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The Denjoy - Wolff Theorem

Abstract. We discuss the asymptotic behavior of one-parameter discrete and continuous semigroups of holomorphic and ρ-nonexpansive mappings in Hilbert and Banach spaces in the spirit of the classical Denjoy-Wolff Theorem

Let D be a domain in a complex Banach space X. By $\mathrm{Hol}(D, X)$ we denote the set of all holomorphic mappings on D with values in X. The subset of all holomorphic self-mappings of D will be denoted by $\mathrm{Hol}(D)$. This set is a semigroup with respect to composition. A mapping $F \in \mathrm{Hol}(D)$ is called an automorphism of D if it has an inverse F^{-1} which also belongs to $\mathrm{Hol}(D)$. The set of all automorphisms of D is a subgroup of $\mathrm{Hol}(D)$ which will be denoted by $\mathrm{Aut}(D)$.

The main goal of our paper is to discuss the asymptotic behavior of one-parameter discrete and continuous semigroups in $\mathrm{Hol}(D)$ in the spirit of Denjoy and Wolff.

Theorem (Denjoy-Wolff). *Let Δ be the open unit disk in the complex plane $\mathbb{C}$. If $F \in \mathrm{Hol}(\Delta)$ is not the identity and is not an automorphism of Δ with exactly one fixed point in Δ, then there is a unique point a in the closed unit disk $\bar{\Delta}$ such that the iterates $\{F^n\}_{n=1}^{\infty}$ of F converge to a, uniformly on compact subsets of Δ.*

Over the last twenty years this result has been developed in at least three directions. The first one concerns increasing the dimension of the underlying space. Finite dimensional extensions are to be found, for instance, in the papers by Kubota [42], MacCluer [47], Chen [11], Abate [1], [2] and Mercer [51]. In this connection see also [6], [4], [46], [28] and [68].

Infinite-dimensional generalizations are due, for example, to Fan [18], [19], Włodarczyk [74], [75], [76], Goebel [20], Vesentini [69], [70], Sine [65] and Mellon [50]. These authors used a variety of approaches and assumed diverse conditions on the mappings and the domains.

The family $\{F^n\}_{n=1}^{\infty}$ of the iterates of $F \in \mathrm{Hol}(D)$ can be considered a one-parameter discrete subsemigroup of $\mathrm{Hol}(D)$. Therefore the second direction is concerned with analogues of the Denjoy-Wolff Theorem for continuous semigroups of holomorphic self-mappings of D. This approach has been used by several mathematicians to study the asymptotic behavior of solutions to Cauchy problems (see, for example, [8], [3], [5], [13] and [57]). Berkson and Porta [8] also apply their continuous analogue of the Denjoy-Wolff Theorem to the study of the eigenvalue problem for composition operators in Hardy spaces (see also [13]).

Several talks in the present conference spurred a renewed interest in an implicit method for finding fixed points of a nonexpansive mapping which uses the so-called "approximating curves". This method also works effectively for the class of holomorphic mappings (see, for example, [22], [40], [61], [49], [37], [31], [32], [58]). These results can also be considered implicit continuous analogues of the Denjoy-Wolff Theorem. It is remarkable that the asymptotic behavior of the approximating curves is actually nicer than that of the usual iterative process.

Moreover, this implicit method can be useful not only for a self-mapping of a domain, but also for a wider class of mappings in $\mathrm{Hol}(D, X)$ which satisfy certain "one-sided" estimates (see, for example, [32] and Section 2.2 below).

Finally, note that it is well known that each element of $\mathrm{Hol}(D)$ is a nonexpansive mapping with respect to any metric assigned to the domain D by a Schwarz-Pick system (see, for example, [24]).

Therefore the third direction yields extensions of the Denjoy-Wolff Theorem to the wider class of such nonexpansive mappings (which are not necessarily holomorphic). See, for example, [20], [23], [22], [61], [53], [41].

We begin with some historical information to trace common features and differences in these developments.

1. Discrete iterations. The classical Denjoy-Wolff Theorem deals with the convergence of the iterates of a holomorphic self-mapping of the open unit disk. Therefore we will say in general that a mapping $F \in \text{Hol}(D)$ is power convergent to a mapping $R \in \text{Hol}(D, X)$ if the sequence of iterates $\{F^n\}_{n=1}^{\infty}$ converges to R uniformly on each ball strictly inside D. If $R \equiv a$ is a constant mapping, then the point $a \in \bar{D}$ will be called *attractive*.

1.1 The one-dimensional case. In this section we let D be the open unit disk Δ in the complex place $\mathbb{C}$. Recall that an automorphism $F \in \text{Aut}(\Delta)$ is said to be elliptic if it has exactly one fixed point in Δ.

So, the Denjoy-Wolff Theorem asserts that $F \in \text{Hol}(\Delta)$ *is power convergent if and only if it is not an elliptic automorphism.*

This result is, in fact, a summary of the following three assertions due to Denjoy and Wolff [14], [77], [78]. Each one of them has been extended to different situations.

Theorem 1.1 (The Wolff-Schwarz Lemma). *If $F \in \text{Hol}(\Delta)$ has no fixed point in Δ, then there is a unique unimodular point $a \in \partial\Delta$ such that every disk D_a in Δ, internally tangent to $\partial\Delta$ at a, is F-invariant, i.e.,*

$$F(D_a) \subset D_a. \tag{1.1}$$

Theorem 1.2. *If $F \in \text{Hol}(\Delta)$ has no fixed point in Δ, then there is a unique unimodular point $b \in \partial\Delta$ such that the sequence $\{F^n\}_{n=1}^{\infty}$ converges to b, uniformly on compact subsets of Δ.*

Theorem 1.3. *If $F \in \text{Hol}(\Delta)$ is not an automorphism of Δ and has a fixed point c in Δ, then this point is unique in Δ, and the sequence $\{F^n\}_{n=1}^{\infty}$ converges to c uniformly on compact subsets of Δ.*

The limit point of Theorem 1.2 will be called the Denjoy-Wolff point of F.

The point a in Theorem 1.1 and the point b in Theorem 1.2 are, of course, one and the same. However, this is not always the case in higher dimensional situations.

Therefore, in the general case, the point a of Theorem 1.1 will usually be called *the sink point of F*. So, the sink point is the Denjoy-Wolff point if it is also attractive.

In fact, there are many situations in the higher dimensional case when a holomorphic fixed point free mapping has a sink point, but is not power convergent (see section 1.2).

Returning to the one-dimensional case, we refer the reader to the paper by R. Burckel [9] and to the book by C. Cowen and B. MacCluer [13] for a modern interpretation of the Denjoy-Wolff Theorem and its applications. Here we mention only three observations concerning this case.

Remark 1.1. The Denjoy-Wolff Theorem directly implies that if $F \in \mathrm{Hol}(\Delta)$ is power convergent to a nonconstant mapping, then it is an automorphism.

We will see below that this fact does not hold in general.

Remark 1.2 By using the Schwarz Lemma, Theorem 1.3 can be rephrased in the following manner:

Theorem 1.4. *Let $F \in \mathrm{Hol}(\Delta)$ have a fixed point $a \in \Delta$. If F is not the identity, then F is power convergent if and only if $|F'(a)| < 1$.*

It turns out that by using the notion of the derivative and its spectral properties, one can also study power convergent mappings in higher dimensional spaces. See, for example, [69], [35], [70], [1], [2], [36].

Remark 1.3. The proofs of Theorems 1.1 and 1.2 are based on the Schwarz Lemma, Montel's Theorem and a geometrical description of the set D_a.

For $\xi \in \partial\Delta$ the set

$$D_\xi = D(\xi, R) = \left\{ z \in \Delta : \frac{|1 - z\bar{\xi}|^2}{1 - |z|^2} < R \right\} \tag{1.2}$$

is called a horocycle at ξ with radius R.

So, Theorem 1.1 actually asserts the following:

Theorem 1.5. *If $F \in \mathrm{Hol}(\Delta)$ has no fixed point in Δ, then there is a unique point $a \in \partial\Delta$ such that*

$$\frac{|1 - F(z)\bar{a}|^2}{1 - |F(z)|^2} \le \frac{|1 - z\bar{a}|^2}{1 - |z|^2} \tag{1.3}$$

for all $z \in \Delta$.

We are now in a position to formulate several generalizations of Theorems 1.1, 1.2 and 1.3.

1.2. The unit Hilbert ball. Let H be a complex Hilbert space with the inner product $(\cdot, \cdot)$, and let B be the open unit ball in H.

The following generalization of the Wolff-Schwarz Lemma (Theorem 1.1) is due to K. Goebel [20]. For the finite dimensional case, $H = \mathbb{C}^n$, this result was independently obtained by B. MacCluer [47] and G. Chen [11].

Theorem 1.6. *If $F \in \text{Hol}(D)$ has no fixed point, then there exists a unique point $a \in \partial B$ such that for each $0 < R < \infty$ the set*

$$E(a,R) = \left\{ x \in B : \frac{|1-(x,a)|^2}{1-\| x \|^2} < R \right\}$$

is F-invariant.

Geometrically, the set $E(a,R)$ is an ellipsoid the closure of which intersects the unit sphere ∂B at the point a. It is a natural analogue of the horocycle $D(a,R)$.

Now the question is whether the sink point a in Theorem 1.6 is also the Denjoy-Wolff point of F, i.e., is it attractive?

For the finite dimensional case (B is then the open Euclidean ball in $H = \mathbb{C}^n$) the affirmative answer was given by B. MacCluer [47].

For infinite dimensional Hilbert balls A. Stachura [66] has given a counterexample to show that the convergence result fails even for biholomorphic self-mappings.

Nevertheless, some restrictions on a mapping from $\text{Hol}(B)$ lead to a generalization of Theorem 1.2.

The following result was obtained by Cho-Ho Chu and P. Mellon [12].

Theorem 1.7. *Let B be the open unit ball in a Hilbert space H, and let $F \in \text{Hol}(B)$ be a compact mapping with no fixed point in B. Then the sink point a in Theorem 1.6 is attractive, i.e., the sequence $\{F^n\}$ of iterates of F converges locally uniformly on B to the constant mapping taking the value a.*

Although this theorem contains Theorem 1.2 (as well as the above-mentioned finite-dimensional result of B. MacCluer), at the same time it loses sight of the automorphisms of B because of the compactness restriction. In this connection we have to mention a result by T.J. Suffridge [67] which complements our information.

Theorem 1.8. *Let B be as above, and let $F \in \text{Hol}(B)$ be an automorphism of B with exactly two fixed points on ∂B. Then one of them is an attractive sink point of F.*

Remark 1.4. It is known that each automorphism of B can be extended to an automorphism of $\bar{B}$ and that it has a fixed point in $\bar{B}$ (see T.L. Hayden and T. J. Suffridge [25]). Moreover, if it does not have a fixed point in B, then it has one or two fixed points on the boundary.

We are not going to discuss now possible generalizations of Theorem 1.3 (that is, the case when F has an interior fixed point), since such a discussion requires an additional local study of such a point, and is also connected to another important topic of fixed point theory, namely the study of holomorphic retracts.

We will consider this question in the sequel in somewhat more general situations. For weak convergence results we refer the reader, for example, to [22]. We also note that Theorem 1.1 can be generalized to the operator ball over a Hilbert space H and more generally to the open unit ball U of a so-called J^*-algebra (see [74], [75], [76], [50]) while Theorem 1.2 fails in general even we assume the compactness of F (see Example 1.1 below). A visible reason for such nonattractiveness of the sink point is the nonsimple structure of the boundary of the domain. Therefore, even for the finite-dimensional case it would be significant to get out of the framework of the Euclidean ball.

1.3. Convex domains in $\mathbb{C}^n$. In 1941 M.H. Heins [27] extended the Denjoy-Wolff Theorem to a finitely connected domain bounded by Jordan curves in $\mathbb{C}$. His approach is specific to the one-dimensional case.

Another look at the Denjoy-Wolff Theorem is provided by a useful result of P. Yang [80] concerning a characterization of the horocycle in terms of the Poincaré hyperbolic metric in Δ. More precisely, he established the following formula:

$$\lim_{\mu\to a}[\rho(\lambda,\mu)-\rho(0,\mu)]=\frac{1}{2}\log\frac{|1-\lambda\bar{a}|^2}{1-|\lambda|^2}.$$

So, in these terms the horocycle D_a in Δ can be described by the formula

$$D(a,R)=\{z\in\Delta:\lim_{\omega\to a}(\rho(z,\omega)-\rho(0,\omega))<\frac{1}{2}\log R\}. \tag{1.4}$$

Since a hyperbolic metric can be defined in each bounded domain in $\mathbb{C}^n$, one can try to extend this formula and use it as a definition of the horosphere in a domain in $\mathbb{C}^n$. Unfortunately, in general the limit in (1.4) does not exist.

Therefore, the new idea of M. Abate [1] was to study two kinds of horospheres. More precisely, he defined the small horosphere $E_{z_0}(x,R)$ of center x, pole z_0 and radius R by the formula

$$E_{z_0}(x,R)=\{z\in D:\limsup_{w\to x}[K_D(z,w)-K_D(z_0,w)]<\frac{1}{2}\log R\},$$

and the big horosphere of center x, pole z_0 and radius R by the formula

$$F_{z_0}(x,R) = \{z \in D : \liminf_{w\to x}[K_D(z,w) - K_D(z_0,w)] < \frac{1}{2}\log R\},$$

where D is a bounded domain in $\mathbb{C}^n$ and K_D is its Kobayashi metric. For the Euclidean ball in $\mathbb{C}^n$, $E_{z_0}(x,R) = F_{z_0}(x,R)$ (see [1], [80]).

Thus each assertion which states for a domain D in $\mathbb{C}^n$ the existence of a point $a \in \partial D$ such that

$$f^n(E_z(a,R)) \subset F_z(a,R)) \tag{1.5}$$

for all $z \in D, R > 0, f \in \text{Hol}(D)$ and $n = 1,2,\ldots$ is a generalization of Wolff's Theorem 1.1.

This is true, for example, for a bounded convex domain in $\mathbb{C}^n$ [1]. A more general result was established by M. Abate in another work [4].

Theorem 1.9. *Let D be a bounded complete hyperbolic domain with a simple boundary, and let $z \in D$. Suppose that for some $f \in \text{Hol}(D)$ the sequence of iterates $\{f^n\}$ is compactly divergent. Then there is $a \in \partial D$ such that for all $z \in D$, $R > 0$ and $n = 1,2,\ldots$ the inclusion (1.5) holds.*

In fact, the assumptions of Theorem 1.9 are not sufficient to ensure a convergence result. Indeed, to generalize the notion of a sink point we give the following definition.

Let D be a domain in a Banach space X and let $F \in \text{Hol}(D)$. We will say that a point $x \in \partial D$ is a boundary sink point for F if there exist two sets of neighborhoods $\{U_\alpha\}$ and $\{V_\alpha\}$, $\alpha \in \mathcal{A}$ (a directed set), in D such that the following conditions hold:

(i) $U_\alpha \subset V_\alpha \subset D$;
(ii) $x \in \bar{U}_\alpha$;
(iii) $\cap_{\alpha\in\mathcal{A}} U_\alpha = \cap_{\alpha\in\mathcal{A}} V_\alpha = \emptyset$;
(iv) $F(U_\alpha) \subset V_\alpha$;
(v) For $\alpha_1 < \alpha_2$, $\alpha_1, \alpha_2 \in \mathcal{A}$, $U_{\alpha_2} \subset U_{\alpha_1}$ and $V_{\alpha_2} \subset V_{\alpha_1}$.

The following example is due to C.H. Chu and P. Mellon [13].

Example 1.1. Let D be the open unit bidisk in $\mathbb{C}^2$, i.e., $D = \Delta \times \Delta$, and let $h \in \text{Hol}(\Delta)$ be a fixed point free mapping with the Denjoy-Wolff point $\zeta \in \partial\Delta$. Consider the mapping $F \in \text{Hol}(D)$ defined as follows:

$$F(z,w) = (e^{i\varphi}z, h(w))$$

where $0 < \varphi < 2\pi$. It is clear that if F has the sink point $a \in \partial D$, then $a = (0,\zeta) \in \partial D$. At the same time, the sequence of iterates $\{F^n\}$ does not converge to any boundary point of D.

Nevertheless, the convergence result does hold for bounded strongly convex C^2 domains, and for strongly pseudo-convex hyperbolic domains with a C^2 boundary [1], [2].

To generalize these facts M. Abate defined the following notion [3]:

A domain $D \subset\subset \mathbb{C}^n$ is said to be F-convex at $x \in \partial D$ if for all $z \in D$ and $R > 0$,

$$\overline{F_z(x, R)} \cap \partial D = \{x\}. \tag{1.6}$$

The domain D is said to be F-convex if (1.6) holds for each $x \in \partial D$.

His result is the following:

Theorem 1.10. *Suppose that in addition to the assumptions of Theorem 1.9 the domain D is F-convex. Then there is a point $a \in \partial D$ which is attractive for f.*

Certain types of domains are known to be F-convex. We mention, for example, strictly pseudo-convex domains with a C^2 boundary, and the so-called m-convex domains (see [1], [2], [50]).

1.4. Domains in Banach spaces. Unfortunately, our knowledge does not include positive results and approaches devoted to extensions of Theorems 1.1 and 1.2 for general Banach spaces. Indeed, a simple example shows that there is a situation where even a sink point does not exist.

Example 1.2. Consider the Banach space c_0 of all complex sequences $x = (x_1, x_2, \ldots, x_n, \ldots)$ such that $x_n \to 0$ as n tends to infinity, with the max norm, and the affine (hence holomorphic) mapping $F : c_0 \to c_0$ defined by $Fx = (a, x_1, x_2, \ldots)$, where $a \neq 0$, $|a| < 1$. It is clear that F maps the open unit ball D of c_0 into itself and that it is continuous on $\bar{D}$. Therefore, if F has a sink point on ∂D, then it must be a fixed point of F in $\bar{D}$. But F has no fixed point there.

However, the question is still open for reflexive Banach spaces. Moreover, if D has a strictly convex boundary and F is compact or, more generally, condensing, then one can expect the convergence result to hold.

Conjecture 1.1. Let X be a strictly convex Banach space, and let F be a holomorphic condensing self-mapping of the open unit ball D of X without a fixed point in D. Then F is power convergent on D.

Remark 1.5. This conjecture has recently been verified for the case when F is assumed to be compact [29].

The situation is more fully understood when F has a fixed point in the domain.

From now on we will assume that D is a bounded domain in a Banach space X and that $F \in \text{Hol}(D)$ has an interior fixed point $a \in D$. Of course, as simple examples show, one cannot expect that a is always an attractive point, even if F is not an automorphism (consider Example 1.1 with $h(w) = w^2$). Nevertheless, rephrasing the Denjoy-Wolff Theorem 1.3 in the form 1.4, one can point out several generalizations of this result.

Proposition 1.1 ([62], [35], [36]). *Let D be a bounded domain in X and let $F \in \text{Hol}(D)$. Suppose that F has an interior fixed point $a \in D$ and let $F'(a)$ be its Fréchet derivative at this point. Then a is an attractive fixed point of F if and only if the spectral radius of $F'(a)$ is strictly less than 1.*

Combining this assertion with the Earle-Hamilton [17] fixed point theorem, one can formulate a geometrical characterization of the attractive fixed point.

Proposition 1.2 ([35], [36]). *Let D be a bounded domain in X and let $F \in \text{Hol}(D)$. Then F has an attractive fixed point in D if and only if there exist a domain $\tilde{D} \subseteq D$ and an integer $n > 0$ such that F^n maps $\tilde{D}$ strictly inside itself.*

If a is an attractive fixed point of F, then by definition it is unique and F is power convergent.

For $D \subset \mathbb{C}$, the converse is also true:

If $F \in \text{Hol}(D)$ is not the identity and has an interior fixed point, and F is power convergent, then a is unique.

However, this is not the case in higher dimensions. Indeed, the following example exhibits a function F, mapping the bidisk B_2 in $\mathbb{C}^2$ into itself, which is power convergent, but its fixed point set consists of an infinite number of points (actually it is a submanifold of D of dimension 1).

Example 1.3. Let $D = B_2 = \Delta \times \Delta$ and $F = (z_1, \frac{1}{2}(z_2^2 + z_1))$. Then F is a power convergent mapping, but all the points of the form $(z_1, 1 - \sqrt{1 - z_1})$ are fixed points of F.

A full description of such a situation was obtained by E. Vesentini [70], [71].

Theorem 1.11. *Let D be a bounded convex domain in a Banach space X, and let F belong to $\mathrm{Hol}(D)$. Suppose that F has a fixed point $a \in D$, and denote the spectrum of the linear operator $F'(a)$ by $\sigma(F'(a))$. Then F is power convergent if and only if the following two conditions hold:*

(i) $\sigma(F'(a)) \subset \Delta \cup \{1\}$; *and*

(ii) *1 is a pole of the resolvent of $F'(a)$ of order at most 1.*

Comments. Condition (ii) is actually equivalent to the condition

$$(*) \qquad \mathrm{Ker}(I - F'(a)) \oplus \mathrm{Im}(I - F'(a)) = X$$

(see, for example, [43], [45]). It is also known that conditions (i) and (ii) are equivalent to $F'(a)$ being power-convergent to a projection P onto $\mathrm{Ker}(I - F'(a))$. So, if $R \in \mathrm{Hol}(D)$ is the limit point of $\{F^n\}$ under these conditions, then $R = a$ is constant if and only if $P = 0$.

In general, there is a neighborhood U of a such that $R^2 = R$ on U, that is, R is an idempotent of the semigroup $\mathrm{Hol}(D)$. In this case the mapping R is said to be a local holomorphic retraction, and its image $R(U)$ is called a holomorphic retract in U.

In our setting this means that $\mathrm{Fix}F \cap U = R(U)$ is a submanifold of U tangent to $\mathrm{Ker}(I - F'(a))$ (see [10]). This fact describes the general structure of the fixed point set. As a matter of fact, only condition (ii) or $(*)$ is sufficient for $\mathrm{Fix}F$ to be a local retract (see, for example, [63], [64], [72], [73], [48], [49]). Moreover, if D is convex, $\mathrm{Fix}F$ is a global retract in D, and hence a connected complex submanifold of D.

However, once again, a retraction onto the fixed point set can be obtained by the simple iteration method only if condition (i) holds, i.e., $F'(a)$ has no spectrum points on the unit circle except, possibly, 1. This brings us to another interesting aspect of fixed point theory: the study of the holomorphic retracts of a domain. This problem has been studied by many mathematicians (see, for example, [60], [26], [47], [7], [69], [70], [71], [1], [2], [3], [72], [73], [48]). Unfortunately a complete account of the relevant results is not possible here.

Returning to the Denjoy-Wolff Theorem we now consider some explicit and implicit continuous versions of this theorem.

2. Continuous versions of the Denjoy-Wolff Theorem.

2.1. One-parameter continuous semigroups (explicit versions). We have already mentioned that for $F \in \mathrm{Hol}(D)$ the family of iterates $S =$

$\{F^n\}_{n=1}^{\infty}$ can be considered a one-parameter discrete-time semigroup. In this case the vector-field $f = I - F$ is referred to as the generator of S (see [30]).

Recall that a family $S = \{F_t\}_{t>0} \subset \mathrm{Hol}(D)$ is called a one-parameter continuous semigroup if it satisfies the semigroup property:

$$F_t \circ F_s = F_{t+s},$$

and for each $x \in D$ there exists the strong limit $\lim_{t\to 0+} F_t(x) = x$.

A natural explicit continuous one-dimensional analogue of the Denjoy-Wolff Theorem was given by E. Berkson and H. Porta [8] and was used by them to study semigroups of composition operators on Hardy spaces.

Theorem 2.1. *Let Δ be the open unit disk in the complex plane $\mathbb{C}$, and let $S = \{F_t\}$ be a one-parameter continuous semigroup of holomorphic self-mappings F_t of Δ. Then F_t converges as $t \to \infty$ to a holomorphic function $h : \Delta \to \mathbb{C}$ if and only if no F_t is an elliptic automorphism of Δ.*

M. Abate generalized this theorem to the n-dimensional case [3], separating the case of a fixed-point free semigroup from the case of a semigroup with an interior common fixed point.

Using the terminology of the theory of dynamical systems we call a common fixed point of a continuous semigroup a stationary point. Thus in this sense the convergence of the semigroup to a stationary point means its asymptotic stability. In such terms the finite-dimensional case was also partially described by Yu. Lyubich [44].

Finally, note that for a bounded domain in a Banach space X and a net $\{F_j\}_{j\in\mathcal{A}}$ of holomorphic self-mappings of D, its pointwise convergence is equivalent to its uniform convergence on compact subsets of D (that is, convergence in the compact open topology on D). If $X = \mathbb{C}^n$, this in turn is the same as uniform convergence on each subset strictly inside D. This is no longer true in the infinite dimensional case. Therefore, in order to achieve certain results, in this situation it may sometimes be more advantageous to consider a finer topology than the compact open topology, namely, the so-called topology of local uniform convergence over D.

Definition 2.1. A net $\{f_j\}_{j\in\mathcal{A}} \subset \mathrm{Hol}(D, X)$ is said to converge to a mapping $f \in \mathrm{Hol}(D, X)$ in the topology of local uniform convergence over D (or briefly T-converge) if for every ball $B \subset\subset D$,

$$\lim_{j\in\mathcal{A}} \sup_{x\in B} \| f_j(x) - f(x) \| = 0.$$

We write in this case $f = T - \lim_{j\in\mathcal{A}} f_j$. For the finite dimensional case this topology coincides with the compact open topology on D.

Definition 2.2. A family $S = \{F_t\}_{t>0} \subset \mathrm{Hol}(D)$ is said to be a locally uniformly continuous one-parameter semigroup (or briefly a T-continuous semigroup) if it satisfies the semigroup property

(i) $F_{s+t} = F_s \circ F_t, \quad s,t > 0,$

and

(ii) $T - \lim_{t\to 0+} F_t = I|_D$.

Proposition 2.1 ([59]). *Let D be a bounded domain in X and let $S = \{F_t\}_{t>0} \subset \mathrm{Hol}(D)$ be a strongly continuous semigroup. The following conditions are equivalent:*

(a) *S is a T-continuous semigroup;*
(b) *The differences $f_t = (I - F_t)/t$ are uniformly bounded on each subset strictly inside D;*
(c) *For each $x \in D$ there exists the strong limit*

$$\lim_{t\to 0+} \frac{1}{t}(I - F_t)(x) = f(x)$$

which is bounded on each subset strictly inside D.

Since it is clear that if (c) holds, then f belongs to $\mathrm{Hol}(D, X)$, it will be called the holomorphic infinitesimal generator of the semigroup S. It is remarkable that actually

$$f = T - \lim_{t\to 0+} \frac{1}{t}(I - F_t),$$

i.e., the convergence in (c) is actually local uniform convergence over D (see [57]).

Moreover, condition (c) means that a T-continuous semigroup is right-differentiable with respect to the parameter t at zero. This implies, in turn, that F_t is differentiable at each point $t \in \mathbb{R}^+$ and that the function $u(t)(x) = F_t(x)$ is the solution of the Cauchy problem $u'(t)(x) = -f(u(t)(x))$, $u(0) = x$ [57].

Since for the finite dimensional case a continuous semigroup of holomorphic self-mappings is T-continuous, it follows that such a semigroup always has a holomorphic infinitesimal generator (see also [2]).

Let us once again assume that D is a domain in a complex Banach space X and let $\mathcal{G} \subset \mathrm{Hol}(D)$ be any semigroup with respect to the composition operation.

Definition 2.3. We will say that the pair $(D, \mathcal{G})$ has the iteration (Denjoy-Wolff) property if the following hypothesis holds:

(I) If $F \in \mathcal{G}$ has no fixed point in D, then the sequence $\{F^n\}$ is T-convergent to a point on the boundary of D.
We will say that the pair $(D, \mathcal{G})$ has the common fixed point property if the following hypothesis holds:

(II) If $\{F_s\}_{s\in\mathcal{A}} \subset \mathcal{G}$ is a net of commuting mappings in $\mathcal{G}$ such that for each $s \in \mathcal{A}$ the mapping F_s has a fixed point in D, then $\cap_{s\in\mathcal{A}} \mathrm{Fix}_D F_s \neq \emptyset$.

Theorem 2.2 ([58]). *Let D be a bounded domain in X and let $\mathcal{G} \subset$ Hol(D) be a semigroup in Hol(D) such that for the pair $(D, \mathcal{G})$ both hypotheses* (I) *and* (II) *are satisfied. Assume that $S = \{F_t\}_{t\geq 0}$ is a T-continuous one-parameter semigroup such that $F_t \in \mathcal{G}$, $t > 0$.*

(1) *The following assertions are equivalent:*
  (a) *S has a stationary point in D, i.e., $\cap_{t\geq 0} Fix_D F_t \neq \emptyset$;*
  (b) *for some $t_0 > 0$ the mapping F_{t_0} has a fixed point in D;*
  (c) *there is $x \in D$ and a sequence $t_n \to \infty$ such that $\{F_{t_n}(x)\}$ is strictly inside D;*
  (d) *for each $x \in D$ there is a sequence $t_n \to \infty$ such that $\{F_{t_n}(x)\}$ is strictly inside D.*

(2) *If S has no stationary point in D, i.e., $\cap_{t\geq 0} Fix F_t = \emptyset$, then there exists a unique point $e \in \partial D$ such that for all $x \in D$, the net $\{F_t\}$ is T-convergent, as $t \to \infty$, to the point e.*

Now let D be a bounded convex domain in $\mathbb{C}^n$. It is proved in [6] that the pair $(D, \mathrm{Hol}(D))$ has the common fixed point property (II) (Definition 2.3). If, in addition, D is a strongly convex C^2-domain in $\mathbb{C}^n$, then the pair $(D, \mathrm{Hol}(D))$ also has the iteration property (I) (see [1], [2]). Hence we have the following consequence of Theorem 2.2 which was proved earlier by M. Abate [3].

Corollary 2.1. *Let D be a bounded strongly convex C^2-domain in $\mathbb{C}^n$, and let $S = \{F_t\}_{t>0} \subset$ Hol(D) be a continuous semigroup. If S has no stationary point in D, then there is a unique point $e \in \partial D$ such that $\{F_t\}$ is T-convergent, as t tends to infinity, to e.*

Other finite dimensional situations when the pair $(D, \mathcal{G})$, $\mathcal{G} \subset \mathrm{Hol}(D)$, satisfies both the hypotheses (I) and (II) of Definition 2.3 can be found in [1], [2].

In the infinite dimensional case, it seems not to be known even for the Hilbert ball B whether the pair $(B, \mathrm{Hol}(B))$ satisfies hypothesis (II) (the common fixed point property), while hypothesis (I) for this pair fails by Stachura's example (see [66]). If we restrict ourselves to the family $\mathcal{G} \subset \mathrm{Hol}(B)$ of all compact holomorphic self-mappings of B, then the iteration property (I) is satisfied by the pair $(D, \mathcal{G})$ according to the theorem of C.H. Chu and P. Mellon (see [12]). However, for T-continuous semigroups this fact can be useful only in the finite-dimensional case. In the infinite dimensional case there is no T-continuous semigroup of holomorphic self-mappings which contains a compact mapping. Indeed, the peculiarity of a T-continuous semigroup $S = \{F_t\}_{t \geq 0}$ on D is that each element F_t of S is a locally biholomorphic mapping on D.

At the same time, Theorem 1.8 is relevant to our considerations. In view of this theorem one can propose that the following assertion holds.

Conjecture 2.1. Let B be the open unit ball in a Hilbert space H, and let $S = \{F_t\}$ be a one-parameter T-continuous semigroup on B. Suppose that for some $t_0 > 0$, $F_{t_0} \in \mathrm{Aut}(B)$ and has two different fixed points on the boundary ∂B of B. Then $\{F_t\}$ converges, as $t \to \infty$, to one of them.

Now we will turn to the asymptotic behavior of those semigroups which have an interior stationary point. It turns out that in such a situation, one can completely describe the T-convergence of the semigroup using the local spectral properties of the linear semigroup obtained by differentiation at the stationary point.

For a linear operator $A : X \to X$, we will denote the spectrum of this operator by $\sigma(A)$.

Theorem 2.3 ([33], [57]). *Let D be a bounded convex domain in a complex Banach space X and let $S = \{F_t\}_{t \geq 0} \subset \mathrm{Hol}(D)$ be a one-parameter T-continuous semigroup such that $\mathcal{F} = \cap_{t>0} Fix_D F_t$ is not empty. Then the following assertions are equivalent:*

(1) *The semigroup $\{F_t\}$ is T-convergent to a holomorphic mapping R which is a retraction of D onto $\mathcal{F}$.*
(2) *There exist $a \in \mathcal{F}$ and $t_0 > 0$ such that $Im(I - (F_{t_0})'(a))$ is closed and $\sigma((F_{t_0})'(a)) \setminus \{1\} \subset \Delta$.*
(3) *There exists $b \in \mathcal{F}$ such that $Im f'(b)$ is closed and $\{\sigma(f'(b))\} \cap \{\lambda \in \mathbb{C} : Re\lambda = 0,\ \lambda \neq 0\} = \emptyset$, where $f = \lim_{t \to 0+} \frac{1}{t}(I - F_t)$.*
(4) *There is a point $c \in \mathcal{F}$ such that the linear semigroup e^{-tA} uniformly converges to a projection on X, where $A = f'(c)$.*

Moreover, if condition (2), (3) or (4) holds for at least one stationary point of S, all of these conditions are satisfied for all points *of $\mathcal{F}$. In addition, in this case $\mathcal{F}$ is a connected complex submanifold of D, tangent to $Ker(I - f'(a))$ at the point $a \in \mathcal{F}$.*

This theorem generalizes finite dimensional results of M. Abate [5].

Note that even in the one-dimensional case, when an interior fixed point exists, both versions of the Denjoy-Wolff Theorem (discrete and continuous) establish the convergence of the corresponding semigroup (discrete-time or continuous) in all cases except for automorphisms with exactly one fixed point. For higher dimensions, the scope of counterexamples is wider, and one needs to find out some special conditions to establish a retraction method for approximating fixed points. Sometimes those conditions have a theoretical character and it is difficult to verify them.

At the same time, it has turned out that for an arbitrary holomorphic self-mapping, one can construct an associated continuous semigroup which has the same fixed point set and which converges to a retraction onto this set under simpler conditions (for the finite dimensional case this always happens).

More precisely, let D be a bounded convex domain in X and let $F \in \mathrm{Hol}(D)$. It was shown in [57] that $f = I - F$ generates a T-continuous one-parameter semigroup $S = \{F_t\}_{t>0} \subset \mathrm{Hol}(D)$, i.e., the Cauchy problem

$$\begin{cases} \frac{dF_t}{dt} = F(F_t) - F_t \\ F_0 = I \end{cases}$$

has a unique solution on $\mathbb{R}^+$. It is clear that

$$\mathrm{Fix}_D F = \bigcap_{t \geq 0} \mathrm{Fix}_D F_t.$$

We refer to this semigroup as being associated with F.

Theorem 2.4 [33]. *Let D be a convex bounded domain in X and let $F \in$ Hol(D). Suppose that F has a fixed-point a in D such that $Im(I - F'(a))$ is closed. Then the associated semigroup $\{F_t\}_{t\geq 0}$ is T-convergent, as $t \to \infty$, to a holomorphic mapping $R \in$ Hol(D) which is a retraction onto $Fix_D F$.*

In fact, this theorem is a consequence of Theorem 2.3, because the semigroup associated with a holomorphic self-mapping always satisfies the spectral condition (2) of that theorem.

Nevertheless, the problem of establishing a retraction onto the stationary point set of a general semigroup (which is not necessarily differentiable) is still open.

2.2. Implicit continuous versions. Several talks in this conference involved an implicit method for finding a fixed point of a nonexpansive mapping by using the so-called "approximating curves". Actually, this method also works effectively for the class of holomorphic self-mappings (see, for example, [22], [41], [61], [49]), and it can also be useful for continuous semigroups of such mappings.

Throughout what follows let D be a bounded convex domain in X and let $F \in \mathrm{Hol}(D)$. The Earle-Hamilton fixed point theorem shows that the equation

$$(**) \qquad z_\alpha = \alpha F(z_\alpha) + (1-\alpha)z$$

can be solved for each $z \in D$ and each $\alpha \in [0,1)$. This solution, viewed as a function from $[0,1)$ into D, is called an approximating curve.

It is clear that if, for some $z \in D$, z_α strongly converges, as $\alpha \to 1-$, to a point $a \in D$, then this point is a fixed point of F. The converse is not so obvious: *If F has a fixed point, does the approximating curve converge as α tends to* $1-$? Many mathematicians have considered this question in different situations (see, for example, [22], [61], [49]).

As a matter of fact, in fixed point theory, approximating curves have been used not only for self-mappings, but also for the so-called "pseudo-contractive" mappings [52], as well as for other classes of mappings. So, in our investigations, one can extend the problem and ask a general question: *What is the class of holomorphic mappings $F : D \to X$ (not necessarily self-mappings) for which the approximating curve exists for all $z \in D$ and what is the asymptotic behavior of such a curve, when α tends to* $1-$?

Note that if $F \in \mathrm{Hol}(D)$, then $z_\alpha \in D$, as defined by (**), depends holomorphically on $z \in D$. Thus equation (**), rewritten in the form

$$(***) \qquad \mathcal{J}_\alpha(z) = \alpha F \mathcal{J}_\alpha(z) + (1-\alpha)z$$

for each $\alpha \in [0,1)$, defines a holomorphic self-mapping $\mathcal{J}_\alpha \in \mathrm{Hol}(D)$. For each fixed $\alpha \in [0,1)$ the mapping $\mathcal{J}_\alpha$ has the same fixed-point set as F.

The following result gives an answer to the first question mentioned above.

Proposition 2.2 ([57]). *Let D be a convex bounded domain in X and let $F \in \mathrm{Hol}(D,X)$. Then equation $(***)$ has a solution $\mathcal{J}_\alpha \in \mathrm{Hol}(D)$ if and only if the mapping $f = I - F$ is a generator of a T-continuous one-parameter semigroup.*

If D is a homogeneous ball in X, a description of such mappings F in terms of one-sided estimates is given in [34] in this volume.

We will denote this class of holomorphic mappings by $AG(D)$.

Theorem 2.5 ([32], [54]). *Let B be the open unit ball in a complex Hilbert space H, and let $F \in AG(D)$.*

If F has no fixed point in D, then there is a unique point $e \in \partial D$ such that:

(1) *For each $\alpha \in (0,1)$ and each $x \in B$, the iterations $\{\mathcal{J}_\alpha^n(x)\}$ strongly converge to e as $n \to \infty$;*

(2) *The net $\{\mathcal{J}_\alpha(x)\}$ strongly converges to e, as $\alpha \to 1-$, for each $x \in B$.*

Theorem 2.6 ([59]). *Let D be a bounded convex domain in an arbitrary complex Banach space X, and let $F \in AG(D)$ have a fixed point $a \in D$ such that $Im(I - F'(a))$ is closed.*

(1) *For each $\alpha \in (0,1)$, the sequence $\{\mathcal{J}_\alpha^n\}_{n=1}^\infty$ is T-convergent to a retraction R onto the fixed-point set of F in D;*

(2) *The net $\{\mathcal{J}_\alpha\}_{\alpha\in(0,1)}$ is T-convergent to the same retraction as $\alpha \to 1-$.*

3. Nonexpansive mappings with respect to the hyperbolic metric. Recall that all metrics assigned to a convex bounded domain D in a Banach space X by Schwarz-Pick systems of pseudometrics [24] coincide [16]. We call this unique metric the hyperbolic metric of D and denote it by ρ_D or simply by ρ.

This section is devoted to a brief discussion of the asymptotic behavior (in the spirit of the Denjoy-Wolff Theorem) of those mappings (not necessarily holomorphic) and semigroups which are nonexpansive with respect to this metric.

We begin with descrete iterations. By an averaged mapping of the first kind in the Hilbert ball B we mean a mapping $U : B \to B$ of the form $(1-c)I \oplus cT$, where $0 < c < 1$, I is the identity, $T : B \to B$ is ρ-nonexpansive, and $(I - c)x \oplus cy$ denotes a metric convex combination of x and y (see [22], p. 103). A mapping of the form $(1 - c)I + cT$, where again $T : B \to B$ is ρ-nonexpansive and $0 < c < 1$, is an averaged mapping of the second kind.

Theorem 3.1 ([53]). *Let U be an averaged mapping of the first or the second kind in the Hilbert ball B.*

(1) *If U has a fixed point, then for each x in B the sequence of iterates $\{U^n x\}$ converges weakly to a fixed point of U.*

(2) *If U is fixed point free, then for each x in B the sequence of iterates $\{U^n x\}$ converges strongly to $e(U)$, a point on the boundary of B.*

There are other classes of ρ-nonexpansive mappings for which the conclusions of Theorem 3.1 hold. Without going into details, we refer, in particular, to the firmly nonexpansive mappings of the first and second kind [21], [22], [55]. Theorem 9 of [39] is an interesting result for ρ-isometries. We also note that A. Kryczka and T. Kuczumow [38] have recently extended the result of C.H. Chu and P. Mellon [12] (Theorem 1.7) to all compact, ρ-nonexpansive and fixed point free self-mappings of B.

Other results on discrete iterations of ρ-nonexpansive mappings for Euclidean balls, as well as for other finite-dimensional domains, can be found in [65] and in [41].

We now turn our attention to ρ-nonexpansive semigroups.

Theorem 3.2 ([54]). *Let $T : C \to C$ be a ρ-nonexpansive self-mapping of a ρ-closed ρ-convex subset C of (B, ρ), and let S be the semigroup corresponding to T.*

(1) *If T has a fixed point, then for each x in C the weak $\lim_{t\to\infty} S(t)x$ exists and is a fixed point of T.*

(2) *If T is fixed point free, then for each x in C the strong $\lim_{t\to\infty} S(t)x = e(T)$, a point on the boundary of B.*

Related results can be found in [56] and in [58]. As for the behavior of the approximating curves associated with ρ-nonexpansive mappings we refer the reader to [22] and to [61], where the open problem posed on p. 135 of [22] has been settled in the affirmative.

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References

[1] Abate, M., *Horospheres and iterates of holomorphic maps*, Math. Z. **198** (1988), 225-238.

[2] ———, *Converging semigroups of holomorphic maps*, Atti Accad. Naz. Lincei **82** (1988), 223-227.

[3] ———, *Iteration Theory of Holomorphic Maps on Taut Manifolds*, Mediterranean Press, Cosenza, 1989.

[4] ———, *Iteration theory, compactly divergent sequences and commuting holomorphic maps*, Ann. Scuola Norm. Sup. Pisa **18** (1991), 167-191.

[5] ———, *The infinitesimal generators of semigroups of holomorphic maps*, Ann. Mat. Pura Appl. **161** (1992), 167-180.

[6] Abate, M. and J.P. Vigué, *Common fixed points in hyperbolic Riemann surfaces and convex domains*, Proc. Amer. Math. Soc. **112** (1991), 503-512.

[7] Bedford, E., *On the automorphism group of a Stein manifold*, Math. Ann. **266** (1983), 215-227.

[8] Berkson, E. and H. Porta, *Semigroups of analytic functions and composition operators*, Michigan Math. J. **25** (1978), 101-115.

[9] Burckel, R. B., *Iterating analytic self-maps of discs*, Amer. Math. Monthly **88** (1981), 396-407.

[10] Cartan, H., *Sur les rétractions d'une variété*, C.R. Acad. Sci. Paris **303** (1986), 715-716.

[11] Chen, G. N., *Iteration for holomorphic maps of the open unit ball and the generalized upper half-plane of* $\mathbb{C}^n$, J. Math. Anal. Appl. **98** (1984), 305-313.

[12] Chu, C. H. and P. Mellon, *Iteration of compact holomorphic maps on a Hilbert ball*, Proc. Amer. Math. Soc. **125** (1997), 1771-1777.

[13] Cowen, C. and B. D. MacCluer, *Composition Operators on Spaces of Analytic Functions*, CRC Press, Boca Raton, FL, 1995.

[14] Denjoy, A., *Sur l'itération des fonctions analytiques*, C. R. Acad. Sci. Paris **182** (1926), 255-257.

[15] Dineen, S., *The Schwarz Lemma*, Clarendon Press, Oxford, 1989.

[16] Dineen, S., R.M. Timoney and J.-P. Vigué, *Pseudodistances invariantes sur les domains d'un espace localement convexe*, Ann. Scuola Norm. Sup. Pisa **12** (1985), 515-529.

[17] Earle, C. J. and R. S. Hamilton, *A fixed point theorem for holomorphic mappings*, vol. 16, Proc. Symp. Pure Math., 1970, 61-65.

[18] Fan, K., *Iteration of analytic functions of operators I*, Math. Z. **179** (1982), 293-298.

[19] ———, *Iteration of analytic functions of operators II*, Linear and Multilinear Algebra **12** (1983), 295-304.

[20] Goebel, K., *Fixed points and invariant domains of holomorphic mappings of the Hilbert ball*, Nonlinear Anal. **6** (1982), 1327-1334.

[21] Goebel, K. and S. Reich, *Iterating holomorphic self-mappings of the Hilbert ball*, Proc. Japan Acad. **58** (1982), 349-352.

[22] ———, *Uniform Convexity, Hyperbolic Geometry and Nonexpansive Mappings*, Marcel Dekker, New York and Basel, 1984.

[23] Goebel, K., T. Sękowski and S. Stachura, *Uniform convexity of the hyperbolic metric and fixed points of holomorphic mappings in the Hilbert ball*, Nonlinear Anal. **4** (1980), 1011-1021.

[24] Harris, L. A., *Schwarz-Pick systems of pseudometrics for domains in normed linear spaces*, Advances in Holomorphy, North Holland, Amsterdam, 1979, 345-406.

[25] Hayden, T. L. and T. J. Suffridge, *Biholomorphic maps in Hilbert space have a fixed point*, Pacific J. Math. **38** (1971), 419-422.

[26] Heath, L. F. and T. J. Suffridge, *Holomorphic retracts in complex n-space*, Illinois J. Math. **25** (1981), 125-135.

[27] Heins, M. H., *On the iteration of functions which are analytic and single-valued in a given multiply-connected region*, Amer. J. Math. **63** (1941), 461-480.

[28] ———, *A theorem of Wolff-Denjoy type*, Complex Analysis, Birkhäuser, Basel, 1988, 81-86.

[29] Kapeluszny, J., T. Kuczumow and S. Reich, *The Denjoy-Wolff theorem in the open unit ball of a strictly convex Banach space, preprint, 1998.*
[30] Kato, T., *Perturbation Theory for Linear Operators*, Springer, Berlin, 1966.
[31] Khatskevich, V., S. Reich and D. Shoikhet, *Fixed point theorems for holomorphic mappings and operator theory in indefinite metric spaces*, Integral Equations Operator Theory **22** (1995), 305-316.
[32] ———, *Null points of holomorphic generators in the Hilbert ball, in Recent Advances in Metric Fixed Point Theory*, University of Seville Press, 1996, 59-72.
[33] ———, *Asymptotic behavior of solutions of evolution equations and the construction of holomorphic retractions*, Math. Nachr. **189** (1998), 171-178..
[34] ———, *Semi-complete holomorphic vector fields on homogeneous open unit balls in Banach spaces, preprint, 1997.*
[35] ———, *Fixed points of analytic operators in a Banach space and applications*, Siberian Math. J. **25** (1984), 188-200.
[36] ———, *Differentiable Operators and Nonlinear Equations*, Birkhäuser, Basel, 1994.
[37] ———, *One version of the implicit function theorem for holomorphic mappings*, C. R. Acad. Sci. Paris **319** (1994), 599-604.
[38] Kryczka, A. and T. Kuczumow, *The Denjoy-Wolff-type theorem for compact K_{B_H}-nonexpansive maps in a Hilbert ball*, preprint, 1997.
[39] Kuczumow, T. and W. O. Ray, *Isometries in the Cartesian product of n unit open Hilbert balls with the hyperbolic metric*, Ann. Mat. Pura Appl. **152** (1988), 359-374.
[40] Kuczumow, T. and A. Stachura,, *Fixed points of holomorphic mappings in the Cartesian product of n unit Hilbert balls*, Canad. Math. Bull. **29** (1986), 281-286.
[41] ———, *Iterates of holomorphic and k_D-nonexpansive mappings in convex domains in $\mathbb{C}^n$*, Adv. Math. **81** (1990), 90-98.
[42] Kubota, Y., *Iteration of holomorphic maps of the unit ball into itself*, Proc. Amer. Math. Soc. **88** (1983), 476-480.
[43] Laursen, K. B. and M. Mbekhta, *Operators with finite chain legnth and the ergodic theorem*, Proc. Amer. Math. Soc. **123** (1995), 3443-3448.
[44] Lyubich, Yu., *A remark on the stability of complex dynamical systems*, Izv. Vyssh. Uchebn. Zaved. Mat. **27** (1983), 49-50.
[45] Lyubich, Yu. and J. Zemanek, *Precompactness in the uniform ergodic theory*, Studia Math. **112** (1994), 89-97.
[46] Ma, D., *On iterates of holomorphic maps*, Math. Z. **207** (1991), 417-428.
[47] MacCluer, B. D., *Iterates of holomorphic self-maps of the unit ball in $\mathbb{C}^n$*, Michigan Math. J. **30** (1983), 97-106.
[48] Mazet, P. and J.-P. Vigué, *Points fixes d'une application holomorphe d'un domaine borné dans lui-même*, Acta Math. **116** (1991), 1-26.
[49] ———, *Convexité de la distance de Carathéodory et points fixes d'applications holomorphes*, Bull. Sci. Math. **116** (1992), 285-305.
[50] Mellon, P., *Another look at results of Wolff and Julia type for J^*-algebras*, J. Math. Anal. Appl. **198** (1996), 444-457.
[51] Mercer, P. R., *Complex geodesics and iterates of holomorphic maps on convex domains in $\mathbb{C}^n$*, Trans. Amer. Math. Soc. **338** (1993), 201-211.
[52] Morales, C., *Strong convergence theorems for pseudo-contractive mappings in Banach space*, Houston J. Math. **16** (1990), 549-557.
[53] Reich, S., *Averaged mappings in the Hilbert ball*, J. Math. Anal. Appl. **109** (1985), 199-206.

[54] ______, *The asymptotic behavior of a class of nonlinear semigroups in the Hilbert ball*, J. Math. Anal. Appl. **157** (1991), 237-242.

[55] Reich, S. and I. Shafrir, *The asymptotic behavior of firmly nonexpansive mappings*, Proc. Amer. Math. Soc. **101** (1987), 246-250.

[56] ______, *Nonexpansive iterations in hyperbolic spaces*, Nonlinear Anal. **15** (1990), 537-558.

[57] Reich, S. and D. Shoikhet, *Generation theory for semigroups of holomorphic mappings in Banach spaces*, Abstr. Appl. Anal. **1** (1996), 1-44..

[58] ______, *Semigroups and generators on convex domains with the huperbolic metric*, Atti Accad. Naz. Lincei, accepted for publication.

[59] ______, *Metric domains, holomorphic mappings and nonlinear semigroups*, preprint, 1998.

[60] Rudin, W., *Function Theory on the unit ball in* $\mathbb{C}^n$, Springer, Berlin, 1980.

[61] Shafrir, I., *Coaccretive operators and firmly nonexpansive mappings in the Hilbert ball*, Nonlinear Analysis **18** (1992), 637-648.

[62] Shoikhet, D., *On some conditions for solvability of equations with an analytic operator*, Preprint, Institute of Physics, Siberian Branch of the Academy of Sciences of the USSR, 198.

[63] ______, *The invariance principle in the fixed point theory for analytic operators*, Preprint, Institute of Physics, Siberian Branch of the Academy of Sciences of the USSR, No. 33M, 1986.

[64] ______, *Some properties of Fredholm operators in Banach analytic manifolds*, Integral Equations Operator Theory **16** (1993), 430-451.

[65] Sine, R., *Behavior of iterates in the Poincaré metric*, Houston J. Math. **15** (1989), 273-289.

[66] Stachura, A., *Iterates of holomorphic self-maps of the unit ball in Hilbert space*, Proc. Amer. Math. Soc. **93** (1985), 88-90..

[67] Suffridge, T., *Common fixed points of commuting holomorphic maps of the hyperball*, Michigan Math. J. **21** (1974), 309-314.

[68] Suzuki, M., *The fixed point set and the iterational limits of a holomorphic self-map*, Kodai Math. J. **10** (1987), 298-306.

[69] Vesentini, E., *Complex geodesics and holomorphic maps*, Sympos. Math. **26** (1982), 211-230.

[70] ______, *Su un teorema di Wolff e Denjoy*, Rend. Sem. Mat. Fis. Milano **53** (1983), 17-26.

[71] ______, *Iterates of holomorphic mappings*, Uspekhi Mat. Nauk **40** (1985), 13-16.

[72] Vigué, J. P., *Sur les points fixes d'applications holomorphes*, C. R. Acad. Sci. Paris **303** (1986), 927-930.

[73] ______, *Fixed points of holomorphic mappings in a bounded convex domain in* $\mathbb{C}^n$, Proc. Sympos. Pure Math. **52** (1991, Part 2), 579-582.

[74] Włodarczyk, K., *Iteration of holomorphic maps of infinite dimensional homogeneous domains*, Monatsh. Math. **99** (1985), 153-160.

[75] ______, *Studies of iterations of holomorphic maps in* J^**-algebras and complex Hilbert spaces*, Quart. J. Math. **37** (1986), 245-256.

[76] ______, *Julia's lemma and Wolff's theorem for* J^**-algebras*, Proc. Amer. Math. Soc. **99** (1987), 472-476.

[77] Wolff, J., *Sur l'itération des fonctions holomorphes dans une région, et dont les valeurs appartiennent à cette région*, C. R. Acad. Sci., Paris **182** (1926), 42-43.
[78] ———, *Sur l'itération des fonctions bornées*, C. R. Acad. Sci., Paris **182** (1926), 200-201.
[79] ———, *Sur une généralisation d'un théorème de Schwarz*, C. R. Acad. Sci. Paris **182** (1926), 918-920.
[80] Yang, P., *Holomorphic curves and boundary regularity of biholomorphic maps of pseudoconvex domains*, preprint, 1978.

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