ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN-POLONIA
VOL. LI.2, 15 SECTIO A 1997

WILLIAM A. KIRK

Remarks on Approximation and Approximate Fixed Points in Metric Fixed Point Theory

ABSTRACT. Some applications of an approach of Ishikawa are discussed. In particular suppose K is a closed convex subset of a Banach space, suppose $T: K \rightarrow K$ is a nonexpansive mapping with $fix(T) \neq \emptyset$, and suppose $f_\alpha = (1-\alpha)I + \alpha T$ for $\alpha \in (0,1)$. It is shown that if $\theta(\|x - T(x)\|) \geq c \operatorname{dist}(x, fix(T))$ for each $x \in K$, where $\theta: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a function which is right continuous at $0 = \theta(0)$, then $\{f_\alpha^n(x)\}$ converges to a point of $fix(T)$ for each $x \in K$. One consequence of this fact is the following. If K is bounded closed and convex, and if $T: K \rightarrow K$ is a nonexpansive mapping which satisfies

$$\|T(x) = T(y)\| \leq k[\|x - T(x)\| + \|y - T(y)\|], \quad (x, y \in K),$$

for some $k > 0$, then T has a unique fixed point p and moreover $f_\alpha^n(x) \rightarrow p$ as $n \rightarrow \infty$ for each $\alpha \in (0,1)$ and $x \in K$.

The paper concludes with a brief discussion of approximate fixed points for Hölder continuous mappings and for mappings which are not necessarily continuous.

1. Ishikawa's theorem and approximation of fixed points.

We begin with a brief historical note. (The results of this section should be

1991 *Mathematics Subject Classification.* Primary 54H25, 47H10.
Key words and phrases. Ishikawa's theorem, metric fixed point theory, approximate fixed points, Hölder conditions.

compared with those in [12].) Let K be a bounded closed convex subset of a uniformly convex Banach space and $T : K \to K$ is a nonexpansive mapping with (nonempty) fixed point set A. It is shown in [24] that if

$$\|x - T(x)\| \geq c \sup\{\|x - y\| : y \in A\} \tag{1.1}$$

for some constant $c > 0$, then the sequence $\{f^n(x)\}$ converges to a point of A for every $x \in K$, where f is the averaged mapping: $f = (1/2)(I + T)$.

It follows from a very simple observation about nonexpansive mappings and a fundamental result of Ishikawa about the behavior of the iterates of f that the above result holds in an arbitrary Banach space. This fact has immediate implications for nonexpansive mappings which satisfy constraints such as

$$\|T(x) - T(y)\| \leq k[\|x - T(x)\| + \|y - T(y)\|] \qquad (x, y \in K).$$

These results are noteworthy in that no geometric assumptions are required on the underlying space.

Let (M, d) be a metric space. A mapping $f : M \to M$ is said to be *nonexpansive* if for each $x, y \in M$, $d(f(x), f(y)) \leq d(x, y)$, and *asymptotically regular* if for each $x \in M$, $d(f^n(x), f^{n+1}(x)) \to 0$ as $n \to \infty$. The result of Ishikawa ([13]) we shall need asserts that if K is a bounded closed convex subset of a Banach space and if $T : K \to K$ is nonexpansive, then for each $\alpha \in (0, 1)$, the mapping $f_\alpha : K \to K$, defined by setting $f_\alpha(x) = (1 - \alpha)x + \alpha T(x)$, $x \in K$, is always asymptotically regular. Since the fixed points of f_α and T coincide, this observation effectively reduces the study of nonexpansive mappings to the much smaller class of asymptotically regular nonexpansive mappings.

We begin with a rather trivial yet basic observation.

Proposition 1.1. *Let M be a complete metric space, suppose $f : M \to M$ is a continuous mapping for which $fix(f) \neq \emptyset$, and suppose f satisfies $d(f(x), p) \leq d(x, p)$ for each $p \in fix(f)$. Then given $x \in M$,*

$$f^n(x) \to p \in fix(f) \Leftrightarrow \text{dist}(f^n(x), fix(f)) \to 0.$$

Proof. The implication $\Rightarrow$ is immediate (for an arbitrary mapping in any metric space). For the reverse implication, let $\{\epsilon_n\}$ be a sequence of positive numbers for which $\epsilon_n \downarrow 0$ and let $x \in M$. By assumption there exists $p_1 \in fix(f)$ and $n_1 \in \mathbb{N}$ such that $d(f^{n_1}(x), p_1) < \epsilon_1$. Also $p_1 \in fix(f)$ implies $d(f^n(x), p_1) \leq d(f^{n_1}(x), p_1) < \epsilon_1$ for all $n \geq n_1$. Similarly, there exists $p_2 \in fix(f)$ and $n_2 > n_1$ such that $d(f^n(x), p_2) < \epsilon_2$ for all $n \leq n_2$.

Note in particular that $n \le n_2 \Rightarrow f^n(x) \in B(p_1;\epsilon_1) \cap B(p_2;\epsilon_2)$. Continue in this manner to obtain for each $k \in \mathbb{N}$ an integer n_k such that $n \le n_k \Rightarrow f^n(x) \in D_k := \cap_{i=1}^k B(p_i;\epsilon_i)$. Since $diam(D_k) \le 2\epsilon_k \xrightarrow{k} 0$, by Cantor's Theorem there exists $p \in M$ such that $\cap_{i=1}^\infty D_i = \{p\}$. Clearly $f^n(x) \to p$. Since f is continuous $fix(f)$ is closed, so $p \in fix(f)$.

□

At this point our strategy becomes one of searching for conditions which will assure that $\text{dist}(f^n(x), fix(f)) \to 0$, and in particular conditions which do not require one to first identify $fix(f)$. This is where the Ishikawa result proves useful.

In the following theorem we use $F_\epsilon(T)$ to denote the set $\{x \in K : \|x - T(x)\| \le \epsilon\}$. The result of [24] is a very special case of this fact.

Theorem 1.2. *Suppose K is a bounded closed and convex subset of a Banach space, suppose $T : K \to K$ is nonexpansive with $fix(T) \neq \emptyset$, and for $\alpha \in (0,1)$, let $f_\alpha := (1-\alpha)I + \alpha T$ Suppose there exits $\epsilon > 0$ such that for all $x \in F_\epsilon(T)$,*

$$\theta(\|x - T(x)\|) \ge \text{dist}(x, fix(T)), \tag{1.2}$$

where $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ is a function which is right continuous at 0 and for which $\theta(0) = 0$. Then for each $x \in K$ and $\alpha \in (0,1)$, the sequence $\{f_\alpha^n(x)\}$ converges to a fixed point of T.

Proof. Note that if $x \in K$ and $\alpha \in (0,1)$ then

$$\|f_\alpha^n(x) - T(f_\alpha^n(x))\| = \alpha \left\|f_\alpha^n(x) - f_\alpha^{n+1}(x)\right\|$$

and $\left\|f^n(x) - f^{n+1}(x)\right\| \to 0$ as $n \to \infty$ by Theorem 2 of Ishikawa [13]. Thus for n sufficiently large

$$\theta(\alpha \left\|f^n(x) - f^{n+1}(x)\right\|) \ge \text{dist}(f^n(x), fix(T)),$$

and by (3.2) we conclude $\text{dist}(f^n(x), fix(f)) \to 0$. The conclusion now follows from Proposition 1.1 and the fact that $fix(f) = fix(T)$.

□

Remark. The above theorem, which is closely related to Theorem 3.1 of [12], also includes Theorem 6.4 of [19] as a special case.

We now examine some particular cases. If $\theta(t) = ct$ for some constant c, then the rate at which $\{f_\alpha^n(x)\}$ converges to a fixed point of T will depend on the rate at which $\{\|f_\alpha^n(x) - f_\alpha^{n+1}(x)\|\}$ converges to 0 *and this rate of convergence is now known* (Baillon and Bruck [3]). In particular if the diameter of K is 1 then for any $x \in K$,

$$\|f_\alpha^n(x) - f_\alpha^{n+1}(x)\| < \frac{1}{\sqrt{\pi\alpha(1-\alpha)n}}.$$

We have not yet addressed whether it is possible to obtain conditions of the type needed without first knowing something about the fixed point set. The answer is that such conditions may be forced by additional information about the mapping. The obvious example is that of a contraction mapping. Note that if $T : K \to K$ is a contraction mapping with Lipshitz constant $c < 1$, then $\|x - T(x)\| \geq (1 - c)\|x - p\|$ where p is the unique fixed point of T. However we can prove the following much more general fact.

Theorem 1.3. *Suppose $K \subseteq X$ is bounded closed and convex, suppose $T : K \to K$ is a nonexpansive mapping, and for fixed $\alpha \in (0,1)$, let $f_\alpha := (1-\alpha)I + \alpha T$. Suppose also that for some $\epsilon > 0$ and $k > 0$ T satisfies the condition*

$$\text{(1.3)} \quad \|T(x) - T(y)\| \leq k[\|x - T(x)\| + \|y - T(y)\|] \quad (x, y \in F_\epsilon(T)).$$

Then T has a unique fixed point p, and for each $x \in K$ and $\alpha \in (0,1)$, the sequence $\{f_\alpha^n(x)\}$ converges to p.

Proof. Since T is nonexpansive and K is bounded and convex it is easy to see that $\inf\{\|x - T(x)\| : x \in K\} = 0$. Thus there exists $\{x_n\} \subseteq K$ such that $\|x_n - T(x_n)\| \to 0$ as $n \to \infty$, and by (1.3)

$$\|T(x_m) - T(x_n)\| \leq k[\|x_m - T(x_m)\| + \|x_n - T(x_n)\|] \to 0$$

as $m, n \to \infty$. Therefore $\{T(x_n)\}$ is a Cauchy sequence which converges to a point $p \in K$ for which $p = T(p)$. The condition (1.3) implies p is unique. Taking $x \in F_\epsilon(T)$ and $y = p$ in (1.3) gives $k\|x - T(x)\| \geq \|T(x) - p\|$ from which $k\|x - T(x)\| + \|x - T(x)\| \geq \|T(x) - p\| + \|x - T(x)\| \geq \|x - p\|$; hence $\|x - T(x)\| \geq (1 + k)^{-1}\|x - p\|$. Theorem 1.2 now applies and we have $f_\alpha^n(x) \to p$ for each $x \in K$ and $\alpha \in (0,1)$.

□

We also have the following variant of Theorem 1.2, which we state without proof. Note that this result does not require an *a priori* assumption that $fix(T) \neq \emptyset$.

Theorem 1.4. *Suppose K is a bounded closed and convex subset of a Banach space and suppose $T : K \to K$ is nonexpansive. Let $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ be any function which is right continuous at 0 and for which $\theta(0) = 0$, and suppose for some $\epsilon > 0$ and for some nonempty compact set $A \subseteq K$,*

$$\theta(\|x - T(x)\|) \geq \operatorname{dist}(x, A) \quad (x \in F_\epsilon(T)).$$

Then for each $x \in K$ and $\alpha \in (0,1)$, the sequence $\{f_\alpha^n(x)\}$ converges to a fixed point of T.

2. Minimal displacement and Hölder conditions. This section contains a summary of the results of [18] which will appear elsewhere. Let K be a bounded closed convex subset of a Banach space X and let $T : K \to K$ be a mapping which satisfies, for some $h > 0$ and $p > 0$, the condition

$$\|T(x) - T(y)\| \leq h \|x - y\|^p, \qquad (x, y \in K). \tag{2.1}$$

These mappings are well understood for various choices of h and p. If $p = 1$ and $h < 1$, then such mappings always have a fixed point because of the celebrated Banach contraction mapping principle discussed above, and if $p = 1$ and $h = 1$ these are the nonexpansive mappings which also are known to have an extensive fixed point theory (see, e.g., [1], [10]). For $p = 1$ and $h > 1$ (the usual Lipschitz condition) fixed points in general do not exist in non-compact settings, but a fair amount (not everything) is known about minimal displacement properties of such mappings (e.g., see [9], [11]).

We now describe the results of [18] which apply to the remaining case, $p < 1$. (Mappings satisfying (2.1) are degenerate if $p > 1$ regardless of the value of h.) The question of minimal displacement for such mappings is taken up in [18], and central to the observations there is the fact that mappings which satisfy the Hölder condition (2.1) for $0 < h, p \leq 1$, also satisfy the following more general condition:

$$\|T(x) - T(y)\| \leq \max\{h, \|x - y\|\}, \qquad (x, y \in K). \tag{2.2}$$

The principal results of [18] apply to mappings of the above class. Specific conditions are identified under which $\inf\{\|x - T(x)\| : x \in K\} < h$ for mappings $T : K \to K$ satisfying (2.2). It is also shown that in certain spaces, including uniformly convex spaces, $\sup[\inf\{\|x - T(x)\| : x \in K\}] < h$ where the supremum ranges over all mappings satisfying (2.2).

It is noteworthy that mappings satisfying (2.2) need not even be continuous. In this sense such results are motivated by a classical result of Victor Klee regarding the *stability* of the fixed point property for continuous mappings in compact convex subsets of normed linear spaces. Let X be a topological space and (M, d) a metric space. A mapping $\varphi : X \to M$ is said to be *ϵ-continuous* for $\epsilon > 0$ provided each point $x \in X$ has a neighborhood U_x such that $diam(\varphi(U_x)) \leq \epsilon$. In [20] Klee proved that if K is a compact convex subset of a normed linear space and if $\varphi : K \to K$ is ϵ-continuous, then for every $\epsilon' > \epsilon$ there exists $z \in K$ such that $\|z - \varphi(z)\| < \epsilon'$. We show here that by essentially adding the assumption $\|\varphi(u) - \varphi(v)\| \leq \|u - v\|$ for $u, v \notin U_x$ the same conclusion holds in noncompact settings, and in many cases (including the compact case) an even stronger conclusion holds. (For the convenience of the reader we append a short proof of Klee's theorem to this paper.)

In a Banach space setting the principal results of [18] are the following. (The setting in [18] is a more abstract.)

Theorem 2.1. *Let K be a weakly compact convex subset of a Banach space and suppose K has normal structure. Suppose $T : K \to K$ satisfies*

$$\|T(x) - T(y)\| \leq \max\{h, \|x - y\|\}, \qquad (x, y \in K).$$

Then there exists $z \in K$ such that $\|z - T(z)\| < h$.

If T satisfies a Hölder condition, then it is possible to say even more.

Theorem 2.2. *Suppose K is a nonempty bounded closed convex subset of a Banach space X, ane suppose $T : K \to K$ satisfies*

$$\|T(x) - T(y)\| \leq h\, \|x - y\|^p, \qquad (x, y \in K),$$

for some $h, p \in (0, 1)$. Then $\inf\{\|x - T(x)\| : x \in K\} \leq h^{1/(1-p)}$.

Recall that a Banach space X is said to have *uniform normal structure* ([8]) if there exists a constant $c \in (0, 1)$ such that each bounded convex subset K of X satisfies

$$\inf_{z \in K} \sup\{\|z - y\| : y \in K\} \leq c\, diam(K).$$

Such spaces are known to be reflexive [22].

Theorem 2.3. *Let K be a bounded closed convex subset of a Banach space X which has uniform normal structure. Suppose $T : K \to K$ satisfies*

$$\|T(x) - T(y)\| \leq \max\{h, \|x - y\|\}, \quad (x, y \in K).$$

Then there exists $z \in K$ such that $\|z - T(z)\| \leq ch$.

As we noted at the outset the preceding results apply to mappings $T : K \to K$ which satisfy the Hölder condition

$$\|T(x) - T(y)\| \leq h\,\|x - y\|^p, \qquad (x, y \in K),$$

where $0 < p$, $h < 1$. In this case one can say a little more.

Theorem 2.4. *Let X be a Banach space which has uniform normal structure with coefficient c, let K be bounded closed convex subset of X, and suppose $T : K \to K$ satisfies*

$$\|T(x) - T(y)\| \leq h\,\|x - y\|^p, \quad (x, y \in K,\ h, p \in (0,1)),$$

Then if p is sufficiently near 1, $\inf\{\|x - T(x)\| : x \in K\} < ch$.

Proof. By Theorem 2.3 there exists $z \in K$ such that $\|z - T(z)\| \leq ch$. Thus

$$\|T(z) - T^2(z)\| \leq h\,\|z - T(z)\|^p \leq c^p h^{1+p}.$$

Since $c < 1$, $c^p h^{1+p} < ch$ for all p sufficiently near 1. □

The above estimate invites comparison with the displacement estimate of Theorem 2.2. Clearly for certain values of p, h and c it will be the case that $ch < h^{1/(1-p)}$; in particular this is always true if $c < h$ and $p \leq 1/2$, and for all $h < 1$ if p is sufficiently small.

The following is another consequence of the results of [18].

Theorem 2.5. *Let K be a closed convex admissible subset of ℓ_∞, and let $T : K \to K$ be h-nonexpansive. Then there exists $z \in K$ such that $\|z - T(z)\| \leq h/2$.*

Most of the results of [18] are qualitative in that no effort was made to determine optimal minimal displacement for the classes of mappings considered. Also there is another question left open:

QUESTION. Let K be a bounded closed convex subset of a Banach space, and suppose $T : K \to K$ is h-nonexpansive for $h > 0$. Under what conditions can one conclude that there exists $z \in K$ such that $\|z - T(z)\| \leq h$?

In view of Theorem 2.1, if K is weakly compact and has normal structure, then one obtains even more, namely, the existence of $z \in K$ for which $\|z - T(z)\| < h$. This leaves the tempting suggestion that weak compactness alone suffices for an affirmative answer to the question. Note that the limiting case of the above (when $h = 0$) involves a question about fixed points of nonexpansive mappings which remains unresolved, although in this case it is known ([2]) that weak compactness alone *does not* suffice. It seems plausible that the two problems may be related.

3. Klee's result. The results of the previous section are obtained by geometric arguments. For sake of comparison we now append a short proof of a slightly sharpened version of Klee's theorem using a partition of unity argument which is standard in topological fixed point theory. (We also remark that even sharper results are known for compact convex $K \subseteq \mathbb{R}^n$; see Cromme and Diener [5]. The results in [5] provide minimal displacement estimates involving certain moduli of continuity for $f : K \to K$ which, for certain f, are smaller than $\inf\{\epsilon : f \text{ is } \epsilon\text{-continuous}\}$.)

The result proved here is the following.

Theorem 3.1. *Suppose K is a compact convex subset of a normed linear space, and suppose $f : K \to K$ is ϵ-continuous. Then there exists a point $x_0 \in K$ such that $\|f(x_0) - x_0\| \leq \epsilon$.*

The proof involves showing that f may be uniformly approximated by a suitable continuous function $\bar{f}$ and then applying Schauder's Theorem. (Klee's approach involves approximating the domain of f with a polyhedral set, using triangulation to approximate f by a piecewise linear function and then applying Brouwer's Theorem.)

Theorem 3.1 is a direct consequence of the following.

Theorem 3.2. *Suppose K is a compact convex subset of a normed linear space X, and suppose $f : K \to K$ is ϵ-continuous. Then there exists continuous mapping $\bar{f} : K \to K$ such that $\|\bar{f}(x) - f(x)\| \leq \epsilon$ for each $x \in K$.*

Before proving the above we need some terminology. If X is a topological space and $\phi : X \to \mathbb{R}$, then the *support* of ϕ is defined to be the closure of the set $\phi^{-1}(\mathbb{R} - \{0\})$. In particular, if $x \notin$(support ϕ) then there is some neighborhood of x on which ϕ vanishes.

Definition 3.1. Let $\{U_1, \cdots, U_n\}$ be an open covering of the topological space X. A family of continuous functions $\phi_i : X \to [0,1]$, $i = 1, \dots, n$, is said to be a partition of unity of X dominated by the family $\{U_i\}$ if
(i) (support ϕ_i) $\subseteq U_i$ for each i; and
(ii) $\sum_{i=1}^{n} \phi_i(x) = 1$ for each $x \in X$.

The topological fact we need is the following. Its proof can be found in any standard topological text (e.g., [6], [14], [23]).

Theorem 3.3 (Partitions of unity). . *Let $\{U_1, \dots, U_n\}$ be a finite open covering of the metric (or, more generally, normal topological) space M. Then there exists a partition of unity of M dominated by the family $\{U_i\}$.*

Proof of Theorem 3.2. Since f is ϵ-continuous, for each $x \in K$ there exists $r_x > 0$ such that $diam(f(U(x; r_x)) \leq \epsilon$. (Recall that $U(x; r_x)$denotes the open ball centered at x with radius r_x.) Since K is compact there exists a finite set $\{x_1, \dots, x_j\} \subseteq K$ such that $K \subseteq \cup_{i=1}^{j} U(x_i; r_{x_i}/2)$. Let $r = \inf\{r_{x_i} : 1 \leq i \leq j\}$. Notice that we now have the following. If $x, y \in K$ and $\|x - y\| < r/2$ then there exists an i such that $x, y \in U(x_i; r_{x_i})$; hence $\|f(x) - f(y)\| \leq \epsilon$. Now again use the fact that K is compact to select $A = \{a_1, \dots, a_n\} \subseteq K$ so that $K \subseteq \cup_{i=1}^{n} U_i$ where $U_i = U(a_i; r/2)$, $i = 1, \dots, n$. Then the family $\{U_i\}$ is a finite open covering of K so there exists a partition of unity $\{\phi_i\}_{i=1}^{n}$ of K dominated by the family $\{U_i\}$. Define the function $\bar{f} : K \to X$ as follows: $\bar{f}(x) = \sum_{i=1}^{n} \phi_i(x) f(a_i)$ for each $x \in K$.

Since $f(A) \subseteq K$ and $\sum_{i=1}^{n} \phi_i(x) = 1$ we see that in fact, $\bar{f}(K) \subseteq \overline{conv}(f(A)) \subseteq K$. Furthermore, since each of the functions ϕ_i is continuous, $\bar{f}$ is continuous.

Observe also that for $x \in K$,

$$\|f(x) - \bar{f}(x)\| = \left\| f(x) - \sum_{i=1}^{n} \phi_i(x) f(a_i) \right\| = \left\| \sum_{i=1}^{n} \phi_i(x)[f(x) - f(a_i)] \right\|$$

$$\leq \sum_{i=1}^{n} \phi_i(x) \|f(x) - f(a_i)\| .$$

Since $\phi_i(x) = 0$ if $x \notin U_i$ while $\|f(x) - f(a_i)\| \leq \epsilon$ if $x \in U_i$, it follows that $\|f(x) - \bar{f}(x)\| \leq \epsilon$. By Schuader's Theorem $\bar{f}$ has a fixed point x_0, whence $\|f(x_0) - x_0\| \leq \epsilon$. □

There is still a little more one can say in this direction. Let χ denote the usual Kuratowski measure of noncompactness. A mapping $T : K \to X$ is said to be *condensing* if T is continuous and if $\chi(T(A)) < \chi(A)$ for each $A \subseteq K$ for which $\chi(A) > 0$. An obvious weakening is to say that $T : K \to X$ is *ϵ-condensing* if T is ϵ-continuous and if $\chi(T(A)) < \chi(A)$ for each $A \subseteq K$ for which $\chi(A) > 0$. This leads to a mild extension of Theorem 3.2.

Theorem 3.4. *Suppose K is a compact convex subset of a normed linear space, and suppose $f : K \to K$ is ϵ-condensing. Then there exists a point $x_0 \in K$ such that $\|f(x_0) - x_0\| \leq \epsilon$.*

Proof. A standard argument using the condensing condition (e.g., see [27; p. 500]) allows one to construct a nonempty *compact* convex subset $H \subseteq K$ such that $T : H \to H$.

□

A similar observation holds for h-nonexpansive mappings.

Theorem 3.5. *Suppose K is a bounded closed convex subset of a Banach space and suppose $T : K \to K$ satisfies:*
(i) *$\chi(T(A)) < \chi(A)$ for each $A \subseteq K$ for which $\chi(A) > 0$;*
(ii) *$\|T(x) - T(y)\| \leq \max\{\epsilon, \|x - y\|\}$ for each $x, y \in K$. Then there exists $x_0 \in K$ such that $\|x_0 - T(x_0)\| < \epsilon$.*

Proof. As above, there exists a nonempty compact convex subset $H \subseteq K$ such that $T : H \to H$. Since H is compact, H is weakly compact and has normal structure. Thus the conclusion follows from Theorem 2.1.

□

Remark. W. Kryszewski has remarked to the author that perhaps the quickest way to see that Klee's result is true is to invoke the set-valued extension of Schauder's theorem due Bohnenblust and Karlin (or, more generally, the extensions due to Fan and Glicksberg; see, e.g., [27, p.452]). Thus under the assumptions of 3.2 introduce the mapping $F : K \to 2^K$ by setting for $x \in K$, $F(x) = B(f(x); \epsilon) \cap K$.

The fact that f is ϵ-continuous is sufficient to ensure that F is upper semi-continuous and hence F has a fixed point in the set valued sense. Thus $x \in F(x)$ for some $x \in K$ from which the conclusion follows.

Such set-valued extensions of Schauder's theorem are also obtained easily via partition of unity arguments.

Acknowledgement. The author thanks Brailey Sims for suggesting the formulation of Theorem 1.2 and for detecting an oversight in the proof of Theorem 3.2.

References

[1] Aksoy, A. G. and M. A. Khamsi, *Nonstandard Methods in Fixed Point Theory*, Springer-Verlag, New York, Berlin, 1990.

[2] Alspach, D. E., *A fixed point free nonexpansive map*, Proc. Amer. Math. Soc. **82** (1981), 423-424.

[3] Baillon, J. B. and R. E. Bruck, *The rate of asymptotic regularity is* $O(\frac{1}{\sqrt{n}})$, *in* Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, (A. G. Kartsatos, ed.), Marcel Dekker, Inc., New York, Basel, Hong Kong, 1996, pp. 51-81.

[4] Bula, I., *On the stability of the Bohl-Brouwer-Schauder theorem*, Nonlinear Anal. **26** (1996), 1859-1868.

[5] Cromme, L. and I. Diener, *Fixed point theorems for discontinuous mappings*, Math. Programming **51** (1991), 257-267.

[6] Dugundji, J., *Topology*, W. C. Brown, Dubuque, 1989.

[7] ———, *Positive definite functions and coincidences*, Fund. Math. **90** (1976), 131-142.

[8] Gillespie, A. and B. Williams, *Fixed point theorem for nonexpansive mappings on Banach spaces with uniformly normal structure*, Appl. Anal. **9** (1979), 121-124.

[9] Goebel, K., *On the minimal displacement of points under lipschitzian mappings*, Pacific J. Math. **45** (1973), 151-163.

[10] Goebel, K. and W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge Univ. Press, Cambridge, 1990, 244 pp.

[11] Goebel, K. and T. Komorowski, *Retracting balls onto spheres and minimal displacement, in Fixed Point Theory and Applications*, (J-B. Baillon and M. A. Théra, eds.), Longman Scientific & Technical, Essex, 1991, pp. 155-172.

[12] Gosh, M. K. and L. Debnath, *Convergence of Ishikawa Iterates of Quasi-Nonexpansive Mappings*, J. Math. Anal. Appl. **207** (1997), 96-103.

[13] Ishikawa, S., *Fixed points and iteration of a nonexpansive mapping in a Banach space*, Proc. Amer. Math. Soc. **59** (1976), 65-71.

[14] Kelley, J. L., *General Topology*, D. Van Nostrand, Princeton, 1955.

[15] Kirk, W. A., *Locally nonexpansive mappings in Banach spaces, in Fixed Point Theory*, (E. Fadell and G. Fournier, eds.), Springer-Verlag, Berlin, Heidelberg, New York, 1981, pp. 178-198.

[16] ———, *An application of a generalized Krasnoselskii-Ishikawa iteration process, in Fixed Point Theory and Applications*, (J-B. Baillon and M. A. Théra, eds.), Longman Scientific & Technical, Essex, 1991, pp. 261-268.

[17] ———, *Continuous mappings in compact hyperconvex metric spaces*, Numer. Funct. Anal. and Optimiz. **17** (1996), 599-603.

[18] ———, *Hölder continuity and minimal displacement (to appear).*

[19] Kirk, W. A. and S. S. Shin, *Fixed Point Theorems in Hyperconvex Spaces*, Houston J. Math. (to appear).

[20] Klee, V., *Stability of the fixed point property*, Colloq. Math. **VIII** (1961), 43-46.

[21] Lacey, H. E., *The Isometric Theory of Classical Banach Spaces*, Springer-Verlag, Berlin, Heidelberg, New York, 1974.

[22] Maluta, E., *Uniformly normal structure and related coefficients*, Pacific J. Math. **111** (1984), 357-369.

[23] Munkres, J. R., *Topology: A First Course*, Prentice-Hall, Englewood Cliffs, 1975.

[24] Outlaw, C. L., *Mean value iteration of nonexpansive mappings in Banach spaces*, Pacific J. Math. **30** (1969), 747-750.

[25] Petryshyn, W. V., *A new fixed point theorem and its applications*, Bull. Amer. Math. Soc. **78** (1972), 225-229.

[26] Sullivan, F., *Ordering and completeness of metric spaces*, Nieuw Arch. Wisk. **24** (1981), 178-193.

[27] Zeidler, E., *Nonlinear Functional Analysis I: Fixed Point Theorems*, Springer-Verlag, New York, Berlin Heidelberg, Tokyo, 1986.

Department of Mathematics
University of Iowa
Iowa City, Iowa 52242-1419
e-mail: kirk@math.uiowa.edu

received September 3, 1997