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Strong Convergence Theorems for Nonexpansive Mappings and Nonexpansive Semigroups

ABSTRACT. Let C be a closed, convex subset of a Banach space, let T be a nonexpansive mapping from C into itself such that the set $F(T)$ of fixed points of T is nonempty and let $\{T(t) : t \geq 0\}$ be a nonexpansive semigroup on C such that the set $\bigcap_{t \geq 0} F(T(t))$ of common fixed points of $\{T(t) : t \geq 0\}$ is nonempty. Let x be an element of C. In this paper, we study strong convergence theorems of sequences generated by x and T. We also study strong convergence theorems of sequences generated by x and $\{T(t) : t \geq 0\}$.

1. Introduction. In 1975 Baillon [2] established the first nonlinear ergodic theorem in a Hilbert space. Bruck [6], [7] extended it as follows:

Theorem A (Bruck). *Let C be a closed, convex subset of a Banach space E and let T be a nonexpansive mapping from C into itself such that the set $F(T)$ of fixed points of T is nonempty. If E is uniformly convex and the norm of E is Fréchet differentiable, then for each $x \in C$, $\{1/(n+1)\sum_{i=0}^{n} T^i x\}$ converges weakly to an element of $F(T)$.*

If $C = E$ and T is linear, $\{1/(n+1)\sum_{i=0}^{n} T^i x\}$ converges strongly. So it has been a problem whether there is a natural strong convergence theorem

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which generalizes linear ergodic theorem. As a strong convergence theorem for a nonexpansive mapping is concerned the following is well known, see [15], [32]:

Theorem B (Reich, Takahashi and Ueda). *Let C, E and T be as in Theorem A. Assume that E is uniformly smooth or that E is uniformly convex Banach space and the norm of E is uniformly Gâteaux differentiable. Then there exists a sunny, nonexpansive retraction P from C onto $F(T)$. Moreover, let $\{a_n\}$ be a real sequence such that $0 < a_n \leq 1$ and $a_n \to 0$. Let x be an element of C and let $\{x_n\}$ be the sequence defined by*

$$x_n = a_n x + (1 - a_n) T x_n \qquad \text{for each } n \in \mathbf{N}.$$

Then $\{x_n\}$ converges strongly to Px.

Since the sequence $\{x_n\}$ converges to a fixed point of T, Halpern [8] and Reich [16] considered the iteration process

$$\text{(1.1)} \qquad y_0 \in C, \quad y_{n+1} = b_n x + (1 - b_n) T y_n \qquad \text{for each } n \in \mathbf{N},$$

and Reich [16] posed the following problem:

Problem (Reich). *Let E be a Banach space. Is there a sequence $\{b_n\}$ such that whenever a weakly compact, convex subset C of E possesses the fixed point property for nonexpansive mappings, then the sequence $\{y_n\}$ defined by (1.1) converges to a fixed point of T for all x in C and all nonexpansive $T : C \to C$?*

On the other hand, Miyadera and Kobayasi [13] obtained the following convergence theorem for a family of nonexpansive mappings:

Theorem C (Miyadera and Kobayasi). *Let C be a closed, convex subset of a uniformly convex Banach space E whose norm is Fréchet differentiable. Let $\{T(t) : t \geq 0\}$ be a nonexpansive semigroup on C such that the set $\bigcap_{t \geq 0} F(T(t))$ of common fixed points of $\{T(t) : t \geq 0\}$ is nonempty. Then for each $x \in C$, $\{1/t \int_0^t T(t)x\,dt\}$ converges weakly to an element of $\bigcap_{t \geq 0} F(T(t))$.*

Generally, $\{1/t \int_0^t T(t)x\,dt\}$ does not converge strongly. So it also has been a problem whether there is a natural iteration process which converges strongly to an element of $\bigcap_{t \geq 0} F(T(t))$.

In this paper we study strong convergence theorems for a nonexpansive mapping and a nonexpansive semigroup. First, we give an answer to Reich's problem which extends Wittmann's result in [33]. Next, using Shimizu and Takahashi's ideas in [19], [20] and the methods employed in the study of nonlinear ergodic theorems [10], [11], [18], [29], 30], we show strong convergence theorems for a nonexpansive semigroup which extend Shimizu and Takahashi's results in [19], [20].

This paper is organized as follows: Section 2 is devoted to some preliminaries. In Section 3 we show our strong convergence theorems and we investigate some corollaries which can be deduced from our results. In the final section we prove our results.

2. Preliminaries. Throughout this paper all vector spaces are real and we denote by $\mathbf{N}$ the set of all nonnegative integers.

Let E be a Banach space, let C be a subset of E and let T be a mapping from C into itself. We denote by $\overline{\text{co}}\, C$ the closed, convex hull of C, and we denote by $F(T)$ the set $\{x \in C : x = Tx\}$. T is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for each $x, y \in C$.

For $r > 0$ we denote by B_r the closed ball in E with center 0 and radius r. E is said to be *uniformly convex* if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\|(x + y)/2\| \leq 1 - \delta$ for each $x, y \in B_1$ with $\|x - y\| \geq \varepsilon$. We know that E is uniformly convex if and only if the function $x \mapsto \|x\|^2$ is *uniformly convex* on each bounded subset of E, i.e., for each $r > 0$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $\|(x+y)/2\|^2 \leq (\|x\|^2 + \|y\|^2)/2 - \delta$ for each $x, y \in B_r$ with $\|x - y\| \geq \varepsilon$; see [28], [34].

Bruck [7] obtained the following nice properties for a nonexpansive mapping in a uniformly convex Banach space:

Proposition (Bruck). *Let D be a bounded, closed, convex subset of a uniformly convex Banach space. Let $N(D)$ be the set of all nonexpansive mappings from D into itself, and for each $\eta > 0$ and $T \in N(D)$ let $F_\eta(T)$ be the set $\{x \in D : \|Tx - x\| \leq \eta\}$. Then*

(i) *for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\overline{\text{co}}\, F_\delta(T) \subset F_\varepsilon(T)$ for all $T \in N(D)$;*

(ii) $$\lim_{n\to\infty} \sup_{\substack{y\in D\\ T\in N(D)}} \left\| \frac{1}{n+1}\sum_{i=0}^{n} T^i y - T\left(\frac{1}{n+1}\sum_{i=0}^{n} T^i y\right)\right\| = 0.$$

Let E^* be the topological dual of E. The value of $x^* \in E^*$ at $x \in E$ will be denoted by $\langle x, x^* \rangle$. We also denote by J the *duality mapping* from

E into 2^{E^*}, i.e., $Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$ for each $x \in E$. We know that $2J$ is the *subdifferential* of the mapping $x \mapsto \|x\|^2$, i.e., $\|y\|^2 \geq \|x\|^2 + 2\langle y - x, Jx\rangle$ for each $x, y \in E$. Let $U = \{x \in E : \|x\| = 1\}$. E is said to be *smooth* if for each $x, y \in U$ the limit

$$(2.1) \qquad \lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists. The norm of E is said to be *uniformly Gâteaux differentiable* if for each $y \in U$ the limit (2.1) exists uniformly for $x \in U$. The norm of E is said to be *Fréchet differentiable* if for each $x \in U$ the limit (2.1) exists uniformly for $y \in U$. E is said to be *uniformly smooth* if the limit (2.1) exists uniformly for $x, y \in U$. We know that if E is smooth then the duality mapping is single-valued and norm to weak star continuous and that if the norm of E is uniformly Gâteaux differentiable then the duality mapping is norm to weak star uniformly continuous on each bounded subset of E.

Let C be a convex subset of E, let K be a nonempty subset of C and let P be a retraction from C onto K, i.e., $Px = x$ for each $x \in K$. P is said to be *sunny* if $P(Px + t(x - Px)) = Px$ for each $x \in C$ and $t \geq 0$ with $Px + t(x - Px) \in C$. We know from [5, Theorem 3] or [14, Lemma 2.7] that if E is smooth, then a retraction P from C onto K is sunny and nonexpansive if and only if

$$\langle x - Px, J(y - Px)\rangle \leq 0 \quad \text{for each } x \in C \text{ and } y \in K$$

and hence there is at most one sunny, nonexpansive retraction from C onto K. We know that in the case when E is a Hilbert space and K is a convex subset of C, P is a sunny, nonexpansive retraction if and only if P is a *metric projection*, i.e., $\|x - Px\| = \min_{y \in K} \|x - y\|$ for each $x \in C$.

Let S be a semigroup. Let $B(S)$ be the space of all bounded real-valued functions defined on S with supremum norm. For $s \in S$ and $f \in B(S)$ we define an element $l_s f$ in $B(S)$ by

$$(l_s f)(t) = f(st) \qquad \text{for each } t \in S.$$

Let X be a subspace of $B(S)$ containing 1 and let X^* be its topological dual. An element μ of X^* is said to be a *mean* on X if $\|\mu\| = \mu(1) = 1$. We know that μ is a mean on X if and only if $\inf f(S) \leq \mu(f) \leq \sup f(S)$ for each $f \in X$. We often write $\mu_t(f(t))$ instead of $\mu(f)$ for $\mu \in X^*$ and $f \in X$. Let X be l_s-invariant, i.e., $l_s(X) \subset X$ for each $s \in S$. A mean μ on X is said to be *left invariant* if $\mu(l_s f) = \mu(f)$ for each $s \in S$ and $f \in X$. A sequence of means $\{\mu_n\}$ on X is said to be *strongly left regular* if $\|\mu_n - l_s^* \mu_n\| \to 0$ for each $s \in S$, where l_s^* is the adjoint operator of l_s. In the case when S is commutative, a left invariant mean is said to be an *invariant mean* and a strongly left regular sequence is said to be a *strongly regular* sequence [10], [12]. We call an invariant mean on $B(\mathrm{N})$ a *Banach limit* [3].

Let E be a reflexive Banach space, let X be a subspace of $B(S)$ containing 1 and let μ be a mean on X. Let f be a function from S into E such that $f(S)$ is bounded and the mapping $t \mapsto \langle f(t), x^*\rangle$ is an element of X for each $x^* \in E^*$. It is easy to see that there exists a unique element $x_0 \in E$ such that $\langle x_0, x^*\rangle = \mu_t\langle f(t), x^*\rangle$ for each $x^* \in E^*$. We remark that this definition is like that of Pettis integral; see [9]. Following [10], we denote such x_0 by $\int f(t)\, d\mu(t)$.

3. Strong convergence theorems.

First, we give an answer to Reich's problem as in [21]. In the case when E is a Hilbert space, this result was obtained by Wittmann in [33].

Theorem 1 (Shioji and Takahashi). *Let C be a closed, convex subset of a Banach space E. Assume that E is uniformly smooth or that E is uniformly convex and the norm of E is uniformly Gâteaux differentiable. Let T be a nonexpansive mapping from C into itself such that $F(T)$ is nonempty and let P be the sunny, nonexpansive retraction from C onto $F(T)$. Let $\{b_n\}$ be a real sequence satisfying $0 \le b_n \le 1$, $b_n \to 0$, $\sum_{n=0}^{\infty} b_n = \infty$ and $\sum_{n=0}^{\infty} |b_{n+1} - b_n| < \infty$. Let x be an element of C and let $\{y_n\}$ be the defined by (1.1). Then $\{y_n\}$ converges strongly to Px.*

If T is linear, $b_n = 1/(n+2)$ and $y_0 = x$, then y_n defined by (1.1) is exactly $1/(n+1)\sum_{i=0}^{n} T^i x$. So this theorem is a natural generalization of a linear ergodic theorem.

We next show strong convergence theorems for a nonexpansive semigroup. Before that, we state definitions of a nonexpansive semigroup and an operator T_μ.

Let S be a semigroup and let C be a closed, convex subset of a reflexive Banach space E. A family $\{T_t \colon t \in S\}$ is said to be a *nonexpansive semigroup on* C if T_t is a nonexpansive mapping from C into itself and $T_{ts} = T_t T_s$ for each $t, s \in S$. Let $\{T_t : t \in S\}$ be a nonexpansive semigroup on C such that $\{T_t u : t \in S\}$ is bounded for some $u \in C$ and let X be a subspace of $B(S)$ such that $1 \in X$ and the mapping $t \mapsto \langle T_t x, x^*\rangle$ is an element of X for each $x \in C$ and $x^* \in E^*$. Following [Rode:ergodic], we also write $T_\mu x$ instead of $\int T_t x\, d\mu(t)$ for a mean μ on X and $x \in C$, i.e., $T_\mu x$ is an element of C satisfying $\langle T_\mu x, x^*\rangle = \mu_t\langle T_t x, x^*\rangle$ for all $x^* \in E^*$. We know that T_μ is a nonexpansive mapping from C into itself and $\bigcap_{t\in S} F(T_t) \subset F(T_\mu)$ for each mean μ on X.

We give typical examples for a semigroup S, a subspace X of $B(S)$, a strongly regular sequence $\{\mu_n\}$ of means, a nonexpansive semigroup $\{T_t : t \in S\}$ on C and an operator T_μ.

Example 1. Let $S = \mathrm{N}$ and let $X = B(\mathrm{N})$. For each $n \in \mathrm{N}$ let μ_n be a mean on $B(\mathrm{N})$ defined by

$$\mu_n(f_0, f_1, \cdots) = \frac{1}{n+1}\sum_{i=0}^{n} f_i \qquad \text{for each } (f_0, f_1, \cdots) \in X.$$

Then $\{\mu_n\}$ is strongly regular. Let T be a nonexpansive mapping from a closed, convex subset C of a reflexive Banach space into itself with $F(T) \neq \emptyset$. Let $\{T_t : t \in \mathrm{N}\} = \{I, T, T^2, \cdots\}$. Then $\{T_t : t \in \mathrm{N}\}$ is a nonexpansive semigroup on C and

$$T_{\mu_n} x = \frac{1}{n+1}\sum_{i=0}^{n} T^i x \qquad \text{for each } x \in C.$$

Example 2. Let $S = [0, \infty)$ and let X be the set of all measurable functions from S into the set of real numbers. From the definition of measurability, we know $1 \in X$ and X is shift invariant, i.e., $l_s(X) \subset X$ for each $s \in S$. For each $n \in \mathrm{N}$ let μ_n be a mean on X defined by

$$(\mu_n)_t(f(t)) = \frac{1}{\gamma_n}\int_0^{\gamma_n} f(t)\,dt \qquad \text{for each } f \in X,$$

where $\{\gamma_n\}$ is a positive real sequence with $\gamma_n \to \infty$. Then $\{\mu_n\}$ is strongly regular. Let $\{T(t) : t \geq 0\}$ be a nonexpansive semigroup on $C = \overline{D(A)}$ generated by $-A$, where A is an m-accretive operator on a uniformly convex Banach space E whose range contains 0. In this case, we know that C is convex and there is an element $u \in C$ whose orbit is bounded. We also know

$$T_{\mu_n} x = \frac{1}{\gamma_n}\int_0^{\gamma_n} T(t)x\,dt \qquad \text{for each } x \in C.$$

We now present a nonexpansive semigroup version of Theorem B which is obtained in [27].

Theorem 2 (Shioji and Takahashi). *Let C be a closed, convex subset of a uniformly convex Banach space E whose norm is uniformly Gâteaux differentiable. Let S be a semigroup and let $\{T_t : t \in S\}$ be a nonexpansive semigroup on C such that $\bigcap_{t\in S} F(T_t)$ is nonempty. Let X be a subspace of $B(S)$ such that $1 \in X$, X is l_s-invariant for each $s \in S$ and the mapping $t \mapsto \langle T_t x, x^* \rangle$ is an element of X for each $x \in C$ and $x^* \in E^*$. If there is a left invariant mean on X, then there is a unique sunny, nonexpansive retraction from C onto $\bigcap_{t\in S} F(T_t)$. Further, let $\{\mu_n\}$ be a strongly left regular sequence of means on X and let P be the sunny, nonexpansive retraction from C onto $\bigcap_{t\in S} F(T_t)$. Let $\{a_n\}$ be a real sequence satisfying $0 < a_n \le 1$, $a_n \to 0$. Let x be an element of C and let $\{x_n\}$ be the sequence defined by*

$$x_n = a_n x + (1 - a_n) T_{\mu_n} x_n \quad \text{for each } n \in \mathbb{N}. \tag{3.1}$$

Then $\{x_n\}$ converges strongly to Px.

Remark 1. By the Banach contraction principle there exists a unique point $x_n \in C$ satisfying (3.1) for each $n \in \mathbb{N}$.

Remark 2. By [31] we know that the condition $\bigcap_{t\in S} F(T_t) \neq \emptyset$ can be replaced by the condition that there exists a bounded orbit, i.e., there exists $u \in C$ such that $\{T_t u : t \in S\}$ is bounded.

We next show another strong convergence theorem for a nonexpansive semigroup. Before that, we need to define a mean to be monotone convergent.

Let S be a semigroup and let X be a subspace of $B(S)$ such that for each bounded subset $\{f_n : n \in \mathbb{N}\}$ of X the mapping $t \mapsto \sup_n f_n(t)$ is an element of X. A mean μ on X is said to be *monotone convergent* if $\mu_t(\lim_n f_n(t)) = \lim_n \mu_t(f_n(t))$ for each bounded sequence $\{f_n : n \in \mathbb{N}\}$ of X such that $0 \le f_1 \le f_2 \le \cdots$. We remark that the space X and each mean μ_n in Example 2 satisfy the conditions mentioned above by the definition of measurability and the standard monotone convergence theorem.

We now show a nonexpansive semigroup version of Theorem 1 which is also obtained in [27].

Theorem 3 (Shioji and Takahashi). *Let C, E, S, $\{T_t : t \in S\}$ and X be as in* Theorem 2. *Assume that for each bounded subset $\{f_n : n \in \mathbb{N}\}$ of X the mapping $t \mapsto \sup_n f_n(t)$ is an element of X. Let $\{\mu_n\}$ be a strongly left regular sequence of monotone convergent means on X and let P be the sunny, nonexpansive retraction from C onto $\bigcap_{t\in S} F(T_t)$. Let $\{b_n\}$ be a real* *sequence satisfying* $0 \le b_n \le 1$, $b_n \to 0$ *and* $\sum_{n=0}^{\infty} b_n = \infty$. *Let* x *be an element of* C *and let* $\{y_n\}$ *be the sequence defined by*

$$y_0 \in C, \quad y_{n+1} = b_n x + (1 - b_n) T_{\mu_n} y_n \qquad \text{for each } n \in \mathbb{N}.$$

Then $\{y_n\}$ *converges strongly to* Px.

Remark 3. In the case when E is a Hilbert space, we don't need either the additional assumption for X or the assumption that each μ_n is monotone convergent; see [24].

As direct consequences of Theorem 2 and Theorem 3 we have the following which are related to Example 1 and Example 2, respectively. These results are obtained in [22], [23], [25].

Corollary 1. *Let* C *be a closed, convex subset of a uniformly convex Banach space* E *whose norm is uniformly Gâteaux differentiable. Let* T *be a nonexpansive mapping from* C *into itself with* $F(T) \neq \emptyset$ *and let* P *be the sunny, nonexpansive retraction from* C *onto* $F(T)$. *Let* $\{a_n\}$ *and* $\{b_n\}$ *be real sequences such that* $0 < a_n \le 1$, $a_n \to 0$, $0 \le b_n \le 1$, $b_n \to 0$ *and* $\sum_{n=0}^{\infty} b_n = \infty$. *Let* x *be an element of* C *and let* $\{x_n\}$ *and* $\{y_n\}$ *be the sequences defined by*

$$x_n = a_n x + (1 - a_n) \frac{1}{n+1} \sum_{j=0}^{n} T^j x_n \quad \text{for each} n \in \mathbb{N},$$

$$y_0 \in C, \quad y_{n+1} = b_n x + (1 - b_n) \frac{1}{n+1} \sum_{j=0}^{n} T^j y_n \qquad \text{for each} n \in \mathbb{N},$$

respectively. Then both $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* Px.

Corollary 2. *Let* C *and* E *be as in* Corollary 1. *Let* $\{T(t) : t \ge 0\}$ *be as in* Example 2. *Then there exists a unique sunny, nonexpansive retraction* P *from* C *onto* $\bigcap_{t \ge 0} F(T(t))$. *Moreover, let* $\{a_n\}$ *and* $\{b_n\}$ *be as in* Corollary 1, *and let* $\{\gamma_n\}$ *be as in* Example 2. *Let* x *be an element of* C *and let* $\{x_n\}$ *and* $\{y_n\}$ *be the sequences defined by*

$$x_n = a_n x + (1 - a_n) \frac{1}{\gamma_n} \int_0^{\gamma_n} T(t) x_n \, dt \qquad \text{for each } n \in \mathbb{N},$$

$$y_0 \in C, \quad y_{n+1} = b_n x + (1 - b_n) \frac{1}{\gamma_n} \int_0^{\gamma_n} T(t) y_n \, dt \qquad \text{for each } n \in \mathbb{N},$$

respectively. Then both $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* Px.

Since we use abstract means in Theorem 2 and Theorem 3, we can also obtain the following

Corollary 3. *Let C, E, $\{T(t) : t \geq 0\}$, P, $\{a_n\}$ and $\{b_n\}$ be as in* Corollary 2. *Let $\{\lambda_n\}$ be a sequence of positive real numbers with $\lambda_n \to 0$. Let x be an element of C and let $\{x_n\}$ and $\{y_n\}$ be the sequences defined by*

$$x_n = a_n x + (1 - a_n)\lambda_n \int_0^\infty e^{-\lambda_n t} T(t)x_n \, dt \qquad \text{for each } n \in \mathbb{N},$$

and

$$y_0 \in C, \quad y_{n+1} = b_n x + (1 - b_n)\lambda_n \int_0^\infty e^{-\lambda_n t} T(t)y_n \, dt \qquad \text{for each } n \in \mathbb{N},$$

respectively. Then both $\{x_n\}$ and $\{y_n\}$ converge strongly to Px.

4. Proofs of Theorems. First, we give the proof of Theorem 1. The proof of next lemma is different from that in [21]. To prove it, we used a lemma concerning a Banach limit in [21]. Here, we prove it directly.

Lemma 1. $\overline{\lim_{n\to\infty}} \langle x - Px, J(y_n - Px)\rangle \leq 0.$

Proof. Let $\{a_m\}$ be a real sequence such that $0 < a_m \leq 1/2$ and $a_m \to 0$. Then there exists a unique point x_m of C satisfying

$$x_m = a_m x + (1 - a_m)Tx_m \text{ for each } m \in \mathbb{N}.$$

We know that $\{x_m\}$ converges strongly to Px by Theorem B. Set

$$R = \sup\big(\{\|Tx_m\|\} \cup \{\|x_m\|\} \cup \{\|Ty_n\|\} \cup \{\|y_n\|\}\big).$$

From $(1 - a_m)(Tx_m - y_n) = (x_m - y_n) - a_m(x - y_n)$, we have

$$\begin{aligned}(1 - a_m)^2\|Tx_m - y_n\|^2 &\geq \|x_m - y_n\|^2 - 2a_m\langle x - y_n, J(x_m - y_n)\rangle \\ &= (1 - 2a_m)\|x_m - y_n\|^2 + 2a_m\langle x - x_m, J(y_n - x_m)\rangle\end{aligned}$$

for each $m, n \in \mathbb{N}$. Then we get

$$\begin{aligned}\langle x - x_m, J(y_n - x_m)\rangle &\leq \frac{1}{2a_m}\big((1 - a_m)^2\|Tx_m - y_n\|^2 - (1 - 2a_m)\|x_m - y_n\|^2\big) \\ &= \frac{1 - 2a_m}{2a_m}\big(\|Tx_m - y_n\|^2 - \|x_m - y_n\|^2\big) + \frac{a_m}{2}\|Tx_m - y_n\|^2 \\ &\leq \frac{1 - 2a_m}{2a_m}\big((\|Tx_m - Ty_n\| + \|Ty_n - y_n\|)^2 - \|x_m - y_n\|^2\big) + 2R^2 a_m \\ &\leq \frac{1 - 2a_m}{2a_m} \cdot 6R\|Ty_n - y_n\| + 2R^2 a_m\end{aligned}$$

for each $m, n \in \mathrm{N}$. Since we can infer $\lim_{n\to\infty} \|Ty_n - y_n\| = 0$ from $\sum_{n=0}^{\infty} |b_{n+1} - b_n| < \infty$, we have $\overline{\lim}_{n\to\infty} \langle x - x_m, J(y_n - x_m)\rangle \leq 2R^2 a_m$ for each $m \in \mathrm{N}$. Since $\{x_m\}$ converges strongly to Px and the norm of E is uniformly Gâteaux differentiable, we obtain the conclusion.

□

We can now prove Theorem 1 by the method employed in [21], [33].

Proof of Theorem 1. Fix $\varepsilon > 0$. By Lemma 1 there exists $m \in \mathrm{N}$ such that $2\langle x - Px, J(y_n - Px)\rangle \leq \varepsilon$ for each $n \geq m$. Since $(1-b_n)(Ty_n - Px) = (y_{n+1} - Px) - b_n(x - Px)$, we have

$$(1 - b_n)^2 \|Ty_n - Px\|^2 \geq \|y_{n+1} - Px\|^2 - 2b_n\langle x - Px, J(y_{n+1} - Px)\rangle$$

for each $n \in \mathrm{N}$. So we get $\|y_{n+1} - Px\|^2 \leq b_n\varepsilon + (1 - b_n)\|y_n - Px\|^2$ for each $n \geq m$. By induction we have

$$\begin{aligned} \|y_{n+m} - Px\|^2 &\leq \left(\prod_{j=0}^{n-1}(1 - b_{m+j})\right)\|y_m - Px\|^2 + \varepsilon \\ &\leq \exp\left(-\sum_{j=0}^{n-1} b_{m+j}\right)\|y_m - Px\|^2 + \varepsilon \end{aligned}$$

for each $n \in \mathrm{N}$. From $\sum_n b_n = \infty$, we get $\overline{\lim}_n \|y_n - Px\|^2 \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $\{y_n\}$ converges strongly to Px.

□

We next give the proofs of Theorem 2 and Theorem 3 as in [27]. Since we gave the proofs for the case of an asymptotically nonexpansive semigroup, the proofs here are simpler than those in [27].

The following lemma is crucial in proving Theorem 2 and Theorem 3. It also plays important roles in [1], [11]. Using this lemma, we solved an open problem on the existence of an ergodic retraction for an amenable semigroup of nonexpansive mappings; see [11]. The idea of the proof is inspired by the existence proof of a Banach limit via nonstandard analysis; see [17].

Lemma 2. *Let C be a closed, convex subset of a uniformly convex Banach space E. Let S be a semigroup and let $\{T_t : t \in S\}$ be a nonexpansive semigroup on C such that $\bigcap_{t\in S} F(T_t)$ is nonempty. Let X be a subspace of $B(S)$ such that $1 \in X$, X is l_s-invariant for each $s \in S$ and the mapping $t \mapsto \langle T_t x, x^*\rangle$ is an element of X for each $x \in C$ and $x^* \in E^*$. Let $\{\mu_n\}$ be* *a strongly left regular sequence of means on X. Then for each $r > 0$ and $t \in S$,*

$$\overline{\lim_{n\to\infty}} \sup_{u \in C\cap B_r} \|T_{\mu_n}u - T_t(T_{\mu_n}u)\| = 0.$$

Proof. Let $r > 0$ and let $t \in S$. Let z be an arbitrary point of $\bigcap_{t\in S} F(T_t)$. Set $D = \{x \in C : \|x - z\| \le r + \|z\|\}$. We remark that $C \cap B_r \subset D$, $T_t(D) \subset D$ and $\|x\| \le r + 2\|z\|$ for each $x \in D$. For $\eta > 0$ we denote by $F_\eta(T_t; D)$ the set $\{x \in D : \|x - T_t x\| \le \eta\}$. Fix $\varepsilon > 0$. By Proposition in Section 2 there exist $\delta > 0$ and $N \in \mathrm{N}$ such that

$$(\overline{\mathrm{co}}\, F_\delta(T_t; D) + B_\delta) \cap D \subset F_\varepsilon(T_t; D) \tag{4.1}$$

and $\left\| \frac{1}{N+1} \sum_{i=0}^{N} (T_t)^i x - T_t\left(\frac{1}{N+1} \sum_{i=0}^{N} (T_t)^i x\right)\right\| \le \delta$ for each $x \in D$.

So we have

$$\left\| \frac{1}{N+1} \sum_{i=0}^{N} (T_t)^i (T_s u) - T_t\left(\frac{1}{N+1} \sum_{i=0}^{N} (T_t)^i (T_s u)\right)\right\| \le \delta$$

for each $s \in S$ and $u \in C \cap B_r$. Hence, for each mean μ on X, we have

$$\int \frac{1}{N+1} \sum_{i=0}^{N} T_{t^i s} u \, d\mu(s) \in \overline{\mathrm{co}}\left\{ \frac{1}{N+1} \sum_{i=0}^{N} T_{t^i s} u : s \in S \right\} \subset \overline{\mathrm{co}} F_\delta(T_t; D) \tag{4.2}$$

for each $u \in C \cap B_r$, where $t^0 s$ represents s. From the strong left regularity of $\{\mu_n\}$ it follows that there exists $n_0 \in \mathrm{N}$ such that $\|\mu_n - l^*_{t^i}\mu_n\| < \delta/(r + 2\|z\|)$ for each $n \ge n_0$ and $i = 1, \ldots, N$. So we have

$$\begin{aligned}
&\left\| T_{\mu_n} u - \int \frac{1}{N+1} \sum_{i=0}^{N} T_{t^i s} u \, d\mu_n(s) \right\| \\
&\qquad = \sup_{\|u^*\|=1} \left| (\mu_n)_s \langle T_s u, u^* \rangle - \frac{1}{N+1} \sum_{i=0}^{N} (\mu_n)_s \langle T_{t^i s} u, u^* \rangle \right| \\
&\qquad \le \frac{1}{N+1} \sum_{i=1}^{N} \sup_{\|u^*\|=1} \left| (\mu_n)_s \langle T_s u, u^* \rangle - (l^*_{t^i}\mu_n)_s \langle T_s u, u^* \rangle \right| \\
&\qquad \le \frac{1}{N+1} \sum_{i=1}^{N} \|\mu_n - l^*_{t^i}\mu_n\| \cdot (r + 2\|z\|) \le \delta
\end{aligned}$$

for each $u \in C \cap B_r$ and $n \ge n_0$. The inequality above, (4.1) and (4.2) yield $T_{\mu_n} u \in F_\varepsilon(T_t; D)$ for each $u \in C \cap B_r$ and $n \ge n_0$, which implies

$\overline{\lim}_n \sup_{u \in C \cap B_r} \|T_{\mu_n} u - T_t(T_{\mu_n} u)\| \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we obtain the conclusion.

□

Till the end of Lemma 5 we assume that C, E, S, $\{T_t : t \in S\}$, X, $\{\mu_n\}$, $\{a_n\}$, x and $\{x_n\}$ are such as in Theorem 2.

Lemma 3. *For each subsequence $\{x_{n_i}\}$ of $\{x_n\}$ there exists an element z of $\bigcap_{t \in S} F(T_t)$ satisfying*

$$\text{(4.3)} \qquad \underline{\lim}_{i \to \infty} \langle y - z, J(x_{n_i} - z) \rangle \leq 0 \qquad \textit{for each } y \in C.$$

Proof. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$. From the uniform convexity of the function $u \mapsto \|u\|^2$ on each bounded subset of E, we conclude that there exists a unique point z of C which satisfies

$$\text{(4.4)} \qquad \overline{\lim}_{i \to \infty} \|x_{n_i} - z\|^2 = \min_{y \in C} \overline{\lim}_{i \to \infty} \|x_{n_i} - y\|^2.$$

We shall show that $z \in \bigcap_{t \in S} F(T_t)$. Suppose this is not true. Then there exists $t \in S$ such that $T_t z \neq z$. From the uniform convexity of the function $u \mapsto \|u\|^2$ on each bounded subset of E and Lemma 2 we have

$$\begin{aligned} \overline{\lim}_{i \to \infty} \left\| x_{n_i} - \frac{T_t z + z}{2} \right\|^2 &< \frac{1}{2}\left(\overline{\lim}_{i \to \infty} \|x_{n_i} - T_t z\|^2 + \overline{\lim}_{i \to \infty} \|x_{n_i} - z\|^2 \right) \\ &\leq \overline{\lim}_{i \to \infty} \|x_{n_i} - z\|^2. \end{aligned}$$

Since z is a unique point of C satisfying (4.4), we get a contradiction. Hence we obtain $z \in \bigcap_{t \in S} F(T_t)$. We next show that z satisfies (4.3). Since

$$\|x_{n_i} - z\|^2 \geq \|x_{n_i} - (\delta y + (1 - \delta) z)\|^2 + 2\delta \langle y - z, J(x_{n_i} - (\delta y + (1 - \delta) z)) \rangle$$

for each $y \in C$, $i \in \mathbb{N}$ and δ with $0 < \delta \leq 1$, this inequality and (4.4) yield

$$\underline{\lim}_{i \to \infty} \langle y - z, J(x_{n_i} - (\delta y + (1 - \delta) z)) \rangle \leq 0$$

for each $y \in C$ and δ with $0 < \delta \leq 1$. Since the norm of E is uniformly Gâteaux differentiable, we obtain (4.3).

Lemma 4. *$\langle x_n - x, J(x_n - z)\rangle \le 0$ for each $n \in \mathbb{N}$ and $z \in \bigcap_{t\in S} F(T_t)$.*

Proof. Let $n \in \mathbb{N}$ and let $z \in \bigcap_{t\in S} F(T_t)$. From

$$a_n(x_n - x) = (1-a_n)(T_{\mu_n}x_n - x_n) \quad \text{and} \quad T_{\mu_n}z = z,$$

we have

$$\begin{aligned}\langle x_n - x, J(x_n - z)\rangle &= \frac{1-a_n}{a_n}\langle T_{\mu_n}x_n - x_n, J(x_n - z)\rangle \\ &= \frac{1-a_n}{a_n}\left(\langle T_{\mu_n}x_n - T_{\mu_n}z, J(x_n - z)\rangle + \langle z - x_n, J(x_n - z)\rangle\right) \\ &\le \frac{1-a_n}{a_n}\left(\|x_n - z\|^2 - \|x_n - z\|^2\right) = 0.\end{aligned}$$

□

Lemma 5. *$\{x_n\}$ converges strongly to an element of $\bigcap_{t\in S} F(T_t)$.*

Proof. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$. By Lemma 3 there exists $z \in \bigcap_{t\in S} F(T_t)$ satisfying $\underline{\lim}_i \langle x - z, J(x_{n_i} - z)\rangle \le 0$. Hence by Lemma 4 we get $\underline{\lim}_{i\to\infty} \|x_{n_i} - z\|^2 \le \underline{\lim}_{i\to\infty} \langle x_{n_i} - x, J(x_{n_i} - z)\rangle \le 0$, which implies that there exists a subsequence of $\{x_{n_i}\}$ converging strongly to z. So each subsequence $\{x_{n_i}\}$ of $\{x_n\}$ contains a subsequence of $\{x_{n_i}\}$ which converges strongly to an element of $\bigcap_{t\in S} F(T_t)$. Let $\{x_{n_i}\}$ and $\{x_{m_i}\}$ be subsequences of $\{x_n\}$ converging strongly to elements z and w of $\bigcap_{t\in S} F(T_t)$, respectively. From Lemma 4 we have $\langle z - x, J(z-w)\rangle \le 0$ and $\langle w - x, J(w - z)\rangle \le 0$. Adding these inequalities we get $z = w$. Consequently $\{x_n\}$ converges strongly to an element of $\bigcap_{t\in S} F(T_t)$.

□

We can now prove Theorem 2.

Proof of Theorem 2. Assume that there is a left invariant mean μ on X. Put $\mu_n = \mu$ for each $n \in \mathbb{N}$. Then $\{\mu_n\}$ is strongly left regular. Let $\{a_n\}$ be a real sequence such that $0 < a_n \le 1$ and $a_n \to 0$. For each $x \in C$, set $Px = \lim_n x_n$, where $\{x_n\}$ is a sequence defined by (3.1). From Lemma 5 and Lemma 4 it follows that P is well defined, $Px \in \bigcap_{t\in S} F(T_t)$ for each $x \in C$ and $\langle x - Px, J(z - Px)\rangle \le 0$ for each $z \in \bigcap_{t\in S} F(T_t)$. So P is a unique sunny, nonexpansive retraction from C onto $\bigcap_{t\in S} F(T_t)$. The latter part of the theorem is obvious by Lemma 5. □

Remark 4. We can also prove that if $\{a_\alpha\}$ is a net of real numbers with $0 < a_\alpha \le 1$ and $a_\alpha \to 0$, $\{\mu_\alpha\}$ is a net of means on X with $\lim_\alpha \|\mu_\alpha - l_s^*\mu_\alpha\| = 0$

for each $s \in S$ and $\{x_\alpha\}$ is a net defined by $x_\alpha = a_\alpha x + (1 - a_\alpha)T_{\mu_\alpha} x_\alpha$ for each α, then $\{x_\alpha\}$ converges strongly to Px.

We next give the proof of Theorem 3. Till the end of Lemma 7 we assume that C, E, S, $\{T_t : t \in S\}$, X, P, $\{\mu_n\}$, $\{b_n\}$, x and $\{y_n\}$ are as in Theorem 3.

Lemma 6. *For each monotone convergent mean μ on X*

$$\overline{\lim_n} \|T_\mu y_n - y_n\| = 0.$$

Proof. Let μ be a monotone convergent mean on X. By a standard measure theory argument, we have that for each bounded sequence $\{f_n : n \in \mathrm{N}\}$ of X, $\overline{\lim}_n f_n \in X$ and $\overline{\lim}_n \mu_t(f_n(t)) \le \mu_t(\overline{\lim}_n f_n(t))$. From Lemma 2 and the definition of $\{y_n\}$ we have $\overline{\lim}_n \|T_t y_n - y_n\| = 0$ for each $t \in S$. Hence we obtain

$$\begin{aligned} \overline{\lim_{n\to\infty}} \|T_\mu y_n - y_n\|^2 &= \overline{\lim_{n\to\infty}} \mu_t \langle T_t y_n - y_n, J(T_\mu y_n - y_n)\rangle \\ &\le \mu_t\big(\overline{\lim_{n\to\infty}} \langle T_t y_n - y_n, J(T_\mu y_n - y_n)\rangle\big) \le 0. \end{aligned}$$

□

The following is crucial to prove Theorem 3.

Lemma 7. $\overline{\lim_{n\to\infty}} \langle x - Px, J(y_n - Px)\rangle \le 0.$

Proof. Let $\{a_m\}$ be a real sequence such that $0 < a_m \le 1/2$ and $a_m \to 0$. By Remark 1 there exists a unique point x_m of C satisfying

$$x_m = a_m x + (1 - a_m)T_{\mu_m} x_m \quad \text{for each } m \in \mathrm{N}.$$

We know that $\{x_m\}$ converges strongly to Px by Theorem 2. Set $R = \sup(\{\|T_{\mu_m} x_m\|\} \cup \{\|x_m\|\} \cup \{\|T_{\mu_m} y_n\|\} \cup \{\|y_n\|\})$. Since each T_{μ_m} is nonexpansive, we can obtain

$$\langle x - x_m, J(y_n - x_m)\rangle \le \frac{1 - 2a_m}{2a_m} \cdot 6R\|T_{\mu_m} y_n - y_n\| + 2R^2 a_m$$

for each $m, n \in \mathrm{N}$ by the same lines as those in the proof of Lemma 1. Hence we obtain the conclusion from Lemma 6 and the uniform Gâteaux differentiability of the norm of E.

□

We can now prove Theorem 3 similarly as in the proof of Theorem 1.

Proof of Theorem 3. Fix $\varepsilon > 0$. By Lemma 7 there exists $m \in \mathbb{N}$ such that $2\langle x - Px, J(y_n - Px)\rangle \leq \varepsilon$ for each $n \geq m$. Then we get $\|y_{n+1} - Px\|^2 \leq b_n\varepsilon + (1 - b_n)\|y_n - Px\|^2$ for each $n \geq m$, which yields $\|y_{n+m} - Px\|^2 \leq \exp(-\sum_{j=0}^{n-1} b_{m+j})\|y_m - Px\|^2 + \varepsilon$ for each $n \in \mathbb{N}$. Thus we get $\overline{\lim}_n \|y_n - Px\|^2 \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $\{y_n\}$ converges strongly to Px.

□

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