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Approximation of Fixed Points for Locally Nonexpansive Mappings

ABSTRACT. Let X and its dual X^* be uniformly convex Banach spaces, D an open and bounded subset of X, and let T be a nonexpansive mapping defined on $\mathrm{cl}(D)$ and taking values in X. If T satisfies the following condition: there exists $z \in D$ such that $\|z - TZ\| < \|x - Tx\|$ for all x on the boundary of D, then the trajectory $t \rightarrow z_t \in D$, $t \in [0,1)$ defined by $z_t = tT(z_t) + (1-t)z$ is continuous and converges strongly to a fixed point of T as $t \rightarrow 1^-$. The paper contains some generalizations of this result.

In 1966, Browder [1] introduced a technique for approximating fixed points of nonexpansive mappings in Hilbert spaces. This idea arises in the context of approximating solutions of nonlinear variational inequalities involving monotone operator theory (Browder [3]). Since then, the problem has been extensively explored in various directions, moreover the diversity of new arguments of proof could be observed.

It is our interest to discuss a formulation of this technique for locally nonexpansive mappings defined in more general Banach spaces. We mainly intend to give an overview of a rather recent work done in this direction. Nevertheless, some new results are also discussed in this context. We should

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also mention that some of these results remain valid for families of operators much more general than those locally nonexpansive one. In fact, pseudo-conctractive mappings fall in this particular category and they will certainly be introduced and defined in the coming discussion.

In the meantime, we mention that throughout the paper we will assume that X is a real Banach space, while its dual will be denoted by X^*. In addition, we shall denote the closure and the boundary of D by $\overline{D}$ and ∂D, respectively, and for $u, v \in X$ we use seg$[u,v]$ to denote the segment $\{tu+(1-t)v : t \in [0,1]\}$. We shall also use $B(x;r)$ and $\overline{B}(x;r)$ to stand for the open ball $\{y \in X : \|x-y\| \leq r\}$, respectively.

Formulation of the problem. Suppose T is a locally nonexpansive mapping defined on a portion of a Banach space X. This means that for each x in the domain $D(T)$ of T there exists a neighbourhood $U(x)$ such that the restriction of T to $U(x)$ is globally nonexpansive (i.e. $\|Tu - Tv\| \leq \|u - v\|$ for all $u, v \in U(x)$).

Let z be an arbitrary point in $D(T)$, and let $t \in [0,1)$. Then we may examine the following questions. Does the implicit equation $x = tT(x) + (1-t)z$ have a solution for each t? And if that is the case, suppose we denote such a solution by x_t. Then the following question appears to be ofinterest, too. Does the net $\{x_t\}$ converge as $t \to 1^-$?

We now begin with the first result of this nature, in our attempt to respond the questions mentioned above, throughout a recollection of some of the work done on this particular subject for the past 30 years. To this end, we start with a result of Browder.

Theorem 1 ([3]). *Let H be a Hilbert space and $T : H \to H$ a nonexpansive mapping. Suppose there exists a bounded closed convex subset C of H that is invariant under T. Let z be an arbitrary point of C. Then the equation*

$$z_t = tT(z_t) + (1-t)z, \tag{1}$$

has a unique solution for each $t \in [0,1)$ and the strong limit $\lim_{t\to 1^-} z_t$ exists and is a fixed point of T.

The very same year, Halpern [7] proved Theorem 1 under the assumption that the nonexpansive self-mapping T should be just defined on the closed unit ball. Although Halpern's result does not seem to be more general, its proof is simpler and more elegant. On the other hand, contrary to Halpern's version, Browder's proof requires that the mapping T should be defined in the whole space X, and this is because of the monotonicity argument used in his proof. Incidentally, this argument does not carry over under Halpern's formulation either.

However, in an attempt to remove the self-mapping assumption and the convexity of the domain of T, Kirk and Morales [11] obtained in 1981 a significant extension of this result for locally nonexpansive mappings which are not necessarily self-mappings, but still valid for Hilbert spaces.

Theorem 2 ([11]). *Let H be a Hilbert space, D a bounded open subset of H, and $T : \overline{D} \to H$ a continuous mapping which is locally nonexpansive on D. Suppose there exists $z \in D$ such that*

$$(2) \qquad \|z - Tz\| < \|x - Tx\| \quad \text{for all } x \in \partial D.$$

Then there exists a unique path $t \to z_t \in D$, $t \in [0,1)$, satisfying $z_t = t\,T(z_t) + (1-t)z$, where the strong limit $\lim_{t\to 1^-} z_t$ exists which a fixed point for T.

We should observe that this is precisely in [11] where the notion of a continuous path, as defined by (1), was explicitly mentioned and proved. Although the notion was implicitly present in a previous work of Kirk and the writer [10]. Shortly after this result was known, Bruck, Kirk and Reich [5] obtained another important extension to a certain class of Banach spaces which include the L^p-spaces and this can be stated as follows.

Theorem 3 ([5]). *Let X be a Banach space that is both uniformly convex and uniformly smooth, D a bounded open subset of X and $T \to \overline{D} \to X$ a continuous mapping which is locally nonexpansive on D. Suppose z is a point of D for which $\|z - Tz\| < \|x - Tx\|$ for all $x \in \partial D$. Then there exists a unique path $t \to z_t \in D$, $t \in [0,1)$, satisfying $z_t = t\,T(z_t) + (1-t)z$, where the strong limit $\lim_{t\to 1^-} z_t$ exists which is a fixed point for T.*

Not until 1990, Morales [14] extended this result to locally pseudo-contractive mappings for the same class of restricted Banach spaces as in the above mentioned Theorem 3. However, local uniform continuity on T was required, which, by the way, is also implicitly imposed in [5].

We now define the notion of pseudo-contractivity which was originally introduced by Browder [2] back in 1967. It is worth mentioning that operators of this type are intimately related to the so-called accretive operators, independently introduced by Browder [2] and Kato [8] in the same year.

Due to the close connection between nonexpansive and pseudo-contractive mappings, we recall the definition of the latter one, since our next result involves this family of operators.

Let X be a real Banach space and let D be a subset of X. An operator $T : D \to X$ is said to be **pseudo-conctractive** if for each $x, y \in D$ there exists $j \in J(x-y)$ such that

$$(3) \qquad < T(x) - T(y), j > \leq \|x - y\|^2,$$

where the mapping $J : X \to 2^{X^*}$ is the normalized duality mapping defined by $J(x) = \{j \in X^* :< x, j >= \|x\|^2, \ \|j\| = \|x\|\}$.

Due to Lemma 1.1 of [8], we may derive from the inequality (3) the original definition of pseudo-contractive mappings T, as given by Browder [2]: $\|x - y\| \leq \|(1 + r)(x - y) - r(Tx - Ty)\|$ holds for all $x, y \in D$ and all $r > 0$.

Furthermore, if the condition (3) holds locally, i.e., if each point $x \in D$ has a neighborhood $U(x)$ such that the restriction of T to $U(x)$ is pseudo-contractive, then T is said to be **locally pseudo-contractive**. Under this framework we have the following result.

Theorem 4 ([14]). *Let X and X^* be uniformly convex Banach spaces, let D be a bounded open subset of X and let $T : \overline{D} \to X$ be a uniformly continuous mapping which is locally pseudo-contractive on D. Suppose there exists $z \in D$ such that $\|z - Tz\| < \|x - Tx\|$ for all $x \in \partial D$. Then there exists a unique path $t \to z_t \in D$, $t \in [0, 1)$, satisfying $z_t = t\,T(z_t) + (1-t)z$, where the strong limit $\lim_{t\to 1^-} z_t$ exists and is a fixed point for T.*

However, recently in 1995, Mutangadura and the author, were able to drop the uniform continuity condition on the operator T, and replace this merely by continuity. We succeeded in proving the following

Theorem 5 ([16]). *Let X and X^* be uniformly convex Banach spaces, let D be a bounded open subset of X, and let $T : \overline{D} \to X$ be a continuous mapping which is locally pseudo-contractive on D. Suppose there exists $z \in D$ such that $\|z - Tz\| < \|x - Tx\|$ for all $x \in \partial D$. Then there exists a unique path $t \to z_t \in D$, $t \in [0, 1)$ satisfying*

$$(4) \qquad z_t = t\,T(z_t) + (1 - t)z,$$

where the strong limit $\lim_{t\to 1^-} z_t$ exists and is a fixed point for T.

As we said earlier, the existence of the path $t \to z_t$ was previously established by Kirk-Morales [10] for general Banach spaces. Therefore it is the strong convergence of this path which is actually at stake.

The proof of Theorem 5 requires a different type of argument and in that process we have discovered the following result which appears to be interesting for its own sake and is at the same time quite useful in the localized theory of accretive operators.

Proposition 1. *Let X be a Banach space, let D be a connected open subset of X, and let $T : \overline{D} \to X$ be a continuous mapping which is locally pseudo-contractive on D. Then the mapping $Fx = 2x - Tx$ is globally one-to-one on D.*

The detailed proof of this proposition can be found in [16]. On the other hand, concerning the convergence of the path described by (4), we should mention that recently in 1996, the author obtained, perhaps one of the most general results of this type for locally nonexpansive mappings, which can be stated as follows:

Theorem 6. *Let X^* be a uniformly convex Banach space, let D be a bound-
ed open subset of X, and let $T : \overline{D} \to X$ be a continuous mapping which is locally nonexpansive on D. Suppose there exists $z \in D$ such that $\|z - Tz\| < \|x - Tx\|$ for all $x \in \partial D$. Then there exists a unique path $t \to z_t \in D$, $t \in [0,1)$ satisfying*

$$z_t = t\,T(z_t) + (1-t)z\,, \tag{4}$$

where the strong limit $\lim_{t\to 1^-} z_t$ exists and is a fixed point for T.

Actually the latter result is used to obtain, under the same framework, an even further extension for locally pseudo-contractive mappings and the reader can find such a discussion in [15]. However, we now turn around and we ask ourselves whether it is possible to obtain a convergence result of this nature for operators defined on domains whose interior is not necessarily non-empty,

and the answer is **tak!**

Theorem 7. *Let X^* be a uniformly convex Banach space. Suppose K is a closed and convex subset of X and let $T : K \to X$ be a nonexpansive mapping on K with $T(K)$ bounded and $T(\partial K) \subset K$. Then for each $z \in K$ there exists a unique path $t \to z_t \in K$, $t \in [0,1)$, satisfying $z_t = t\,Tz_t + (1-t)z$, where the strong limit $\lim_{t\to 1^-} z_t$ exists and is a fixed point of T.*

Nevertheless, in 1990 (see [14]), we obtained a quite significant extension of Theorem 7, for pseudo-contractive mappings under an even weaker boundary condition than $T(\partial K) \subset K$. This condition is known as the weakly inward condition, which can be formulated as follows:

A mapping $T : K \to X$ is said to satisfy the **weakly inward condition** if $Tx \in \overline{I_D(x)}$ for all $x \in D$, where $I_D(x)$ is the inward set relative to D defined as:

$$I_D(x) = \{(1-\lambda)x + \lambda y : \lambda \geq 0,\ y \in D\}.$$

Then we have

Theorem 8 ([14]). *Let X be a Banach space and let X^* be uniformly convex. Suppose K is a bounded closed and convex subset of X, and let $T : K \to X$ be a pseudo-contractive mapping satisfying the weakly inward condition. Then for each $z \in K$ there exists a unique path $t \to z_t \in K$, $t \in [0,1)$, satisfying $z_t = tTz_t + (1-t)z$, where the strong limit $\lim_{t\to 1^-} z_t$ exists and is a fixed point of T.*

Perhaps, we should use this opportunity to mention that [14] has been apparently overlooked by some of the specialists in the area. Recently a result on Hilbert spaces, which is a particular case of Theorem 5 of [14], has been published in [18, Theorem 1], and followed by an additional extension of the same author (see [19]), which is still a particular case of Theorem 5 of [14].

Let us go back to Theorem 6 for a comment concerning its proof. Incidentally, results of this type normally require the existence of a fixed point to prove the convergence of the path described by (4). Nevertheless, the proofs of Theorems 7 and 8 do not explicitly need the existence of a fixed point. However, in the case of Theorem 6, we do require an existence theorem, which can be stated as follows:

Theorem 9. *Let X^* be a uniformly convex Banach space, let D be a bound- ed open subset of X, and let $T : \overline{D} \to X$ be a continuous mapping which is locally nonexpansive on D. Suppose there exists $z \in D$ such that $\|z - Tz\| < \|x - Tx\|$ for all $x \in \partial D$. Then T has a fixed point in D.*

We should point out that Theorem 9 holds true for pseudo-contractive mappings where the closed unit ball of the Banach space X enjoys the F.P.P. for nonexpansive self-mappings. In fact, this is a corollary of Theorem 1 of [15].

As a consequence of Theorem 9 and this observation, we can state and prove the very same result in a slightly more general fashion, as it can be seen in our next result.

Theorem 10. *Let X be a Banach space for which the closed unit ball has the fixed point property for nonexpansive self-mappings, let D be a bounded open subset of X and let $T : \overline{D} \to X$ be a closed mapping which is locally nonexpansive on D. Suppose there exists $z \in D$ such that $\|z - Tz\| < \|x - Tx\|$ for all $x \in \partial D$. Then T has a fixed point in D.*

Proof. Due to Corollary 1 of [17] we derive that $\inf\{\|x - Tx\| : x \in D\} = 0$. Then we may choose $w \in D$ such that $\|w - Tw\| < \|z - Tz\|$.

We now define the set $D_0 = \{x \in D : \|x - Tx\| < \|z - Tz\|\}$. Therefore $D_0 \neq \emptyset$ (since $w \in D_0$) and $\partial D_0 \subset D$. In adition, $\|w - Tw\| < \|x - Tx\|$ for all $x \in \partial D_0$. This means, we are in position of applying Theorem 9 to T defined on ∂D_0. Therefore T has a fixed point in D_0.

Perhaps a more interesting question is, whether under the same assumptions on T as in Theorem 10, the path described by (4) converges to a fixed point of T, while z satisfies the equation (2). The answer is yes, provided the space X enjoys some kind of smoothness.

Theorem 11. *Let X^* be a uniformly convex Banach space, let D be a bounded open subset of X, and let $T : \overline{D} \to X$ be a closed mapping which is locally nonexpansive on D. Then there exists a unique path $t \to z_t \in D$, $t \in [0,1)$, satisfying*

$$z_t = tTz_t + (1-t)z, \tag{5}$$

where the strong limit $\lim_{t\to 1^-} z_t$ exists and is a fixed point of T.

The proof of Theorem 11 rests upon some preliminary facts which are similar to those obtained, when T is assumed to be continuous on all $\overline{D}$. We should also mention that these facts will be stated and proven for pseudo-contractive mappings.

Proposition 2. *Let X be a Banach space, D a bounded open subset of X, and $T : \overline{D} \to X$ be a closed mapping which is continuous and locally pseudo-contractive on D. For a fixed $z \in D$, suppose $sT(z_t) + (1-s)z = z_s$ for some $z_s \in D$ and $s \in (0,1)$, and suppose for $0 < \sigma \leq \|Tz_s - z\|$, $B(z_s;\sigma) \subset D$. Then for each $t \in (0,1)$ satisfying*

$$|t - s| < \sigma s(1-s)/2\|z_s - z\|, \tag{6}$$

there exists a unique point $z_t \in B(z_s;\sigma)$ such that $z_t = tTz_t + (1-t)z$ and moreover, this point satisfies

$$\|z_t - z_s\| \leq \|z_s - z\|\,|t-s|/s(1-t). \tag{7}$$

Proof. Let $B = B(z_s;\sigma)$. Since T is globally pseudo-contractive on $\overline{B}$, we shall show that the mapping $T_t(x) = tTx + (1-t)z$ satisfies the Leray-Schauder condition on ∂B for those t satisfying (6). To see this, suppose that for some $\alpha > 0$ and $x \in \partial B$ we have $T_t(x) - z_s = \alpha(x - z_s)$. Then there exists $j \in J(x - z_s)$ such that $< tT(x) - tT(z_s), j > \leq t\|x - z_s\|^2$, which implies

(8) $(\alpha - t)\|x - z_s\|^2 \leq (1 - ts^{-1}) < z - z_s, j > \leq |t - s|\,\|z - z_s\|\,\|x - z_s\|/s.$

Therefore, $(\alpha - t)\sigma \leq |t - s|\,\|z - z_s\|/s$.

Due to the choice of σ and the fact that t satisfies (6), we derive that $\alpha - t < (1-s)/2$ and thus $\alpha < 1$. Therefore, by Theorem 1 of [12], the mapping T_t has a fixed point in $\overline{B}$. This means equation (5) has a solution which can be denoted by z_t. Now, by selecting $\alpha = 1$ and $x = z_t$ in (8), we easily see that z_t satisfies (7) and the proof is complete.

W should observe that the next proposition is in some sense slightly more general than Lemma 3 of [14], although, for the sake of completion we include its proof.

Proposition 3. *Let X be a Banach space, D be a bounded open subset of X, and $T : \overline{D} \to X$ be a closed mapping, which is continuous and locally pseudo-contractive on D. Suppose there exists $z \in D$ such that $\|z - Tz\| < \|x - Tx\|$ for all $x \in \partial D$. Then there exists a unique continuous path $t \to z_t \in D$, $t \in [0,1)$, satisfying*

$$z_t = tTz_t + (1-t)z. \tag{9}$$

Proof. For $t \in [0,1)$, let $T_t : \overline{D} \to X$ be defined by $T_t(x) = tTx + (1-t)z$. Since T is continuous at z, for an arbitrary $\varepsilon > 0$ we may select $\delta > 0$ so that $T(B(z;\delta)) \subset B(Tz;\varepsilon)$. Hence we may choose $\sigma \in (0,1)$ such that T_t maps the closed ball $\overline{B}(z;\delta)$ into itself for all $t \in [0,\sigma)$. This implies that T_t has a unique fixed point $z_t \in \overline{B}(z;\delta)$ (see Theorem 1 of [12], and also Proposition 1 of [10]) satisfying the equation (9) for $t \in [0,\sigma)$. In the light of Proposition 2, it remains to show that (9) has a solution for $t = \sigma$. To see this, let $t_n \to \sigma$ such that $z_{t_n} = t_nTz_{t_n} + (1-t_n)z$. Since T is globally pseudo-contractive and continuous on $\partial B(z;\delta)$ (see Proposition 1 of [10]), there exists $j \in J(z_{t_n} - z_{t_m})$ such that

$$\begin{aligned}< z_{t_n} - z_{t_m}, j > &= < t_nTz_{t_n} - t_mTz_{t_m} + (t_m - t_n)z, j > \\ &= (t_n - t_m) < Tz_{t_n} - z, j > + t_m < Tz_{t_n} - Tz_{t_n} - Tz_{t_m}, j > \\ &\leq \|j\| \left[|t_n - t_m|\,\|Tz_{t_n} - z\| + t_m\|z_{t_n} - z_{t_m}\| \right].\end{aligned}$$

Therefore $\|z_{t_n} - z_{t_m}\| \leq |t_n - t_m| \, \|Tz_{t_n} - z\|/(1-\sigma)$ and since the sequence $\{Tz_{t_n} - z\}$ is bounded, we infer that $\{z_{t_n}\}$ is a Cauchy sequence, which must converge to a point z_σ. Since $\overline{B}(z;\delta) \subset D$ and T is continuous on D, we conclude that $z_\sigma = \sigma T(z_\sigma) + (1-\sigma)z$.

Proof of Theorem 11. As we point out in the proof of Theorem 10, we allude to Corollary 1 of [17] to derive that $\inf\{\|x - Tx\| : x \in D\} = 0$. Then we may choose $w \in D$ such that

$$\|w - Tw\| < \|z - Tz\|. \tag{10}$$

We now define the set $D_0 = \{x \in D : \|x - Tx\| < \|z - Tz\|\}$. Therefore $D_0 \neq \emptyset$ (since $w \in D_0$) and $\partial D_0 \subset D$. In addition, $\|w - Tw\| < \|x - Tx\|$ for $x \in \partial D_0$. This means that T is continuous on $\overline{D}_0$, and therefore by Theorem 6, the path $t \to w_t$ (defined by (9) after substituting z by w) for which w satisfies (10) exists, and is uniquely defined on $[0,1)$, and hence the strong $\lim_{t\to 1^-} w_t$ also exists. Since this holds for every $w \in D_0$ which satisfies (10), we choose a sequence $\{z^n\}$ in D_0 such that $z^n \to z$.
Then for each z^n, the corresponding path can be written as $z_t^n = t\,T(z_t^n) + (1-t)z^n$, $t \in [0,1]$. From here we follow the argument used in the proof of Theorem 1 of [16] to prove the convergence of the path $t \to z_t$, which satisfies (9), to a fixed point of T, as $t \to 1^-$.

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