{S_m x : m \geq n\}$ consists of at most one point, where $U = \cap_{n=1}^{\infty} F(T_n)$.*

Lemma 4.4 [27]. *Let E be a uniformly convex Banach space, let $\{t_n\}$ be a real sequence such that $0 < b \leq t_n \leq c < 1$ for $n \geq 1$ and let $a \geq 0$. Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences of E such that $\limsup_{n\to\infty}\|x_n\| \leq a$, $\limsup_{n\to\infty}\|y_n\| \leq a$ and $\lim_{n\to\infty}\|t_n x_n + (1 - t_n)y_n\| = a$. Then $\lim_{n\to\infty}\|x_n - y_n\| = 0$.*

Takahashi and Kim [40] also proved the following

Theorem 4.5 [40]. *Let E be a uniformly convex Banach space E which satisfies Opial's condition, or whose norm is Fréchet differentiable, let C be a nonempty closed convex subset of E, and let $T : C \to C$ be a nonexpansive mapping with a fixed point. Suppose $x_1 \in C$, and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n T[\beta_n T x_n + (1 - \beta_n)x_n] + (1 - \alpha_n)x_n \quad \text{for all } n \geq 1,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are chosen so that $\alpha_n \in [a, b]$ and $\beta_n \in [0, b]$ or $\alpha_n \in [a, 1]$ and $\beta_n \in [a, b]$ for some a, b with $0 < a \leq b < 1$. Then $\{x_n\}$ converges weakly to a fixed point of T.

Motivated by Theorems 3.2 and 3.5, Suzuki and Takahashi [29] obtained the following theorem:

Theorem 4.6 [29]. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E which satisfies Opial's condition, or whose norm is Fréchet differentiable. Let T be a nonexpansive mapping from C into itself with a fixed point. Suppose that $\{x_n\}$ is given by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n T[\beta_n T x_n + (1-\beta_n)x_n] + (1-\alpha_n)x_n \quad \text{for all } n \geq 1,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ such that $\Sigma_{n=1}^{\infty}\alpha_n(1-\alpha_n) = \infty$ and $\limsup_{n\to\infty}\beta_n < 1$, or $\Sigma_{n=1}^{\infty}\alpha_n\beta_n = \infty$ and $\limsup_{n\to\infty}\beta_n < 1$. Then $\{x_n\}$ converges weakly to a fixed point of T.

To prove Theorem 4.6, Suzuki and Takahashi [29] used the following two lemmas. Before stating them, we give two notations. We denote by $\mathbf{N}$ the set of positive integers. Let I be an infinite subset of $\mathbf{N}$. If $\{\lambda_n\}$ is a sequence of nonnegative numbers, then we denote by $\{\lambda_i : i \in I\}$ the subsequence of $\{\lambda_n\}$.

Lemma 4.7 [29]. *Let $\{\lambda_n\}$ and $\{\mu_n\}$ be sequences of nonnegative numbers such that $\Sigma_{n=1}^{\infty}\lambda_n = \infty$ and $\Sigma_{n=1}^{\infty}\lambda_n\mu_n < \infty$. Then for $\varepsilon > 0$, there exists an infinite subset I of N such that $\Sigma\{\lambda_j : j \in \mathrm{N}\backslash I\} \leq \varepsilon$ and the subsequence $\{\mu_i : i \in I\}$ of $\{\mu_n\}$ converges to 0.*

Lemma 4.8 [29]. *Let $\{\lambda_n\}$ and $\{\mu_n\}$ be sequences of nonnegative numbers such that $\lambda_{n+1} \leq \lambda_n + \mu_n$ for all $n \in \mathrm{N}$. Suppose there exists a subsequence $\{\mu_i : i \in I\}$ of $\{\mu_n\}$ such that $\mu_i \to 0$, $\lambda_i \to \alpha$ and $\Sigma\{\mu_j : j \in \mathrm{N}\backslash I\} < \infty$. Then $\lambda_n \to \alpha$.*

Compare Theorem 4.6 with Theorem 4.2 of Tan and Xu [44]. This indicates that the assumption $\Sigma_{n=1}^{\infty}\beta_n(1-\alpha_n) < \infty$ in Theorem 4.2 is superfluous. We do not know whether the assumptions $\Sigma_{n=1}^{\infty}\alpha_n\beta_n = \infty$ and $\limsup_{n\to\infty}\beta_n < 1$ in Theorem 4.6 are replaced by $\Sigma_{n=1}^{\infty}\beta_n(1-\beta_n) = \infty$ and $\liminf_{n\to\infty}\alpha_n > 0$. We also know the following strong convergence theorem which is connected with Rhoades [25], Tan and Xu [44], and Takahashi and Kim [40].

Theorem 4.9 [28]. *Let E be a strictly convex Banach space, let C be a nonempty closed convex subset of E, and let $T : C \to C$ be a nonexpansive mapping for which $T(C)$ is contained in a compact subset of C. Suppose $x_1 \in C$, and $\{x_n\} \subset C$ is given by*

$$x_{n+1} = \alpha_n T[\beta_n T x_n + (1-\beta_n)x_n] + (1-\alpha_n)x_n \quad \text{for} \quad n \geq 1,$$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are chosen so that* $\Sigma_{n=1}^{\infty}\alpha_n(1-\alpha_n)=\infty$ *and* $\limsup_{n\to\infty}\beta_n<1$, *or* $\Sigma_{n=1}^{\infty}\beta_n(1-\beta_n)=\infty$ *and* $\liminf_{n\to\infty}\alpha_n>0$. *Then* $\{x_n\}$ *converges strongly to a fixed point of* T.

Let C be a closed convex subset of a Banach space E, and let T,S be selfmaps on C. Then Das and Debata [5] considered the following iteration scheme: $x_1\in C$, and

$$x_{n+1}=\alpha_n S[\beta_n Tx_n+(1-\beta_n)x_n]+(1-\alpha_n)x_n \quad \text{for } n\geq 1,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0,1]$. They proved a strong convergence theorem concerning Roades' result [25]. Takahashi and Tamura [43] obtained the following weak convergence theorem.

Theorem 4.10 [43]. *Let* E *be a uniformly convex Banach space* E *which satisfies Opial's condition, or whose norm is Fréchet differentiable, let* C *be a nonempty closed convex subset of* E, *and let* $S,T:C\to C$ *be nonexpansive mappings such that* $F(S)\cap F(T)$ *is nonempty. Suppose* $x_1\in C$, *and* $\{x_n\}$ *is given by*

$$x_{n+1}=\alpha_n S[\beta_n Tx_n+(1-\beta_n)x_n]+(1-\alpha_n)x_n \quad \text{for } n\geq 1,$$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are chosen so that* $\alpha_n,\beta_n\in[a,b]$ *for some* a,b *with* $0<a\leq b<1$. *Then* $\{x_n\}$ *converges weakly to a common fixed point of* S *and* T.

Further, Takahashi and Tamura [43] obtained the following

Theorem 4.11 [43]. *Let* C *be a nonempty closed convex subset of a uniformly convex Banach space* E, *and let* $S,T:C\to C$ *be nonexpansive mappings such that* $F(S)\cap F(T)$ *is nonempty. Let* P *be the metric projection of* E *onto* $F(S)\cap F(T)$, *suppose* $x_1\in C$ *and* $\{x_n\}$ *is given by*

$$x_{n+1}=\alpha_n S[\beta_n Tx_n+(1-\beta_n)x_n]+(1-\alpha_n)x_n \quad \text{for } n\geq 1,$$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are real sequences in* $[0,1]$. *Then* $\{Px_n\}$ *converges strongly to a common fixed point of* S *and* T.

To apply convergence theorems of Mann's type to the problem of image recovery, we need to extend Theorem 4.10 to the following

Theorem 4.12 [38]. *Let E be a uniformly convex Banach space E which satisfies Opial's condition, or whose norm is Fréchet differentiable, let C be a nonempty closed convex subset of E, and let $\{T_1, T_2, \dots, T_m\}$ be finite nonexpansive mappings of C into itself such that $\cap_{n=1}^m F(T_n)$ is nonempty. Let a, b be real numbers with $0 < a \le b < 1$ and suppose $x_1 \in C$, and $\{x_n\}$ is given by $x_{n+1} = U_{nm} x_n$ for $n \ge 1$, where*

$$\begin{aligned} U_{n1} x_n &= \alpha_{n1} T_1 x_n + (1 - \alpha_{n1}) x_n, \\ U_{n2} x_n &= \alpha_{n2} T_2 U_{n1} x_n + (1 - \alpha_{n2}) x_n, \\ &\vdots \\ U_{nm-1} x_n &= \alpha_{nm-1} T_{m-1} U_{nm-2} x_n + (1 - \alpha_{nm-1}) x_n, \\ U_{nm} x_n &= \alpha_{nm} T_m U_{nm-1} x_n + (1 - \alpha_{nm}) x_n, \end{aligned}$$

and $\alpha_{n1}, \alpha_{n2}, \dots, \alpha_{nm} \in [a, b]$ for $n \ge 1$. Then $\{x_n\}$ converges weakly to a common fixed point of $T_1, T_2, \dots, T_m$.

5. Applications.

In this section, we first deal with weak convergence theorems which are connected with the problem of image recovery in a Banach space setting. Using Theorem 3.9, or Lemma 4.3, we can prove the following result.

Theorem 5.1 [14]. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm, and let C be a nonempty closed convex subset of E. Let T be an asymptotically regular nonexpansive mapping of C into itself with $F(T) \neq \phi$. Then, for each $x \in C$, $\{T^n x\}$ converges weakly to an element of $F(T)$.*

We can also prove the following lemma by using Edelstein and O'Brien [6].

Lemma 5.2. *Let E be a Banach space and let C be a nonempty convex subset of E. Let S be a mapping on C given by $S = \beta_0 I + \sum_{i=1}^r \beta_i S_i$, $0 < \beta_i < 1$, $i = 0, 1, \dots, r$, $\sum_{i=0}^r \beta_i = 1$, such that each S_i is nonexpansive on C and $\cap_{i=1}^r F(S_i)$ is nonempty. Then S is asymptotically regular on C.*

Using Theorem 5.1 and Lemma 5.2, we can prove the first weak convergence theorem for nonexpansive mappings given by convex combinations of retractions.

Theorem 5.3 [42]. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let C be a nonempty convex subset of E. Let $C_1, C_2, \dots, C_r$ be nonexpansive retracts of C such that $\cap_{i=1}^r C_i \neq \phi$. Let T be a mapping on C given by $T = \Sigma_{i=1}^r \alpha_i T_i$, $0 < \alpha_i < 1$, $\Sigma_{i=1}^r \alpha_i = 1$, such that for each i, $T_i = (1-\lambda_i)I + \lambda_i P_i$, $0 < \lambda_i < 1$, where P_i is a nonexpansive retraction of C onto C_i. Then, $F(T) = \cap_{i=1}^r C_i$ and further, for each $x \in C$, $\{T^n x\}$ converges weakly to an element of $\cap_{i=1}^r C_i$.*

In this section, we also consider the problem of image recovery to the situation where the constraints are inconsistent.

Theorem 5.4 [42]. *Let E be a reflexive Banach space and let C be a nonempty closed convex subset of E which has normal structure. Let $C_1, C_2, \dots, C_r$ be nonempty bounded nonexpansive retracts of C. Let T be a mapping on C given by $T = \Sigma_{i=1}^r \alpha_i T_i$, $0 < \alpha_i < 1$, $i = 1, \dots, r$, $\Sigma_{i=1}^r \alpha_i = 1$, such that for each i, $T_i = (1-\lambda_i)I + \lambda_i P_i$, $0 < \lambda_i < 1$, where P_i is a nonexpansive retraction of C onto C_i. Then $F(T)$ is nonempty. Further, assume that E is strictly convex and $\cap_{i=1}^r C_i = \phi$. Then $F(T) \cap C_i = \phi$ for some i.*

Let C and D be nonempty convex subsets of a Banach space E. Then we denote by $i_C D$ the set of $z \in D$ such that for any $x \in C$, there exists $\lambda \in (0,1)$ with $\lambda x + (1-\lambda)z \in D$ and by $\partial_C D$ the set of $z \in D$ such that there exists $x \in C$ with $\lambda x + (1-\lambda)z \notin D$ for all $\lambda \in (0,1)$.

Theorem 5.5 [42]. *Let E be a strictly convex and reflexive Banach space and let C be a nonempty closed convex subset of E which has normal structure. Let $C_1, C_2, \dots, C_r$ be nonempty bounded sunny nonexpansive retracts of C such that for each i, an element of $\partial_C C_i$ is an extreme point of C_i. Let T be a mapping on C given by $T = \Sigma_{i=1}^r \alpha_i T_i$, $0 < \alpha_i < 1$, $i = 1, \dots, r$, $\Sigma_{i=1}^r \alpha_i = 1$, such that for each i, $T_i = (1-\lambda_i)I + \lambda_i P_i$, $0 < \lambda_i < 1$, where P_i is a sunny nonexpansive retraction of C onto C_i. If $\cap_{i=1}^r C_i$ is empty, then $F(T)$ consists of one point.*

Finally, we discuss the problem of image recovery by using Theorem 4.12. Before doing it, we give a definition. Let C be a closed convex subset of a Banach space E. Let $T_1, T_2, \dots, T_r$ be finite mappings of C into itself and let $\alpha_1, \alpha_2, \dots, \alpha_r$ be real numbers such that $0 \leq \alpha_i \leq 1$ for every $i = 1, 2, \dots, r$. Then, we define a mapping W of C into itself as follows:

$$U_1 = \alpha_1 T_1 + (1-\alpha_1)I,$$
$$U_1 = \alpha_2 T_2 U_1 + (1-\alpha_2)I,$$

$$U_{r-1} = \alpha_{r-1}T_{r-1}U_{r-2} + (1-\alpha_{r-1})I,$$
$$W = \alpha_r T_r U_{r-1} + (1-\alpha_r)I.$$

Such a mapping W is called the *W-mapping* generated by $T_1, T_2, \dots, T_r$ and $\alpha_1, \alpha_2, \dots, \alpha_r$.

Lemma 5.6 [38]]. *Let C be a closed convex subset of a Banach space E. Let $T_1, T_2, \dots, T_r$ be nonexpansive mappings of C into itself such that $\cap_{i=1}^r F(T_i)$ is nonempty and let $\alpha_1, \alpha_2, \dots, \alpha_r$ be real numbers such that $0 < \alpha_i < 1$ for every $i = 1, 2, \dots, r$. Let W be the W-mapping of C into itself generated by $T_1, T_2, \dots, T_r$ and $\alpha_1, \alpha_2, \dots, \alpha_r$. Then W is asymptotically regular. Further, if E is strictly convex, $F(W) = \cap_{i=1}^r F(T_i)$.*

Using Lemma 5.6 and Theorem 5.1, we have the following theorem.

Theorem 5.7 [38]. *Let E be a uniformly convex Banach space E with a Fréchet differentiable norm and let C be a closed convex subset of E. Let $T_1, T_2, \dots, T_r$ be nonexpansive mappings of C into itself and let $\alpha_1, \alpha_2, \dots, \alpha_r$ be real numbers such that $0 < \alpha_i < 1$ for every $i = 1, 2, \dots, r$. Let W be the W-mapping of C into itself generated by $T_1, T_2, \dots, T_r$ and $\alpha_1, \alpha_2, \dots, \alpha_r$. Then, for each $x \in C$, $\{W^n x\}$ converges weakly to an element of $\bigcap_{i=1}^r F(T_i)$.*

As a direct consequence of Theorem 5.7, we have the following theorem.

Theorem 5.8 [38]. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm. Let $C_1, C_2, \dots, C_r$ be nonexpansive retracts of C such that $\cap_{i=1}^r C_i \neq \phi$. Let W be the W-mapping of C into itself generated by $P_1, P_2, \dots, P_r$ and $\alpha_1, \alpha_2, \dots, \alpha_r$, where P_i is a nonexpansive retraction of C onto C_i and $0 < \alpha_i < 1$ for every $i = 1, 2, \dots, r$. Then for each $x \in C$, $\{W^n x\}$ converges weakly to an element of $\cap_{i=1}^r C_i$.*

References

[1] Baillon, J. B., *Un théorème de type ergodic pour les contractions non linéaires dans un espace de Hilbert*, C. R. Acad. Sci. Paris **280** (1976), 1511-1514.

[2] ———, *Quelques properiétés de convergence asymptotique pour les semigroupes de contractions impaires*, C. R. Acad. Sci. Paris **283** (1976), 75-78.

[3] Bruck, R. E., *A simple proof of the mean ergodic theorem for nonlinear contractions in Banach spaces*, Israel J. Math. **32** (1979), 107-116.

[4] Crombez, G., *Image recovery by convex combinations of projections*, J. Math. Anal. Appl. **155** (1991), 413-419.

[5] Das, G. and J. P. Debata, *Fixed points of quasinonexpansive mappings*, Indian J. Pure Appl. Math. **17** (1986), 1263-1269.

[6] Edelstein, M. and R. C. O'Brien, *Nonexpansive mappings, asymptotic regularity and successive approximations*, J. London Math. Soc. **17** (1978), 547-554.

[7] Groetsch, C. W., *A note on segmenting Mann iterates*, J. Math. Anal. Appl. **40** (1972), 369-372.

[8] Hirano, N., K. Kido and W. Takahashi, *Nonexpansive retractions and nonlinear ergodic theorems in Banach spaces*, Nonlinear Anal. **12** (1988), 1269-1281.

[9] Ishikawa, S., *Fixed points and iteration of a nonexpansive mapping in a Banach space*, Proc. Amer. Math. Soc. **59** (1976), 65-71.

[10] Kada, O. and W. Takahashi, *Nonlinear ergodic theorems of almost nonexpansive curves for commutative semigroups*, Topol. Methods Nonlinear Anal. **5** (1995), 305-324.

[11] ______, *Strong convergence and nonlinear ergodic theorems for commutative semigroups of nonexpansive mappings*, Nonlinear Anal. **28** (1997), 495-511.

[12] Kada, O., A. T. Lau and W. Takahashi, *Asymptotically invariant net and fixed point set for semigroup of nonexpansive mappings*, Nonlinear Anal. **29** (1997), 539-550.

[13] Kirk, W. A., *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Monthly **72** (1965), 1004-1006.

[14] Kitahara, S. and W. Takahashi, *Image recovery by convex combinations of sunny nonexpansive retractions*, Topol. Methods Nonlinear Anal. **2** (1993), 333-342.

[15] Lau, A. T., K. Nishiura and W. Takahashi, *Nonlinear ergodic theorems for semigroups of nonexpansive mappings and left ideals*, Nonlinear Anal. **26** (1996), 1411-1427.

[16] Lau, A. T., N. Shioji and W. Takahashi, *Existence of nonexpansive retractions for amenable semigroups of nonexpansive mappings and nonlinear ergodic theorems in Banach spaces*, to appear.

[17] Lau, A. T. and W. Takahashi, *Weak convergence and non-linear ergodic theorems for reversible semigroups of nonexpansive mappings*, Pacific J. Math. **126** (1987), 277-294.

[18] ______, *Invariant means and fixed point properties for non-expansive representations of topological semigroups*, Topol. Methods Nonlinear Anal. **5** (1995), 39-57.

[19] ______, *Invariant submeans and semigroups of nonexpansive mappings on Banach spaces with normal structure*, J. Funct. Anal. **142** (1996), 79-88.

[20] Mann, W. R., *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506-510.

[21] Mitchell, T., *Topological semigroups and fixed points*, Illinois J. Math. **14** (1970), 630-641.

[22] Mizoguchi, N. and W. Takahashi, *On the existence of fixed points and ergodic retractions for Lipschitzian semigroups in Hilbert spaces*, Nonlinear Anal. **14** (1990), 69-80.

[23] Opial, Z., *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc. **73** (1967), 591-597.

[24] Reich, S., *Weak convergence theorems for nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl. **67** (1979), 274-276.

[25] Rhoades, B. E., *Comments on two fixed point iteration methods*, J. Math. Anal. Appl. **56** (1976), 741-750.

[26] Rodé, G., *An ergodic theorem for semigroups of nonexpansive mappings in a Hilbert space*, J. Math. Anal. Appl. **85** (1982), 172-178.

[27] Schu, J., *Weak and strong convergence to fixed points of asymptotically nonexpansive mappings*, Bull. Austral. Math. Soc. **43** (1991), 153-159.

[28] Shioji, N., N. Tsukiyama and W. Takahashi, *A strong convergence theorem for nonexpansive mappings by the Ishikawa iteration process*, to appear.

[29] Suzuki, T. and W. Takahashi, *On weak convergence to fixed points of nonexpansive mappings in Banach spaces*, to appear.

[30] Takahashi, W., *Fixed point theorem for amenable semigroups of non-expansive mappings*, Kōdai Math. **21** (1969), 383-386.

[31] ______, *Recent results in fixed point theory*, SEA Bull. Math., 4 (1981), 59-85.

[32] ______, *A nonlinear ergodic theorem for an amenable semigroup of nonexpansive mappings in a Hilbert space*, Proc. Amer. Math. Soc. **81** (1981), 253-256.

[33] ______, *Fixed point theorems for families of nonexpansive mappings on unbounded sets*, J. Math. Soc. Japon. **36** (1984), 543-553.

[34] ______, *A nonlinear ergodic theorem for a reversible semigroup of nonexpansive mappings in a Hilbert space*, Proc. Amer. Math. Soc. **96** (1986), 55-58.

[35] ______, *Nonlinear Functional Analysis*, (Japanese), Kindai-kagakusha, Tokyo 1988.

[36] ______, *Fixed point theorem and nonlinear ergodic theorem for nonexpansive semigroups without convexity*, Canadian J. Math. **44** (1992), 880-887.

[37] ______, *Fixed point theorems and nonlinear ergodic theorems for nonlinear semigroups and their applications*, Proceedings of the Second World Congress on Nonlinear Analysts, Athens, Greece, 1996(to appear).

[38] ______, *Convergence theorems for nonexpansive mappings and feasibility problems*, (to appear).

[39] Takahashi, W. and D. H. Jeong, *Fixed Point theorem for nonexpansive semigroups on Banach space*, Proc. Amer. Math. Soc. **122** (1994), 1175-1179.

[40] Takahashi, W. and G. E. Kim, *Approximating fixed points of nonexpansive mappings in Banach spaces*, Math. Japon. (to appear).

[41] Takahashi, W. and J. Y. Park, *On the asymptotic behavior of almost orbits of commutative semigroups in Banach spaces*, Nonlinear and Convex Analysis (B. L. Lin and S. Simons, Eds.), Lecture Notes in Pure and Appl. Math., Marcel Dekker, Inc., New York, 1987, 271-293.

[42] Takahashi, W. and T. Tamura, *Limit theorems of operators by convex combinations of nonexpansive retractions in Banach spaces*, J. Approx. Theory (to appear).

[43] Takahashi, W. and T. Tamura, *Convergence theorems for a pair of nonexpansive mappings*, J. Convex Analysis (to appear).

[44] Tan, K. K. and H. K. Xu, *Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process*, J. Math. Anal. Appl. **178** (1993), 301-308.

Department of Mathematical
and Computing Sciences
Tokyo Institute of Technology
Ohokayama, Meguro-ku, Tokyo 152, JAPAN
e-mail: wataru@is.titech.ac.jp

received September 29, 1997