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VICTOR KHATSKEVICH, SIMEON REICH
and DAVID SHOIKHET

Semi-complete Holomorphic Vector Fields on Homogeneous Open Unit Balls in Banach Spaces

Abstract. We present characterizations of complex dynamical systems on homogeneous open unit balls in Banach spaces. More precisely, using holomorphic fixed point theory and a Hille-Yosida type characterization of holomorphic generators of one-parameter semigroups on convex bounded domains, we establish a criterion for holomorphic mappings on homogeneous balls to be semi-complete vector fields in terms of one-sided interior estimates.

Let D be a domain in a complex Banach space X, and let $\mathrm{Hol}(D,X)$ denote the set of holomorphic mappings from D into X.

Definition. A mapping $f \in \mathrm{Hol}(D,X)$ is said to be a semi-complete (respectively, complete) vector field on D if the Cauchy problem

$$(1)\qquad \begin{cases} \frac{\partial u(t,x)}{\partial t} + f(u(t,x)) = 0 \\ u(0,x) = x \end{cases}$$

has a solution $u(t,x) \in D$ defined on $\mathbb{R}^+ \times D$ (respectively, $\mathbb{R} \times D$).

Note that if $f \in \mathrm{Hol}(D,X)$ is semi-complete, then a solution of (1) is unique and determines a one-parameter semigroup (flow) $F_t(x)$ $(= u(t,x))$

of holomorphic self-mappings F_t, $t \in \mathbb{R}^+$, of D. The mapping f is the infinitesimal generator of this flow, i.e.,

(2) $$\lim_{t \to 0^+} \frac{x - F_t(x)}{t} = f(x)$$

for all $x \in D$, where the limit in (2) is taken with respect to the norm of X. The class of semi-complete vector fields on D will be denoted by hol(D).

If f is a complete vector field on D, then its flow F_t (defined on $\mathbb{R}$) is a one-parameter group of automorphisms on D, i.e., $F_t \in \text{Aut}(D)$, $t \in \mathbb{R}$. In this case we will write that $f \in \text{aut}(D)$.

The motivation to investigate the class of semi-complete vector fields comes, for example, from the theory of stochastic branching processes [HTE], [SBA], fixed point theory [A-R-S], quantum mechanics [UH], optimization and control theory [H-M], and evolution equations [B-P],[BH2]. One of the important questions that often arises can be formulated as follows: What are the conditions for $f \in \text{hol}(D)$ to actually be in aut(D)?

If D is a bounded symmetric circular domain, then the class aut(D) of complete vector fields has been studied intensively. See, for example, [VJP], [I-S], [UH], and [SD]. In particular, it is known that aut(D) is a real Banach Lie algebra, while hol(D) is only a real cone [R-S1], [R-S2].

Our main purpose in this paper is to describe the class hol(D) of semi-complete vector fields in terms of one-sided estimates [K-Z], [SM]. This will also provide an answer to the above-mentioned question.

To motivate our approach we briefly review several previous results. For the one-dimensional case, namely $D = \Delta$, the unit disk in $\mathbb{C}$, an implicit condition which characterizes hol(Δ) was obtained by Berkson and Porta [B-P]. It was later shown by Abate [AM] that their condition can be rewritten in the form

(3) $$\text{Re} f(x)\bar{x} \geq -\frac{1}{2}\text{Re} f'(x)(1 - |x|^2), \quad x \in \Delta.$$

In addition, Abate gives a generalization of this condition (in a more complicated form) to the case of the Euclidean ball D in $\mathbb{C}^n$.

On the other hand, it follows directly from (2) that if f has a holomorphic extension to $\bar{D}$, then f satisfies the so-called "flow invariance" boundary condition

(4) $$\text{Re}\langle f(x), x\rangle \geq 0, \quad x \in \partial D,$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{C}^n$. This condition is sometimes called a one-sided estimate. Such conditions play an important role in the derivation of existence theorems for nonlinear equations (see [K-Z], [SM], [BFE],

[BH1]). Unfortunately, even for the one-dimensional case it is not clear how to get (4) from (3) in such a situation.

At the same time, for $n = 1$ it can be shown, by using the maximum principle for harmonic functions, that (4) implies the following interior condition:

$$\text{(5)} \qquad \operatorname{Re} f(x)\bar{x} \geq \operatorname{Re} f(0)\bar{x}(1 - |x|^2)\,, \quad x \in \Delta.$$

We claim that even if $f \in \operatorname{Hol}(\Delta, \mathbb{C})$ does not extend continuously to $\partial\Delta$, condition (5) is necessary and sufficient for f to be a semi-complete vector field. Moreover, this fact can be generalized in the same form to all infinite dimensional Banach spaces X with a homogeneous open unit ball D. Recall that a domain D is said to be homogeneous if for each two points x and y in D there is $f \in \operatorname{Aut}(D)$ such that $f(x) = y$. Examples of Banach spaces with a homogeneous open unit ball include Hilbert spaces, the space $L(H, K)$ of bounded linear operators between two Hilbert spaces H and K, those subspaces of $L(H, K)$ which are J^* algebras [HLA], and more generally, those Banach spaces which can be realized as JB^* triple systems [UH].

Let X' denote the dual space of X. We use the pairing $\langle x, x' \rangle$ to denote the value of a linear functional $x' \in X'$ at the element $x \in X$. The mapping $J : X \to 2^{X'}$ defined by $J(x) = \{x' \in X' : \langle x, x' \rangle = \| x \|^2 = \| x' \|^2\}$ is called the (normalized) duality mapping of X.

Theorem. *Let X be a complex Banach space with a homogeneous open unit ball D. Then*

1. *If $f \in \operatorname{Hol}(D, X)$ is a semi-complete vector field, then for each $x \in D$, and each $x' \in J(x)$ the following condition holds:*

$$\text{(6)} \qquad \operatorname{Re}\langle f(x), x' \rangle \geq Re\langle f(0), x' \rangle (1 - \| x \|^2).$$

2. *Conversely, if $f \in \operatorname{Hol}(D, X)$ is bounded on each ball strictly inside D, and for each $x \in D$ there is $x' \in J(x)$ such that condition (6) holds, then f is a semi-complete vector field on D.*

The proof of the Theorem is based on several results which are of independent interest.

The first of these results is an analogue of the Hille-Yosida theorem.

Proposition 1. *Let D be a bounded convex domain in X and let $f \in \mathrm{Hol}(D,X)$. Then $f \in \mathrm{hol}(D)$ if and only if for each $\lambda > 0$ the nonlinear resolvent $R(\lambda, f) = (I + \lambda f)^{-1}$ is well-defined and belongs to $\mathrm{Hol}\,(D,D)$. In addition, for each $t \geq 0$ there exists the local uniform limit*

$$u(t, \bullet) = \lim_{n\to\infty} R^n(\frac{t}{n}, f)$$

and it is the solution of the Cauchy problem (1).

Proposition 2. *Let D be a bounded domain in a complex Banach space X, and let $\{F_t\}_{t\geq 0} \subset \mathrm{Hol}(D,D)$ be a one-parameter semigroup on D. Then $F_t : R^+ \to \mathrm{Hol}(D,D)$ is differentiable in $t \geq 0$ if and only if it is locally uniformly continuous on $\mathbb{R}^+ = [0,\infty)$. In addition, in this case the limit in* (2),

$$(2') \qquad \lim_{t\to 0^+} \frac{I - F_t}{t} = f$$

is locally uniform on D and $f \in \mathrm{Hol}(D,X)$ is a semi-complete vector field which is bounded on each ball strictly inside D.

Using Yosida approximations, the Earle-Hamilton fixed point theorem [E-H] and Propositions 1 and 2, one can establish that the class $\mathrm{hol}(D)$ is a real cone. This fact, in turn, combined with the representation of $\mathrm{aut}(D)$ on bounded symmetric domains [UH], implies the following assertion.

Proposition 3. *Let D be a bounded symmetric circular domain in X. Then $\mathrm{hol}(D)$ admits the direct sum decomposition*

$$\mathrm{hol}(D) = G_+ \bigoplus G_0,$$

where $G_+ = \{h \in \mathrm{hol}(D),\ h(0) = 0\}$ and $G_0 = \Big\{g_a \in \mathrm{aut}(D) : g_a(x) = a - P_a(x)$, where $a \in X$, $P_a(x)$ is a homogeneous polynomial of degree 2 such that $P_{ia} = -iP_a\Big\}$.

Now it is easy to show that if D is a homogeneous ball, then for each $g_a \in G_0$, $x \in D$, and $x' \in J(x)$, the following equality holds:

$$\mathrm{Re}\langle g_a(x), x'\rangle = \mathrm{Re}\langle a, x'\rangle(1 - \| x \|^2).$$

In addition, it follows by the Schwarz Lemma and (2) that for each $h \in G_+$ and $x \in D$,

$$\mathrm{Re}\langle h(x), x' \rangle \geq 0, \quad x' \in J(x).$$

Thus for $f \in \mathrm{hol}(D)$, setting $a = f(0)$ and using Proposition 3 we obtain the first assertion of the Theorem.

Applying Proposition 1 and arguments similar to those used in [A-R-S] to show the existence of null points, we can now establish the second assertion of the Theorem.

Corollary. *Let D be a homogeneous ball in X. Then a semi-complete vector field f is complete if and only if the linear operator $A = if'(0)$ is Hermitian.*

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International College of Technology
ORT Braude College Campus
P. O. Box 78, 20101 Karmiel, Israel

received July 13, 1998

Department of Mathematics
The Technion-Israel Institute of Technology
32000 Haifa, Israel
e-mail: sreich@techunix.technion.ac.il

Department of Applied Mathematics
International College of Technology
P. O. Box 78, 20101 Karmiel, Israel
e-mail: davs@tx.technion.ac.il