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Approximating Fixed Points of Nonlinear Mappings in Banach Spaces

ABSTRACT. Let C be a nonempty bounded closed convex subset of a Banach space X and $T : C \to C$ asymptotically nonexpansive in the intermediate sense, i.e. T is uniformly continuous and

$$\limsup_{n\to\infty} \sup\{\|T^n x - T^n y\| - \|x-y\| : x, y \in C\} \le 0.$$

Then, under certain conditions on X, $\{n_i\}$, $\{\alpha_i\}$ and $\{\beta_i\}$, the sequence generated by $x_{i+1} := \alpha_i T^{n_i}[\beta_i T^{n_i} x_i + (1-\beta_i)x_i] + (1-\alpha_i)x_i$ starting at $x_1 \in C$, converges weakly to a fixed point of T. Convergence of fixed point sets of multivalued nonexpansive mappings is also established under both Hausdorff metric and the Mosco sense in restricted Banach spaces.

1. Introduction. Let X be a real Banach space, C a subset of X (not necessarily convex) and $T : C \to C$ a self-mapping of C. There appear in the literature two definitions of an asymptotically nonexpansive mapping. The weaker definition (cf. Kirk [12]) requires that

$$\limsup_{n\to\infty} \sup_{y\in C} (\|T^n x - T^n y\| - \|x-y\|) \le 0$$

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for every $x \in C$ and that T^N is continuous for some $N \geq 1$. The stronger definition (briefly called *asymptotically nonexpansive* as in [5]) requires each iterate T^n to be Lipschitzian with Lipschitz constants $L_n \to 1$ as $n \to \infty$. For further generalization of an averaging iteration of Schu [21], Bruck et al. [2] introduced a definition somewhere between these two: T is *asymptotically nonexpansive in the intermediate sense* provided T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0.$$

On the other hand, let C be a nonempty closed convex subset of X and $T : C \to C$ a (single-valued) nonexpansive mapping (i.e., $\|Tx - Ty\| \leq \|x - y\|$ for all $x,\ y \in C$). Given $u \in C$ and $t \in (0,1)$, we can define a contraction $T_t : C \to C$ by

$$T_t x = tTx + (1-t)u, \quad x \in C. \tag{1}$$

Then, by Banach's contraction principle, T_t has a unique fixed point x_t in C, that is, we have

$$x_t = tTx_t + (1-t)u. \tag{2}$$

The convergence of $\{x_t\}$ as $t \to 1$ to a fixed point of T has been investigated by several authors. In fact, the strong convergence of $\{x_t\}$ as $t \to 1$ for T acting on a bounded C was proved in a Hilbert space independently by Browder [1] and Halpern [10] and in a uniformly smooth Banach space by Reich [20] (cf. [9]). This result was also extended to Ishikawa iteration scheme (cf. Ishikawa [11]) by Tan and Xu [25] and very recently by Takahashi and Kim [27]. For recent progress for nonexpansive nonself-mappings, the reader is referred to [15], [24] and [29].

Recently, López Acedo and Xu [13] studied the convergence of $\{x_t\}$ for multivalued nonexpansive mappings T in a Hilbert space as follows: Let C be a nonempty closed convex subset of a Hilbert space, $K(C)$ the family of all nonempty compact subsets of C, $T : C \to K(C)$ a multivalued nonexpansive mapping with a unique fixed point z, and $T_t : C \to K(C)$ a multivalued contraction defined by (1). Suppose in addition that $Tz = \{z\}$. Then $H(F(T_t), F(T))$ converges to 0 as $t \to 1$, where H is the Hausdorff metric, and $F(T_t)$ and $F(T)$ denote the sets of all fixed points of T_t and T, respectively.

In this paper, we first show how to construct (in a uniformly convex Banach space which either satisfies the Opial property, or has a Fréchet differentiable norm) a fixed point of a mapping which is asymptotically nonexpansive in the intermediate sense as the weak limit of a sequence $\{x_i\}$ defined by an iteration of the form

$$x_{i+1} = \alpha_i T^{n_i}[\beta_i T^{n_i} x_i + (1-\beta_i)x_i] + (1-\alpha_i)x_i,$$

where $\{\alpha_i\}$ and $\{\beta_i\}$ are chosen so that $\alpha_i \in [a,b]$ and $\beta_i \in [0,b]$ or $\alpha_i \in [a,1]$ and $\beta_i \in [a,b]$ for some a,b with $0 < a \leq b < 1$, and $\{n_i\}$ is a sequence of nonnegative integers. Bruck et al. [2] have considered the above iteration only in the case when $\beta_i = 0$ for all $i \geq 1$, which generalizes an averaging iteration of Schu [21]. In particular, our results reduce to those due to [26] for asymptotically nonexpansive mappings. Second, we shall carry over the above result due to López Acedo and Xu [13] in Hilbert spaces to Banach space settings, that is, we prove that $H(F(T_t), F(T))$ converges to 0 as $t \to 1$ in a smooth Banach space with a weakly sequentially continuous duality mapping.

2. Preliminaries. Let X be a real Banach space with norm $\|\cdot\|$ and let X^* be its dual. The value of $x^* \in X^*$ at $x \in X$ will be denoted by $\langle x, x^*\rangle$. When $\{x_n\}$ is a sequence in X, then $x_n \to x$ (resp. $x_n \rightharpoonup x$, $x_n \overset{*}{\rightharpoonup} x$) will denote strong (resp. weak, weak*) convergence of the sequence $\{x_n\}$ to x.

A Banach space X is said to be *uniformly convex* if $\delta(\epsilon) > 0$ for every $\epsilon > 0$, where the modulus $\delta(\epsilon)$ of convexity of X is defined by

$$\delta(\epsilon) = \inf\{1 - \|\frac{x+y}{2}\| : \|x\| \leq 1,\ \|y\| \leq 1,\ \|x-y\| \geq \epsilon\}.$$

Let $S(X) = \{x \in X : \|x\| = 1\}$. Then the norm of X is said to be *Gâteaux differentiable* (and X is said to be *smooth*) if

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{3}$$

exists for each x, y in $S(X)$. It is said to be *Fréchet differentiable* if for each $x \in S(X)$, the limit in (3) is attained uniformly for $y \in S(X)$. We associate with each $x \in X$ the set

$$J_\phi(x) = \{x^* \in X^* : \langle x, x^*\rangle = \|x\|\|x^*\| \text{ and } \|x^*\| = \phi(\|x\|)\},$$

where $\phi : [0,\infty) \to [0,\infty)$ is a continuous and strictly increasing function with $\phi(0) = 0$ and $\lim_{t\to\infty}\phi(t) = \infty$. Then $J_\phi : X \to 2^{X^*}$ is said to be the *duality* mapping. Suppose that J_ϕ is single-valued. Then J_ϕ is said to be *weakly sequentially continuous* if for each $\{x_n\} \subset X$ with $x_n \rightharpoonup x$,

$J_\phi(x_n) \stackrel{*}{\rightharpoonup} J_\phi(x)$. For brevity, we set $J := J_\phi$. In our proof we assume without loss of generality that J is normalized. It is well known that if X is smooth, then the duality mapping J is single-valued and strong-weak* continuous; for more details, see [3].

A Banach space X is said to satisfy the *Opial property* [17] if for any sequence $\{x_n\}$ in X, $x_n \rightharpoonup x$ implies

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in X$ with $y \neq x$. Spaces satisfying this property include all Hilbert spaces and l^p for $1 < p < \infty$. Also it is known [7] that if X admits a weakly sequentially continuous duality mapping, then X satisfies the Opial property. For more details on the Opial property, see also [6].

Finally, we say that X satisfies the *uniform Opial property* [19] if $r_X(c) > 0$ for all $c > 0$, where r_X denotes the Opial modulus of X, i.e.,

$$r_X(c) = \inf\left\{\limsup_{n\to\infty} \|x_n - x\| - 1\right\},$$

where $c \geq 0$ and the infimum is taken over all $x \in X$ with $\|x\| \geq c$ such that $x_n \rightharpoonup 0$ and $\limsup_{n\to\infty} \|x_n\| \geq 1$. It is easy to see that $r_X(0) \leq 0$, and that r_X is continuous and nondecreasing (see Lin et al. [14]).

3. Convergence theorems of nonlipschitzian mappings. Schu [21] considered the averaging iteration $x_{i+1} = \alpha_i T^i x_i + (1 - \alpha_i)x_i$ when $T : C \to C$ is asymptotically nonexpansive and $\{\alpha_i\}$ is a sequence in $(0, 1)$ which is bounded away from 0 and 1. Throughout this section we shall consider, instead of this a more general iteration

$$x_{i+1} = \alpha_i T^{n_i} y_i + (1 - \alpha_i)x_i, \tag{4}$$

$$y_i = \beta_i T^{n_i} x_i + (1 - \beta_i)x_i, \tag{5}$$

where $\{\alpha_i\}$ and $\{\beta_i\}$ are chosen so that $\alpha_i \in [a, b]$ and $\beta_i \in [0, b]$ or $\alpha_i \in [a, 1]$ and $\beta_i \in [a, b]$ for some a, b with $0 < a \leq b < 1$, and $\{n_i\}$ is a sequence of nonnegative integers (which need not be increasing). A strictly increasing sequence $\{m_i\}$ of positive integers will be called *quasi-periodic* [2] if the sequence $\{m_{i+1} - m_i\}$ is bounded (equivalently, if there exists $b > 0$ such that any block of b consecutive positive integers contains a term of the sequence).

We begin with the following easy observation.

Lemma 3.1 [2]. *Suppose $\{r_k\}$ is a bounded sequence of real numbers and $\{a_{k,m}\}$ is a doubly-indexed sequence of real numbers which satisfy*

$$\limsup_{k\to\infty}\limsup_{m\to\infty} a_{k,m} \le 0, \quad r_{k+m} \le r_k + a_{k,m} \quad \text{for each } k, m \ge 1.$$

Then $\{r_k\}$ converges to an $r \in \mathbb{R}$; if $a_{k,m}$ can be taken to be independent of k, $a_{k,m} \equiv a_m$, then $r \le r_k$ for each k.

Using Lemma 3.1, we have the following result which is crucial for our argument.

Lemma 3.2. *Suppose X is a uniformly convex Banach space, C is a bounded convex subset of X, and $T : C \to C$ is asymptotically nonexpansive in the intermediate sense. Put*

$$c_n = \max(0, \sup_{x,y\in C}(\|T^n x - T^n y\| - \|x - y\|)),$$

so that $\lim_{n\to\infty} c_n = 0$. Suppose that for any $x_1 \in C$, $\{x_i\}$ is generated by (4)-(5) for $i \ge 1$ and $\sum_{i=1}^{\infty} c_{n_i} < +\infty$. Then for every $w_1, w_2 \in F(T)$ and $0 < t < 1$, $\lim_{i\to\infty} \|tx_i + (1-t)w_1 - w_2\|$ exists.

Proof. The proof still follows the lines of the proof in [2]. We have not assumed C is closed, but since T is uniformly continuous, it (and its iterates) can be extended to the norm closure C with the same modulus of uniform continuity and the same constants c_n, so it does no harm to assume that C itself is closed. By a theorem of Kirk [12], T has at least one fixed point w in C.

We begin with showing that for $w \in F(T)$, the limit $\lim_{i\to\infty} \|x_i - w\|$ exists. Since $\|y_k - w\| \le \beta_k c_{n_k} + \|x_k - w\|$, this together with (4) and (5) implies

$$\begin{aligned}
(6)\qquad \|x_{k+1} - w\| &\le \alpha_k\|T^{n_k} y_k - w\| + (1-\alpha_k)\|x_k - w\| \\
&= \alpha_k\|T^{n_k} y_k - T^{n_k} w\| + (1-\alpha_k)\|x_k - w\| \\
&\le \alpha_k(\|y_k - w\| + c_{n_k}) + (1-\alpha_k)\|x_k - w\| \\
&\le \alpha_k(\|x_k - w\| + c_{n_k}(1+\beta_k)) + (1-\alpha_k)\|x_k - w\| \\
&\le \|x_k - w\| + c_{n_k}(1+\beta_k),
\end{aligned}$$

and hence

$$(7)\qquad \|x_{k+m} - w\| \le \|x_k - w\| + 2\sum_{i=k}^{k+m-1} c_{n_i}.$$

Applying Lemma 3.1 with $r_k = \|x_k - w\|$ and $a_{k,m} = 2\sum_{i=k}^{k+m-1} c_{n_i}$, we see that $\lim_{i\to\infty} \|x_i - w\| (= r)$ exists for every $w \in F(T)$.

Now putting $T_i := \alpha_i T^{n_i}[\beta_i T^{n_i} + (1-\beta_i)I] + (1-\alpha_i)I$ (I denotes the identity mapping of X) for each $i \in \mathrm{N}$ and, for $k \geq j$, $S(k,j) := T_{k-1}T_{k-2}\cdots T_j$, it is easily seen that $x_k = S(k,j)x_j$ and $F(T_i) \supseteq F(T)$. Since

$$\|T_i x - T_i y\| \leq \alpha_i c_{n_i}(1+\beta_i) + \|x-y\| \leq 2c_{n_i} + \|x-y\|$$

for all $x, y \in C$, we have for $k \geq j$

$$\|S(k,j)x - S(k,j)y\| \leq 2\sum_{i=j}^{k-1} c_{n_i} + \|x-y\| \quad \text{for all } x, y \in C. \tag{8}$$

Let $w \in F(T)$ and $0 < t < 1$. We show that

$$\lim_{j\to\infty} \sup_{k\geq j} \|S(k,j)[tx_j + (1-t)w] - tx_k - (1-t)w\| = 0. \tag{9}$$

To this end, if $r = 0$, then using (8) repetitiously we have for $k \geq j$,

$$\begin{aligned} &\|S(k,j)[tx_j + (1-t)w] - tx_k - (1-t)w\| \\ &\leq \|S(k,j)[tx_j + (1-t)w] - w\| + t\|x_k - w\| \\ &\leq 2\sum_{i=j}^{k-1} c_{n_i} + t\|x_j - w\| + t\|S(k,j)x_j - w\| \\ &< 4\sum_{i=j}^{\infty} c_{n_i} + 2\|x_j - w\| \to 0 \qquad \text{as } j \to \infty, \end{aligned}$$

which gives (9). Now let $r > 0$ and suppose (9) does not hold, i.e., there are some $\epsilon_0 > 0$ and a subsequence $\{m_j\}$ of N with $m_j \geq j$ such that

$$\sup_{k\geq m_j} \|S(k,m_j)[tx_{m_j} + (1-t)w] - tx_k - (1-t)w\| \geq 2\epsilon_0, \tag{10}$$

for each $j \geq \mathrm{N}$. By uniform convexity of X, we can also choose $d > 0$ so small that $(r+d)\left[1 - 2t(1-t)\delta\left(\frac{\epsilon_0}{r+d}\right)\right] := r_0 < r$, where δ is the modulus of convexity of X. For $\rho > 0$ with $\rho < \min\{d/2, r - r_0\}$, there exists $j_0 \in \mathrm{N}$ such that for $j \geq j_0$

$$r_0 \leq r - \rho < \|x_j - w\| < r + \rho, \qquad \sum_{i=j_0}^{\infty} c_{n_i} < t(1-t)\rho.$$

Let j ($\geq j_0$) be fixed. By (10), we can choose a k ($\geq m_j$) such that $\|S(k,m_j)[tx_{m_j}+(1-t)w]-tx_k-(1-t)w\|\geq\epsilon_0$. Put $z:=tx_{m_j}+(1-t)w$, $u:=(1-t)[S(k,m_j)z-w]$ and $v:=t[S(k,m_j)x_{m_j}-S(k,m_j)z]$. Then, it follows that

$$\|u\|=(1-t)\|S(k,m_j)z-w\|\leq(1-t)\Big(\sum_{i=m_j}^{k-1}c_{n_i}+\|z-w\|\Big)$$

$$<(1-t)\Big(t\rho+t\|x_{m_j}-w\|\Big)<t(1-t)(r+2\rho)<t(1-t)(r+d)$$

and also

$$\|v\|=t\|S(k,m_j)x_{m_j}-S(k,m_j)z\|\leq t\Big(\sum_{i=m_j}^{k-1}c_{n_i}+\|x_{m_j}-z\|\Big)$$

$$<t\Big((1-t)\rho+(1-t)\|x_{m_j}-w\|\Big)<t(1-t)(r+2\rho)<t(1-t)(r+d).$$

We also have

$$\|u-v\|=\|S(k,m_j)z-tx_k-(1-t)w\|\geq\epsilon_0$$

and $tu+(1-t)v=t(1-t)[S(k,m_j)x_{m_j}-w]$. By Lemma in [8], we have

$$\begin{aligned}t(1-t)\|S(k,m_j)x_{m_j}-w\|&=\|tu+(1-t)v\|\\&\leq t(1-t)(r+d)\Big[1-2t(1-t)\delta\Big(\frac{\epsilon_0}{r+d}\Big)\Big]\\&\leq t(1-t)r_0,\end{aligned}$$

and hence $\|S(k,m_j)x_{m_j}-w\|\leq r_0$. This implies that

$$r_0\leq r-\rho<\|x_k-w\|=\|S(k,m_j)x_{m_j}-w\|\leq r_0,$$

which gives a contradiction. This proves that (9) holds for $w\in F(T)$ and $0<t<1$.

Now let $w_1,w_2\in F(T)$ and $0<t<1$. For $k\geq j$, since

$$\begin{aligned}\|tx_k+(1-t)w_1-w_2\|&\leq\|tx_k+(1-t)w_1-S(k,j)[tx_j+(1-t)w_1]\|\\&\quad+\|S(k,j)[tx_j+(1-t)w_1]-w_2\|\\&\leq\|S(k,j)[tx_j+(1-t)w_1]-tx_k-(1-t)w_1\|\\&\quad+2\sum_{i=j}^{k-1}c_{n_i}+\|tx_j+(1-t)w_1-w_2\|\\&\leq\sup_{k\geq j}\|S(k,j)[tx_j+(1-t)w_1]-tx_k-(1-t)w_1\|\\&\quad+2\sum_{i=j}^{\infty}c_{n_i}+\|tx_j+(1-t)w_1-w_2\|,\end{aligned}$$

we obtain from (9) (replacing w by w_1) and the condition $\sum_{i=1}^{\infty} c_{n_i} < +\infty$ that

$$\limsup_{k\to\infty} \|tx_k + (1-t)w_1 - w_2\| \le \liminf_{j\to\infty} \|tx_j + (1-t)w_1 - w_2\|,$$

by first taking lim sup as $k \to \infty$ and next lim inf as $j \to \infty$. □

Lemma 3.3 [4], [22]. *Let X be a uniformly convex Banach space, $0 < b \le t_n \le c < 1$ for all $n \ge 1$, $r \ge 0$. Suppose that $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are sequences of X such that $\limsup_{n\to\infty} \|x_n\| \le r$, $\limsup_{n\to\infty} \|y_n\| \le r$, and $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = r$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Using Lemma 3.2 and 3.3, we have the following:

Theorem 3.1. *Suppose that X is a uniformly convex Banach space, C is a bounded convex subset of X and $T : C \to C$ is asymptotically nonexpansive in the intermediate sense. Put*

$$c_n = \max(0, \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|)),$$

so that $\lim_{n\to\infty} c_n = 0$. Suppose that $\{n_i\}$ is a sequence of nonnegative integers such that $\sum_{i=1}^{\infty} c_{n_i} < +\infty$ and such that $\mathcal{O} = \{i : n_{i+1} = 1 + n_i\}$ is quasi-periodic. Then for any $x_1 \in C$ and $\{x_i\}$ generated by (4)-(5) for $i \ge 1$, we have $\lim_{i\to\infty} \|x_i - Tx_i\| = 0$.

Proof. As in the proof of Lemma 3.2, we show that for $w \in F(T)$ the limit $\lim_{i\to\infty} \|x_i - w\|$ ($\equiv r$) exists. If $r = 0$, we immediately obtain

$$\|Tx_i - x_i\| \le \|Tx_i - w\| + \|w - x_i\| = \|Tx_i - Tw\| + \|w - x_i\|,$$

and hence, by the uniform continuity of T, $\lim_{i\to\infty} \|x_i - Tx_i\| = 0$.

Suppose $r > 0$. If $0 < a \le \alpha_i \le b < 1$ and $0 \le \beta_i \le b < 1$, then

$$\|T^{n_i} y_i - w\| \le \|y_i - w\| + c_{n_i} \le (1 + \beta_i)c_{n_i} + \|x_i - w\|,$$

and hence $\limsup_{i\to\infty} \|T^{n_i} y_i - w\| \le r$. Furthermore, we have

$$\lim_{i\to\infty} \|\alpha_i(T^{n_i} y_i - w) + (1-\alpha_i)(x_i - w)\| = \lim_{i\to\infty} \|x_{i+1} - w\| = r.$$

By Lemma 3.3, we have

$$\lim_{i\to\infty} \|T^{n_i} y_i - x_i\| = 0. \tag{11}$$

This is equivalent to

$$\lim_{i\to\infty} \|x_i - x_{i+1}\| = 0. \tag{12}$$

Also, since

$$\begin{aligned}\|T^{n_i}x_i - x_i\| &\le \|T^{n_i}x_i - T^{n_i}y_i\| + \|T^{n_i}y_i - x_i\| \\ &\le c_{n_i} + \|x_i - y_i\| + \|T^{n_i}y_i - x_i\| \\ &= c_{n_i} + \beta_i\|T^{n_i}x_i - x_i\| + \|T^{n_i}y_i - x_i\|,\end{aligned}$$

we have

$$(1-b)\|T^{n_i}x_i - x_i\| \le (1-\beta_i)\|T^{n_i}x_i - x_i\| \le c_{n_i} + \|T^{n_i}y_i - x_i\| \to 0$$

by taking the lim sup on both sides as $i \to \infty$. This yields

$$\lim_{i\to\infty} \|T^{n_i}x_i - x_i\| = 0. \tag{13}$$

On the other hand, we have, for all $i \ge 1$,

$$\begin{aligned}\|x_{i+1} - w\| &\le \alpha_i\|T^{n_i}y_i - w\| + (1-\alpha_i)\|x_i - w\| \\ &\le \alpha_i(\|y_i - w\| + c_{n_i}) + (1-\alpha_i)\|x_i - w\|\end{aligned}$$

and hence

$$\frac{\|x_{i+1} - w\| - \|x_i - w\|}{\alpha_i} \le \|y_i - w\| + c_{n_i} - \|x_i - w\|.$$

If $0 < a \le \alpha_i \le 1$ and $0 < a \le \beta_i \le b < 1$, we have

$$\begin{aligned}r &\le \liminf_{i\to\infty} \|y_i - w\| \le \limsup_{i\to\infty} \|y_i - w\| \le \limsup_{i\to\infty}(\beta_i c_{n_i} + \|x_i - w\|) \\ &\le \limsup_{i\to\infty}(bc_{n_i} + \|x_i - w\|) = \limsup_{i\to\infty} \|x_i - w\| = r\end{aligned}$$

and hence

$$r = \lim_{i\to\infty} \|y_i - w\| = \lim_{i\to\infty} \|\beta_i(T^{n_i}x_i - w) + (1-\beta_i)(x_i - w)\|.$$

Using Lemma 3.3 again, we obtain (13).

For the remaining part of the proof, it is now possible to imitate the steps of the original argument in [2] and so the conclusion follows similarly. □

Remark 3.1 (a) Under the assumptions of Theorem 3.1, in particular taking $\beta_i = 0$ for all $i \ge 1$ in (5), this result reduces to the original one due to Bruck et. al [2].

(b) We don't know whether Theorem 3.1 still holds in case $\{\alpha_i\}$ is a sequence in $(0, 1)$ which is bounded away from 0 and 1 and $\{\beta_i\}$ is chosen so that $\limsup_{i\to\infty} \beta_i = 1$.

As a direct observation of Theorem 1 in [2], we have the following:

Theorem 3.2. *Suppose a Banach space X has the uniform Opial property, C is a nonempty weakly compact subset of X and $T : C \to C$ is asymptotically nonexpansive in the weak sense. If $\{x_n\}$ is a sequence in C such that $\lim_{n\to\infty} \|x_n - w\|$ exists for each fixed point w of T, and if $\{x_n - T^k x_n\}$ is weakly convergent to 0 for each $k \geq 1$, then $\{x_n\}$ is weakly convergent to a fixed point of T.*

It is known [30] that if X is uniformly convex and has the Opial property, then X has the uniform Opial property. Thus, combining Theorem 3.1 and Theorem 3.2, we immediately have the following:

Theorem 3.3. *Under the assumptions of Theorem 3.1, if X has the Opial property and C is closed, then the sequence $\{x_i\}$ generated by (4)-(5) with starting $x_1 \in C$ is weakly convergent to a fixed point of T.*

Proof. By Theorem 3.1, $\lim_{i\to\infty} \|x_i - Tx_i\| = 0$. Since T is uniformly continuous, we have for each $k \in \mathbb{N}$, $\lim_{i\to\infty} \|x_i - T^k x_i\| = 0$, which in turn implies $x_i - T^k x_i \rightharpoonup 0$. The conclusion now follows from Theorem 3.2. □

Theorem 3.4. *Under the assumptions of Theorem 3.1, assume that X has a Fréchet differentiable norm and C is closed. If $\omega_w(x_i) \subseteq F(T)$, then the sequence $\{x_i\}$ generated by (4)-(5) with starting $x_1 \in C$ is weakly convergent to a fixed point of T, where $\omega_w(x_i)$ denotes the weak ω-lim set of sequence $\{x_i\}$, i.e., the set $\{w \in X : w = \text{w-}\lim_{j\to\infty} x_{i_j} \text{ for some } i_j \uparrow \infty\}$.*

Proof. Using Lemma 3.2, it is easy to see that the limit $\lim_{i\to\infty}\langle x_i, J(w_1 - w_2)\rangle$ exists for all $w_1, w_2 \in F(T)$ (for details, see [25] or [2]). In particular, this implies that

$$\langle p - q, J(w_1 - w_2)\rangle = 0 \quad \text{for all } p, q \text{ in } \omega_w(x_i). \tag{14}$$

Replacing w_1 and w_2 in (14) by q and p, respectively, we have

$$0 = \langle p - q, J(q - p)\rangle = -\|p - q\|^2,$$

for all $p, q \in \omega_w(x_i)$. This proves the uniqueness of weak subsequential limits of $\{x_i\}$ and completes the proof that $\{x_i\}$ converges weakly. □

Remark 3.2. If $I - T$ is demiclosed at 0, i.e., for any sequence $\{x_i\}$ in C, the conditions $x_i \rightharpoonup w$ and $x_i - Tx_i \to 0$ imply $w - Tw = 0$, it easily follows from Theorem 3.1 that $\omega_w(x_i) \subseteq F(T)$.

It is well known [28] that if $T : C \to C$ is asymptotically nonexpansive, then $I - T$ is demiclosed at 0. As a direct consequence of Theorem 3.3 and 3.4, we have the following:

Corollary 3.1. *Let X be a uniformly convex Banach space which satisfies the Opial property or has a Fréchet differentiable norm, C is a nonempty bounded closed convex subset of X and $T : C \to C$ is anasymptotically nonexpansive mapping. Suppose $\{n_i\}$ is a sequence of nonnegative integers such that $\sum_{i=1}^{\infty}(L_{n_i} - 1) < +\infty$ and such that $\mathcal{O} = \{i : n_{i+1} = 1 + n_i\}$ is quasi-periodic. Then the sequence $\{x_i\}$ generated by (4)-(5) with starting $x_1 \in C$ is weakly convergent to a fixed point of T.*

Remark 3.3. If we take $n_i \equiv i$ for all $i \geq 1$ and if $\{\alpha_i\}$ and $\{\beta_i\}$ in (4)-(5) are chosen so that $\alpha_i \in [a, b]$ and $\beta_i \in [0, b]$ for some a, b with $0 < a \leq b < 1$, then Corollary 3.1 reduces to Theorem 3.2. due to [26]. Recently, it is known [27] that, under the assumptions of Corollary 3.1 (with $F(T) \neq \emptyset$ instead of the boundedness of C), if $T : C \to C$ is nonexpansive, then the sequence $\{x_i\}$ generated by an iteration of the form

$$x_{i+1} = \alpha_i T[\beta_i T x_i + (1 - \beta_i)x_i] + (1 - \alpha_i)x_i \tag{15}$$

starting $x_1 \in C$ is weakly convergent to a fixed point of T, where $\{\alpha_i\}$ and $\{\beta_i\}$ are chosen so that $\alpha_i \in [a, b]$ and $\beta_i \in [0, b]$ or $\alpha_i \in [a, 1]$ and $\beta_i \in [a, b]$ for some a, b with $0 < a \leq b < 1$. Compare this with Tan and Xu's result [25].

Theorem 3.5. *Under the assumptions of Theorem 3.1, if T has a precompact range, then the sequence $\{x_i\}$ generated by (4)-(5) with starting $x_1 \in C$ is strongly convergent to a fixed point of T.*

Proof. It follows from the proof of Theorem 1.5 in [21], that there exists $w \in C$ and a subsequence $\{x_{i_j}\}$ of $\{x_i\}$ which converges strongly to w. But T is continuous and $\lim_{i\to\infty} \|x_i - Tx_i\| = 0$ by Theorem 3.1. Thus w is a fixed point of T. As in the proof of Lemma 3.2 again, we observe that

$$\|x_{i_j+1} - w\| \leq \|x_{i_j} - w\| + c_{n_{i_j}}(1 + \beta_{i_j})$$

and for $m \geq 1$, we have

$$\|x_{i_j+m} - w\| \leq \|x_{i_j} - w\| + 2 \sum_{k=i_j}^{i_j+m-1} c_{n_k}.$$

Since $\lim_{j\to\infty} \|x_{i_j} - w\| = 0$ and $\sum_{i=1}^{\infty} c_{n_i} < +\infty$, we see that the whole sequence $\{x_i\}$ converges to w. □

Remark 3.4. We don't know whether Theorem 3.5 still remains true under the weak condition of X (that is, strict convexity). For a nonexpansive mapping $T : C \to C$ and the sequence $\{x_i\}$ defined by (15), see [27].

Recall that a mapping $T : C \to C$ is said to satisfy Condition A [23] if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that $\|x - Tx\| \geq f\big(d(x, F(T))\big)$ for all $x \in C$, where $d(x, F(T)) = \inf_{z \in F(T)} \|x - z\|$.

Theorem 3.6. *Under the assumptions of Theorem 3.1, if T satisfies Condition A, then the sequence $\{x_i\}$ generated by (4)-(5) with starting $x_1 \in C$ is strongly convergent to a fixed point of T.*

Proof. By Condition A, we have $\|x_i - Tx_i\| \geq f\big(d(x_i, F(T))\big)$ for all $i \geq 1$. In the proof of Lemma 3.2, since $\|T_i x - T_i y\| \leq 2c_{n_i} + \|x - y\|$ for all $x, y \in C$ and $i \geq 1$, we have

$$\|x_{i+1} - z\| = \|T_i x_i - T_i z\| \leq 2c_{n_i} + \|x_i - z\| \tag{16}$$

for all $z \in F(T)$ and so $d(x_{i+1}, F(T)) \leq 2c_{n_i} + d(x_i, F(T))$ for all $i \geq 1$. By Lemma 3.1 (or see [25; Lemma 1]), the limit $\lim_{i\to\infty} d(x_i, F(T))$ exists. We claim that $\lim_{i\to\infty} d(x_i, F(T)) = 0$. To this end, if not, i.e., $d := \lim_{i\to\infty} d(x_i, F(T)) > 0$, then we can choose a $k \in \mathrm{N}$ such that for all $i \geq k$, $0 < \frac{d}{2} < d(x_i, F(T))$. Then it follows from Condition (A) and Theorem 3.1 that

$$0 < f(d/2) \leq f\big(d(x_i, F(T))\big) \leq \|x_i - Tx_i\| \to 0$$

as $i \to \infty$. This is a contradiction, which shows that $d = 0$. We can thus choose a subsequence $\{x_{i_j}\}$ of $\{x_i\}$ such that $\|x_{i_j} - z_j\| \leq 2^{-j}$ for all $j \geq 1$ and some sequence $\{z_j\}$ in $F(T)$. Replacing i and z in (16) by i_j and z_j, respectively, we have

$$\|x_{i_j+1} - z_j\| \leq 2c_{n_{i_j}} + \|x_{i_j} - z_j\| \leq 2c_{n_{i_j}} + 2^{-j},$$

and hence

$$\begin{aligned}\|z_{j+1} - z_j\| &\leq \|z_{j+1} - x_{i_j+1}\| + \|x_{i_j+1} - z_j\| \\ &\leq 2^{-(j+1)} + 2c_{n_{i_j}} + 2^{-j} < 2(2^{-j} + c_{n_{i_j}}),\end{aligned}$$

which shows that $\{z_j\}$ is Cauchy and therefore converges strongly to a point z in $F(T)$, since $F(T)$ is closed. Now it is readily seen that $\{x_{i_j}\}$ converges strongly to z. Since the limit $\lim_{i\to\infty} \|x_i - z\|$ exists as in the proof of Lemma 3.2, $\{x_i\}$ itself converges strongly to $z \in F(T)$. □

Remark 3.5. If $T : C \to C$ is nonexpansive, Theorem 3.6 reduces to Theorem 3 due to Tan-Xu [25].

Finally we give a simple example of an asymptotically nonexpansive mapping in the intermediate sense for which the averaging iteration $\{x_i\}$ generated by (4)-(5) converges strongly to a unique fixed point of T.

Example 3.1. Consider $C := [0,1] \subseteq X := \mathbb{R}$ and let $a_n = 2^{1-n}$ for each $n \geq 1$. Then we construct a continuous mapping T as follows. On each subinterval $[a_{n+1}, a_n]$, the graph of T consists of the sides of the isosceles triangle with base $[a_{n+1}, a_n]$ and height a_n. Then $Ta_n = 0$, and if x_n denotes the middle of $[a_{n+1}, a_n]$, then $Tx_n = a_{n+1}$. If we further define $T0 = 0$, then $T : C \to C$ is asymptotically nonexpansive in the intermediate sense but it has no Lipschitz bound at 0. Obviously, $c_i < 2^{-i}$ for $n \geq 1$ in Theorem 3.1 and $\sum_{i=1}^{\infty} c_{n_i} < +\infty$. It also follows from Theorem 3.5 that for any $x_1 \in [0,1]$, the sequence $\{x_i\}$ generated by (4)-(5) for $i \geq 1$ converges strongly to a unique fixed point 0 of T.

4. Convergence theorem for multivalued nonexpansive mappings. For a metric space (X, d), we denote by $CB(X)$ the family of all nonempty bounded closed subsets of X, by $K(X)$ the family of all nonempty compact subsets of X and by H the Hausdorff metric on $CB(X)$ induced by the metric d of X, that is, for $A,\ B \in CB(X)$,

$$H(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b.A)\},$$

where $d(x,D) = \inf\{d(x,y) : y \in D\}$ is the distance from a point $x \in X$ to a subset $D \subset X$. Now recall that a multivalued mapping $T : X \to CB(X)$ is said to be *nonexpansive* if $H(Tx,Ty) \leq d(x,y)$, $x, y \in X$. Recall also that a sequence $\{A_n\}$ in $CB(X)$ is said to converge to an element $A \in CB(X)$ under *the Mosco sense* if

$$\text{w-}\limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n = A,$$

where $\text{w-}\limsup_{n\to\infty} A_n := \{x \in X\colon$ there are subsequences $\{n_k\}$ and $\{x_{n_k}\}$ with $x_{n_k} \in A_{n_k}$ such that $x_{n_k} \rightharpoonup x\}$ and $\liminf_{n\to\infty} A_n := \{x \in X :$ there exists $x_n \in A_n$ for each n such that $x_n \to x\}$. It is easy to see that if $H(A_n, A) \to 0$ ($A_n,\ A \in CB(X)$), then $A_n \to A$ under the sense of Mosco.

Let C be a nonempty bounded closed convex subset of a Banach space X and $T : C \to K(C)$ nonexpansive. For each fixed $u \in C$ and $t \in (0,1)$, we define the mapping $T_t : C \to K(C)$ by the same formula (1) as before. Then T_t is a multivalued contraction and hence it has a (nonunique, in general) fixed point $x_t \in C$ (see [16]): that is,

$$x_t \in tTx_t + (1-t)u. \tag{17}$$

Let $y_t \in Tx_t$ be such that

$$x_t = ty_t + (1-t)u. \tag{18}$$

Now a natural question arises whether Browder's theorem can be extended to the multivalued case. A simple example of Pietramala [18] shows that the answer is negative even if X is Euclidean.

Example 4.1 [18] Let $C = [0,1] \times [0,1]$ be the square in the real plane and $T : C \to K(C)$ defined by $T(a,b) =$ the triangle with vertices $(0,0)$, $(a,0)$, $(0,b), (a,b) \in C$. Then it is easy to see that for any $(a_i, b_i) \in C$, $i = 1, 2$,

$$H(T(a_1,b_1), T(a_2,b_2)) = \max\{|a_1 - a_2|, |b_1 - b_2|\} \le \|(a_1,b_1) - (a_2,b_2)\|,$$

showing that T is nonexpansive. It is also easy to see that the fixed point set of T is $F(T) = \{(a,0) : 0 \le a \le 1\} \cup \{(0,b) : 0 \le b \le 1\}$. Let $u = (1,0)$. Then the mapping T_t defined by (1) has the fixed point set

$$F(T_t) = \{(a,0) : 1 - t \le a \le 1\}.$$

Let

$$x_t = \begin{cases} (\frac{1}{n}, 0), & \text{if } t = 1 - \frac{1}{n} \\ (1,0) & \text{otherwise.} \end{cases}$$

Then $\{x_t\}$ satisfies (17) but does not converge.

The same example also shows that the net $\{F(T_t)\}$ of fixed point sets of the T_t's does not converge as $t \to 1$ to the fixed point set $F(T)$ of T under either the Hausdorff metric or the Mosco sense. However, López Acedo and Xu [13] gave under some restriction on $F(T)$ the following result which will be used in the proof of the main theorem.

Lemma 4.1 [13]. *Let C be a nonempty closed bounded convex subset of a Banach space X satisfying the Opial property and $T : C \to K(C)$ a nonexpansive mapping such that $F(T) = \{z\}$. Then for any $u \in C$, the net $\{F(T_t)\}$ of fixed point sets of the T_t's weakly converges as $t \to 1$ to the fixed point set $F(T)$ of T, that is,*

$$w\text{-}\limsup_{t \to 1} F(T_t) = w\text{-}\liminf_{t \to 1} F(T_t) = F(T).$$

Now we establish the following strong convergence theorem for multivalued nonexpansive mappings under assumption that the unique fixed point z of T is such that $Tz = \{z\}$.

Theorem 4.1. *Suppose X is a smooth Banach space with a weakly sequentially continuous duality mapping $J : X \to X^*$, C is a nonempty closed convex subset of E and $T : C \to K(C)$ a nonexpansive mapping with a unique fixed point z. Suppose in addition that $Tz = \{z\}$. Then $H(F(T_t), F(T)) \to 0$ as $t \to 1$.*

Proof. First we observe that $\{F(T_t)\}$ is uniformly bounded. In fact, given any $x_t \in F(T_t)$, we have some $y_t \in Tx_t$ such that $x_t = ty_t + (1-t)u$. Since

$$\|y_t - z\| = d(y_t, Tz) \leq H(Tx_t, Tz) \leq \|x_t - z\|,$$

we have $\|x_t - z\| \leq t\|y_t - z\| + (1-t)\|u - z\| \leq t\|x_t - z\| + (1-t)\|u - z\|$. This implies that $\|x_t - z\| \leq \|u - z\|$ and $\{x_t\}$ is uniformly bounded. Now choose $x_t \in F(T_t)$ such that

$$H(F(T_t), F(T)) = \sup_{x \in F(T_t)} \|x - z\| < \|x_t - z\| + 1 - t.$$

We show that $\|x_t - z\| \to 0$ as $t \to 1$. Indeed, we have $y_t \in Tx_t$ satisfying (18). Since $\|y_t - z\| = d(y_t, Tz) \leq H(Tx_t, Tz) \leq \|x_t - z\|$, we have

$$\begin{aligned}\langle \frac{1}{t}x_t - (\frac{1}{t} - 1)u - z, J(z - x_t)\rangle &= \langle y_t - z, J(z - x_t)\rangle \\ &\geq -\|z - x_t\|\|J(z - x_t)\| = -\|z - x_t\|^2 = \langle x_t - z, J(z - x_t)\rangle\end{aligned}$$

and hence $\langle (\frac{1}{t} - 1)(x_t - u), J(z - x_t)\rangle \geq 0$. So, we have $\langle x_t - u, J(z - x_t)\rangle \geq 0$. This immediately implies that

$$\langle z - u, J(z - x_t)\rangle = \langle z - x_t, J(z - x_t)\rangle + \langle x_t - u, J(z - x_t)\rangle \geq \|z - x_t\|^2.$$

Since $x_t \rightharpoonup z$ as $t \to 1$ by Lemma 4.1 and J is weakly sequentially continuous, we have $\|x_t - z\| \to 0$ as $t \to 1$. This completes the proof. □

Corollary 4.1. *Let the assumptions of Theorem 4.1 be satisfied. Then*

$$w\text{-}\limsup_{t \to 1} F(T_t) = \|\cdot\|\text{-}\liminf_{t \to 1} F(T_t) = F(T).$$

Corollary 4.2 [13]. *Let H be a real Hilbert space, C a nonempty closed convex subset of H, and $T : C \to K(C)$ a nonexpansive mapping with unique fixed point z. Suppose in addition that $Tz = \{z\}$. Then*

$$H(F(T_t), F(T)) \to 0 \quad as \quad t \to 1.$$

Remark 4.1. (1) Corollary 4.1 is an extension of Theorem 1 of Pietramala [18] (Corollary 1 of López Acedo and Xu [13]) to a Banach space setting.

(2) It is an open question whether the assumption $Tz = \{z\}$ in Theorem 1 can be omitted. We also do not know if Theorem 1 is valid in a Banach space with a Fréchet differentiable norm.

(3) We wish to point out that the Banach space X in Theorem 4.1 is not reflexive.

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