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The Denjoy - Wolff - Type Theorem for Compact k_{B_H} - Nonexpansive Maps on a Hilbert Ball

ABSTRACT. In this note we establish the metric character of the Denjoy-Wolff -type theorem for compact maps on a Hilbert ball.

1. Introduction. In [2], C.-H. Chu and P. Mellon proved the Denjoy-Wolff-type theorem for compact holomorphic maps on a Hilbert ball. In our short note we show that the above mentioned result has a strictly metric character.

2. Basic facts. Let (X,d) be a metric space. Then (X,d) is called finitely compact if each nonempty, bounded and closed subset of X is compact. We say that $f: X \to X$ is nonexpansive if $d(f(x), f(y)) \leq d(x,y)$ for each $x, y \in X$. The basic result due to A. Całka determines the behavior of a sequence of iterates of a nonexpansive mapping in a finitely compact space X.

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Theorem 2.1 [1]. *Let f be a nonexpansive mapping of a finitely compact metric space X into itself. If for some $x_0 \in X$ the sequence $\{f^n(x_0)\}$ contains a bounded subsequence, then for every $x \in X$ the sequence $\{f^n(x)\}$ is bounded.*

We recall now a few facts about the Kobayashi distance k_{B_H} on the Hilbert ball B_H. It is known that $k_{B_H}(w,z) = \operatorname{ar\,tanh}(1 - \sigma(w,z))^{1/2}$ for $w, z \in B_H$, where

$$\sigma(w,z) = \left[\left(1 - \|w\|^2\right)\left(1 - \|z\|^2\right)\right] \Big/ \left(|1 - (w,z)|\right)^2,$$

[7], [8]. Directly from the above formula for k_{B_H} we get

Lemma 2.2. *If $w_n, z_n \in B_H$ for $n = 1, 2, \ldots$, $\lim_n \|w_n\| = 1$ and $\sup_n k_{B_H}(w_n, z_n) < \infty$, then $\|w_n - z_n\| \to 0$.*

The most important result for k_{B_H}-nonexpansive mappings on B_H is due to K. Goebel, T. Sękowski and A. Stachura.

Theorem 2.3 [7], [8]. *A k_{B_H}-nonexpansive mapping $f : B_H \to B_H$ has a fixed point if and only if there exists $z \in B_H$ with $\sup_n \|f^n(z)\| < 1$.*

We also have

Theorem 2.4 [5], [7]. *If a k_{B_H}-nonexpansive mapping $f : B_H \to B_H$ is fixed-point free, then there exists a unique point ξ of norm one such that all "ellipsoids"*

$$E(\xi, \lambda) = \left\{ z \in B_H : \frac{|1 - (z,\xi)|}{1 - \|z\|^2} < \lambda \right\},$$

$\lambda > 0$, are invariant under f and $\overline{E(\xi,\lambda)} \cap \partial B_H = \{\xi\}$ (here $\overline{E(\xi,\lambda)}$ denotes the norm closure of $E(\xi,\lambda)$). Moreover, for every $z \in B_H$ there exists $\lambda > 0$ such that $z \in E(\xi, \lambda)$.

If we consider the unit open ball B in a Banach space X and if k_B is the Kobayashi distance on B, then the following facts are important:
i) the following formula

$$k_B(0,z) = \operatorname{ar\,tanh} \|z\| \tag{2.1}$$

is valid for each $z \in B$ [7];

ii) for $z_1, z_2, w_1, w_2 \in B$, $0 \le t \le 1$, and $r \ge 0$, inequalities $k_B(z_1, z_2) \le r$ and $k_B(w_1, w_2) \le r$ imply

$$k_B((1-t)w_1 + tz_1, (1-t)w_2 + tz_2) \le r \tag{2.2}$$

[10];

iii) every holomorphic self-mapping of B is nonexpansive in k_B [7].

Finally, we recall

Theorem 2.5 (The weakened version of the Earle-Hamilton theorem) [3]. *For every $0 \le t < 1$ and for each k_B-nonexpansive mapping $f : B \to B$ the mapping $tf : B \to B$ is a k_B-contraction and therefore has a unique fixed point.*

3. Iterates of compact k_B-nonexpansive maps with fixed points. In this part of our note we prove the theorem analogous to Theorem 2.3 for a compact k_B-nonexpansive self-map on the unit open ball B in a Banach space X. We say that the mapping $f : B \to B$ is compact if $\overline{f(B)}$ is compact in X.

Theorem 3.1. *Let B be the open unit ball in a Banach space X and let $f : B \to B$ be a compact k_B-nonexpansive mapping. The following statements are equivalent*

i) *f has a fixed point;*

ii) *there exists $z \in B$ and a subsequence of its iterates $\{f^{n_i}(z)\}$ such that $\sup_i \|f^{n_i}(z)\| < 1$;*

iii) *there exists $z \in B$ such that $\sup_n \|f^n(z)\| < 1$;*

iv) *for each $z \in B$ we have $\sup_n \|f^n(z)\| < 1$;*

v) *there exists a nonempty, closed, convex and f-invariant subset A of B such that $\sup_{z \in A} \|z\| < 1$;*

vi) *there exists a nonempty f-invariant subset A of B such that $\sup_{z \in A} \|z\| < 1$;*

vii) *there exists a sequence $\{z_n\}$ such that $z_n - f(z_n) \to 0$ and $\sup_n \|z_n\| < 1$.*

Proof. The implication i)$\to$ ii) is obvious.

ii)$\to$ iii). By (2.1) the assumption $\sup_i \|f^{n_i}(z)\| < 1$ implies

$$\sup_i k_B(0, f^{n_i}(z)) = \sup_i \operatorname{ar\,tanh} \|f^{n_i}(z)\| < \infty.$$

By the finite compactness of $\left(\overline{f(B)} \cap B, k_B\right)$ we can apply Theorem 2.1 and therefore $\sup_n k_B(0, f^n(z)) = \sup_n \operatorname{ar\,tanh} \|f^n(z)\| < \infty$. Hence $\sup_n \|f^n(z)\| < 1$.

iii)→ iv). Let us take an arbitrary $w \in B$. Then we have

$$\begin{aligned}\sup_n \operatorname{ar\,tanh} \|f^n(w)\| &= \sup_n k_B(0, f^n(w)) \\ &\leq \sup_n [k_B(0, f^n(z)) + k_B(f^n(z), f^n(w))] \\ &\leq \sup_n k_B(0, f^n(z)) + k_B(z, w) < \infty\end{aligned}$$

which gives $\sup_n \|f^n(w)\| < 1$.

iv)→ v) Let us take an arbitrary $z \in B$. By iv) $\sup_n k_B(0, f^n(z)) < \infty$. It allows us to apply the method of an asymptotic center [4], [6], [7]. For every $w \in B$ the number $r(w) = \limsup_n k_B(f^n(z), w)$ is called an asymptotic radius of $\{f^n(z)\}$ at w and the number

$$r = \inf_{w \in \overline{conv\, f(B)} \cap B} r(w)$$

is an asymptotic radius of $\{f^n(z)\}$ with respect to $\overline{conv\, f(B)} \cap B$. Finally, the set $A = \left\{ w \in \overline{conv\, f(B)} \cap B : r(w) = r \right\}$ is an asymptotic center of $\{f^n(z)\}$ in $\overline{conv\, f(B)} \cap B$. First we show that A is nonempty, compact and convex subset of B. Indeed, for each $\epsilon > 0$ the set

$$A(\epsilon) = \left\{ w \in \overline{conv\, f(B)} \cap B : r(w) \leq r + \epsilon \right\}$$

is nonempty, k_B-closed and by (2.2) it is also convex. $A(\epsilon)$ lies strictly inside B because $\operatorname{ar\,tanh} \|w\| = k_B(0, w) \leq r(0) + r(w) \leq r(0) + r + \epsilon$ for every $w \in A(\epsilon)$. Hence $A(\epsilon)$ is compact for each $\epsilon > 0$ and $A = \bigcap_{\epsilon > 0} A(\epsilon)$ is nonempty, compact and convex. Next we have $f(A) \subset A$.

The implications v)→ vi) and vi)→ iii) are obvious.

v)→ i). Since we have $\sup_{z \in A} \|z\| < 1$ the set $\overline{conv\, f(A)}$ is compact and f-invariant. After applying either the Schauder theorem [11] or Theorem 2.5 we get an existence of a fixed point of f in B.

i)→ vii) . Obvious.

vii)→ i). Since f is compact, the sequence $\{f(z_n)\}$ contains a subsequence $\{f(z_{n_m})\}$ which is convergent to $z \in B$. The point z is a fixed point of f.

■

Remark. The assumption that f is a compact map is essential because there exists a Banach space X with the open unit ball B and a fixed-point-free holomorphic map $f : B \to B$ with $\sup_n \|f^n(z)\| < 1$ for each $z \in B$ (see [9]).

4. Denjoy-Wolff-type theorem. Now we are ready to prove

Theorem 4.1. *Let H be a Hilbert space with the open unit ball B_H and let k_{B_H} be the Kobayashi distance on B_H. For each compact, k_{B_H}-nonexpansive and fixed-point-free mapping $f : B_H \to B_H$ there exists $\xi \in \partial B_H$ such that the sequence $\{f^n\}$ of iterates of f converges locally uniformly on B_H to the constant map taking the value ξ.*

Proof. Let us choose $z \in B_H$ and next $\lambda > 0$ such that $f^n(z) \in E(\xi, \lambda)$ for $n = 1, 2, \ldots$. The mapping f is fixed-point-free and therefore Theorem 3.1 implies $\lim_n \|f^n(z)\| = 1$. Now it is sufficient to apply Theorem 2.4 to get $\lim_n f^n(z) = \xi$. By Lemma 2.2 we obtain locally uniform convergence of $\{f^n\}$ to ξ. ■

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