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An Application of Opial's Modulus to the Fixed Point Theory of Semigroups of Lipschitzian Mappings

ABSTRACT. In this paper we present a new theorem concerning the existence of common fixed points of asymptotically regular and uniformly Lipschitzian semigroups.

1. Introduction. Let $(X, \|\cdot\|)$ be a Banach space and let Λ be a family of sequences in X. The family Λ is called a family of convergent sequences [12], [19] if Λ satisfies the following conditions

(i) Λ is a linear space,
(ii) each $\{x_n\} \in \Lambda$ is bounded,
(iii) if $\{x_n\} \in \Lambda$, then each one of its subsequences $\{x_{n_i}\}$ also belongs to Λ,
(iv) there exists a limit function Λ-$\lim : \Lambda \longrightarrow X$ which is linear,
(v) if $x_n = x$ for $n = 1, 2, \ldots$, then $\{x_n\} \in \Lambda$ and Λ-$\lim x_n = x$,
(vi) if $\{x_n\} \in \Lambda$ and Λ-$\lim x_n = x$, then Λ-$\lim x_{n_i} = x$ for every subsequence $\{x_{n_i}\}$ of $\{x_n\}$,

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(vii) each norm convergent sequence $\{x_n\}$ has a subsequence $\{x_{n_i}\}$ such that $\{x_{n_i}\} \in \Lambda$ and $\lim x_{n_i} = \Lambda\text{-}\lim x_{n_i}$,

(viii) the norm $\|\cdot\|$ is lower semicontinuous with respect to Λ, i.e.,

$$\|\Lambda\text{-}\lim x_n\| \leq \liminf_n \|x_n\|$$

for each $\{x_n\} \in \Lambda$,

(ix) Λ has sequences which are not norm convergent.

We say that a nonempty bounded subset C of X is sequentially Λ - compact if C is closed with respect to Λ-lim and every sequence $\{x_n\}$ in C has a subsequence $\{x_{n_i}\}$ which belongs to Λ.

We will use the following notation. If $\{x_n\}_{n\geq 1}$ is a bounded sequence and $x \in X$, then

$$r(x, \{x_n\}) = \limsup_{n\to\infty} \|x - x_n\|.$$

A Banach space X is said to satisfy the non-strict Λ-Opial condition [14], [22] if whenever a sequence $\{x_n\} \in \Lambda$ and $\Lambda\text{-}\lim x_n = x$, then

$$\liminf_{n\to\infty} \|x - x_n\| \leq \liminf_{n\to\infty} \|y - x_n\|$$

for every $y \in X$.

Now we define the Opial modulus $r_{X,\Lambda}$ of X with respect to the family Λ [21], [23] by $r_{X,\Lambda}(c) = \inf\{\liminf_{n\to\infty} \|x_n + x\| - 1\}$, where $c \geq 0$ and the infimum is taken over all $x \in X$ with $\|x\| \geq c$ and all sequences $\{x_n\} \in \Lambda$ such that $\Lambda\text{-}\lim x_n = 0$ and $\liminf_{n\to\infty} \|x_n\| \geq 1$. The function $r_{X,\Lambda}$ is continuous and nondecreasing [21], [24].

If for $s > 0$ and $c \geq 0$ we denote $\inf\{\liminf_{n\to\infty} \|x_n + x\| - s\}$ by $r_{X,\Lambda,S}(c)$, where the infimum is taken over all $x \in X$ with $\|x\| \geq c$ and all sequences $\{x_n\} \in \Lambda$ such that $\Lambda\text{-}\lim x_n = 0$ and $\liminf_{n\to\infty} \|x_n\| \geq s$, then we have

$$s + r_{X,\Lambda,s}(c) = s\left(1 + r_{X,\Lambda}\left(\frac{c}{s}\right)\right). \tag{1.1}$$

Now let (X, d) be a metric space and $T : X \to X$. We use the symbol $|T|$ to denote the exact Lipschitz constant of T, i.e.,

$$|T| = \inf\{k \in [0, \infty] : d(Tx, Ty) \leq k d(x, y) \text{ for all } x, y \in X\}.$$

If G is an unbounded subset of $[0, \infty)$ satisfying $t + h \in G$ for all $t, h \in G$, $t - h \in G$ for all $t, h \in G$ with $t \geq h$, and $\Xi = \{T_t : t \in G\}$ is a family of self-mappings on X such that $T_{s+t}x = T_sT_tx$ for all $s,t \in G$ and $x \in C$, then Ξ is called a semigroup of mappings on X.

Ξ is said to be uniformly Lipschitzian if there exists $k \in \mathbb{R}_+$ such that $|T_t| \leq k$ for each $t \in G$ [13], [14].

We also use the following notation:

$$\sigma(\Xi) = \liminf_{n\to\infty} |T_t|.$$

If Ξ satisfies, in addition, $\lim_{t\to\infty} d(T_{t+h}x, T_tx) = 0$ for each $x \in \Xi$ and $h \in G$, then Ξ is said to be asymptotically regular [4].

2. Existence of common fixed points of semigroups of mappings. Our main result is the following theorem.

Theorem 2.1. *Let X be a Banach space with $r_{X,\Lambda}(1) > 0$ and with the non-strict Λ-Opial property. Let C be a sequentially Λ- compact subset of X and $\Xi = \{T_t : t \in G\}$ an asymptotically regular semigroup with*

$$\sigma(\Xi) = k < 1 + r_{X,\Lambda}(1). \tag{2.1}$$

Then there exists z in C such that $T_tz = z$ for all $t \in G$.

Proof. Let us select a sequence $\{t_n\}$ and $0 < c < 1$ such that $\sigma(\Xi) = k = \lim_{n\to\infty} |T_{t_n}|$, $t_n \to \infty$, and

$$\sup_n |T_{t_n}| < 1 + r_{X,\Lambda}(c) < 1 + r_{X,\Lambda}(1). \tag{2.2}$$

This is possible by (2.1) and the continuity of $r_{X,\Lambda}$. First we claim that if for $x \in C$ a subsequence $\{T_{t_{n_i}}x\}$ of the sequence $\{T_{t_n}x\}$ is Λ-convergent to y, $\{T_{t_{n_i}}y\}$ is Λ-convergent to z and all the limits

$$\begin{aligned} r(y, \{T_{t_{n_i}}x\}) &= \lim_{i\to\infty} \|y - T_{t_{n_i}}x\|, \\ r(y, \{T_{t_{n_i}}y\}) &= \lim_{i\to\infty} \|y - T_{t_{n_i}}y\|, \\ r(z, \{T_{t_{n_i}}y\}) &= \lim_{i\to\infty} \|z - T_{t_{n_i}}y\| \end{aligned} \tag{2.3}$$

exist, then

$$\begin{aligned} r(z, \{T_{t_{n_i}}y\}) &= \lim_{i\to\infty} \|z - T_{t_{n_i}}y\| \leq c\, r(y, \{T_{t_{n_i}}x\}) \\ &= c \lim_{i\to\infty} \|y - T_{t_{n_i}}x\|. \end{aligned} \tag{2.4}$$

Let us suppose this were not the case. Then, after deleting a finite number of indices if necessary - see the limit which appears in (2.3) - we have

$$\inf_j \left\| z - T_{t_{n_j}} y \right\| > cr\left(y, \left\{T_{t_{n_i}} x\right\}\right). \tag{2.5}$$

Let us observe that $r\left(y, \left\{T_{t_{n_i}} x\right\}\right) = 0$ leads to

$$\begin{aligned} \left\| y - T_{t_{n_j}} y \right\| &\leq r\left(y, \left\{T_{t_{n_i}} x\right\}\right) + r\left(T_{t_{n_j}} y, \left\{T_{t_{n_i}} x\right\}\right) \\ &\leq \left| T_{t_{n_j}} \right| \cdot r\left(y, \left\{T_{t_{n_i} - t_{n_j}} x\right\}\right) \end{aligned}$$

and by the asymptotic regularity of Ξ we obtain $\left\| y - T_{t_{n_j}} y \right\| = 0$ for $j = 1, 2, \ldots$, and therefore $y = z$. But this contradicts (2.5). Hence

$$r\left(y, \left\{T_{t_{n_i}} x\right\}\right) > 0. \tag{2.6}$$

The asymptotic regularity of Ξ, the non-strict Opial property, the monotonicity and the continuity of $r_{X,\Lambda}$, and the application of (1.1), (2.2), (2.3), (2.5) and (2.6) now yield the following contradiction:

$$\begin{aligned} \left[1 + r_{X,\Lambda}(c)\right] \cdot r\left(y, \left\{T_{t_{n_i}} x\right\}\right) &> \sigma(\Xi) \cdot r\left(y, \left\{T_{t_{n_i}} x\right\}\right) \\ &\geq \limsup_{j \to \infty} r\left(T_{t_{n_j}} y, \left\{T_{t_{n_i}} x\right\}\right) \\ &\geq \limsup_{j \to \infty} \left[1 + r_{X,\Lambda}\left(\frac{\left\| y - T_{t_{n_j}} y \right\|}{r\left(y, \left\{T_{t_{n_i}} x\right\}\right)}\right)\right] \cdot r\left(y, \left\{T_{t_{n_i}} x\right\}\right) \\ &= \left[1 + r_{X,\Lambda}\left(\frac{\lim_{j \to \infty} \left\| y - T_{t_{n_j}} y \right\|}{r\left(y, \left\{T_{t_{n_i}} x\right\}\right)}\right)\right] \cdot r\left(y, \left\{T_{t_{n_i}} x\right\}\right) \\ &\geq \left[1 + r_{X,\Lambda}\left(\frac{\lim_{j \to \infty} \left\| z - T_{t_{n_j}} y \right\|}{r\left(y, \left\{T_{t_{n_i}} x\right\}\right)}\right)\right] \cdot r\left(y, \left\{T_{t_{n_i}} x\right\}\right) \\ &\geq \left[1 + r_{X,\Lambda}(c)\right] \cdot r\left(y, \left\{T_{t_{n_i}} x\right\}\right). \end{aligned}$$

Therefore the inequality (2.4) is valid. Now using the standard diagonalization procedure we can construct a sequence $\{x_l\} \subset C$ in the following way: $x_0 \in C$ arbitrary, $x_l = \Lambda\text{-}\lim T_{t_n} x_{l-1}$ for $l = 1, 2, \ldots$, where all the limits

$$r\left(x_{l+1}, \left\{T_{t_{n_i}} x_l\right\}\right) = \lim_{i \to \infty} \left\| x_{l+1} - T_{t_{n_i}} x_l \right\|$$

and

$$r\left(x_{l+1}, \left\{T_{t_{n_i}} x_{l+1}\right\}\right) = \lim_{i \to \infty} \left\| x_{l+1} - T_{t_{n_i}} x_{l+1} \right\|$$

for $l = 0, 1, \ldots$ exist. By (2.4) we have

$$r\left(x_{l+1}, \{T_{t_{n_i}} x_l\}\right) \le c^l r\left(x_1, \{T_{t_{n_i}} x_0\}\right) \tag{2.7}$$

for $l = 0, 1, \ldots$. Next by the Λ-lower semicontinuity of the norm, the asymptotic regularity of Ξ and (2.7) we obtain

$$\begin{aligned} \|x_{l+1} - x_l\| &\le r\left(x_{l+1}, \{T_{t_{n_i}} x_l\}\right) + r\left(x_l, \{T_{t_{n_i}} x_l\}\right) \le r\left(x_{l+1}, \{T_{t_{n_i}} x_l\}\right) \\ &+ \limsup_{i\to\infty} \limsup_{j\to\infty} \left\|T_{t_{n_j}} x_{l-1} - T_{t_{n_i}} x_l\right\| \le r\left(x_{l+1}, \{T_{t_{n_i}} x_l\}\right) \\ &+ \limsup_{i\to\infty} \limsup_{j\to\infty} \left[\left\|T_{t_{n_j}} x_{l-1} - T_{t_{n_i}+t_{n_j}} x_{l-1}\right\|\right. \\ &+ \left.\left\|T_{t_{n_i}+t_{n_j}} x_{l-1} - T_{t_{n_i}} x_l\right\|\right] \le r\left(x_{l+1}, \{T_{t_{n_i}} x_l\}\right) \\ &+ k \cdot r\left(x_l, \left\{T_{t_{n_j}} x_{l-1}\right\}\right) \le c^{l-1} \cdot (c+k) \cdot r\left(x_1, \{T_{t_{n_i}} x_0\}\right) \end{aligned}$$

for $l = 1, 2 \ldots$, which shows that $\{x_l\}$ is strongly convergent to $\overline{x}$. By (2.7) for this $\overline{x}$ we get

$$\begin{aligned} r\left(\overline{x}, \{T_{t_{n_i}} \overline{x}\}\right) \le \lim_{l\to\infty} \lim_{i\to\infty} &\left[\|\overline{x} - x_{l+1}\| + \|x_{l+1} - T_{t_{n_i}} x_l\|\right. \\ &+ \left.\left|T_{t_{n_i}}\right| \cdot \|x_l - \overline{x}\|\right] = 0. \end{aligned}$$

The asymptotic regularity of Ξ and $\left|T_{t_{n_i}}\right| < \infty$ imply that $T_{t_{n_i}} \overline{x} = \overline{x}$ for $i = 1, 2, \ldots$. Now we apply the asymptotic regularity of Ξ once more to obtain

$$\|T_t \overline{x} - \overline{x}\| = \lim_{i\to\infty} \|T_{t+t_{n_i}} \overline{x} - T_{t_{n_i}} \overline{x}\| = 0$$

for each $t \in G$. The proof is complete. ■

Remark 2.1. Theorem 2.1 is a generalization of Theorem 3.2 in [20].

3. The case of $L^1(0,1)$ with the topology of pointwise convergence.

H. Brezis and E. Lieb [3] (see also [2]) proved that in $L^1(0,1)$ if $f_n \to f$ a.e. and the sequence $\{f_n\}$ is bounded in $L^1(0,1)$, then

$$\lim_n (\|f_n\| - \|f_n - f\|) = \|f\|.$$

This implies that for the family Λ of pointwise convergent and bounded in $L^1(0,1)$ sequences we have $r_{L^1(0,1),\Lambda}(c) = c$ and

$$1 + r_{L^1(0,1),\Lambda}(1) = 2. \tag{3.1}$$

We obtain the same result if we consider the family $\widetilde{\Lambda}$ of convergent in measure and bounded in $L^1(0,1)$ sequences [16]. It is obvious that $\Lambda \subsetneqq \widetilde{\Lambda}$, but the family of all sequentially Λ-compact sets and the family of all sequentially $\widetilde{\Lambda}$-compact sets coincide in $L^1(0,1)$ [11]. In view of the equality (3.1), Theorem 2.1 is especially interesting if we recall Alspach's example [1] of a fixed point free nonexpansive selfmapping of a convex weakly compact subset of $L^1(0,1)$.

4. Common fixed points of commuting asymptotically regular and uniformly Lipschitzian mappings. In this section we establish the existence of common fixed points of commuting asymptotically regular and uniformly Lipschitzian mappings.

Theorem 4.1. *Let (X,d) be a metric space and let $k > 0$ be a constant such that every asymptotically regular and uniformly Lipschitzian selfmapping $T : X \to X$ with $\sup_n |T^n| < k$ has a fixed point. If $T_1,\ T_2 : X \to X$ are two commuting asymptotically regular and uniformly Lipschitzian mappings such that $\sup_n |T_1^n| < k_1$, $\sup_n |T_2^n| < k_2$ and $k_1 \cdot k_2 < k$, then T_1 and T_2 have a common fixed point.*

Proof. First we observe that $T = T_2 \circ T_1$ is an asymptotically regular and uniformly Lipschitzian mapping. Indeed, for each $x, y \in X$ and $n = 1, 2, \ldots$ we have

$$d(T^n x, T^n y) = d(T_2^n T_1^n x, T_2^n T_1^n y) \le (k_2 \cdot k_1) \cdot d(x,y),$$

and

$$\begin{aligned} d\left(T^{n+1}x, T^n x\right) &= d\left(T_2^{n+1}T_1^{n+1}x, T_2^n T_1^n x\right) \\ &\le d\left(T_2^{n+1}T_1^{n+1}x, T_2^{n+1}T_1^n x\right) + d\left(T_2^{n+1}T_1^n x, T_2^n T_1^n x\right) \\ &\le k_2 d\left(T_1^{n+1}x, T_1^n x\right) + d\left(T_1^n T_2^{n+1}x, T_1^n T_2^n x\right) \\ &\le k_2 d\left(T_1^{n+1}x, T_1^n x\right) + k_1 d\left(T_2^{n+1}x, T_2^n x\right). \end{aligned}$$

Hence $|T^n| \le k_1 k_2 < k$ for all n and $\lim_n d\left(T^{n+1}x, T^n x\right) = 0$.

By assumption there exists a fixed point of T. Now we show that every such point x_0 is a common fixed point of T_1 and T_2. To this end, we observe that

$$d(T_1x_0, x_0) = d(T_1T^nx_0, T^nx_0) = d\left(T_2^nT_1^{n+1}x_0, T_2^nT_1^nx_0\right)$$
$$\leq k_2 d\left(T_1^{n+1}x_0, T_1^nx_0\right) \underset{n}{\rightarrow} 0.$$

Hence $T_1x_0 = x_0$. Similarly we prove that $T_2x_0 = x_0$. ■

Remark 4.1. For common fixed point results for nonexpansive mappings see [5], [6], [17] and [18].

Remark 4.2. For up-to-date references about fixed points of asymptotically regular and uniformly Lipschitzian mappings see [7], [8], [9], [10], [15] and [20].

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