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ADAM IDZIK and DARIUSZ MIKLASZEWSKI

Borsuk–Ulam Type Theorems I

ABSTRACT. Two new theorems are proved. They make plausible a more general conjecture: there exist p pairwise disjoint faces of a convex compact set C in $R^{(p-1)(n+1)}$ with nonempty interior such that the intersection of their images under an admissible map $\varphi : C \to R^n$ is nonempty.

Let m, n be natural numbers and let R^n denote n-dimensional Euclidean space, K^n - the unit ball and its boundary $\partial K^n = S^{n-1}$ - the unit sphere. Given a subset X of R^n, let $\mathcal{P}(X)$ denote the family of all nonempty subsets of X. We say that a map $\varphi : X \to \mathcal{P}(R^m)$ is upper semicontinuous (u.s.c.) on X if the set $\{x \in X : \varphi(x) \subset V\}$ is open in X whenever V is an open subset of R^m. The unit simplex in R^{n+1} is denoted by Δ^n.

In his paper Górniewicz [G] introduced a new class of multi-valued functions called admissible maps. The class of admissible maps contains acyclic maps (e.g. contractible or convex valued) and it is essentially larger. The composition of two admissible maps is again an admissible map (for definition and properties see also [GG]).

To formulate our result we introduce some notions. Given a convex compact set $C \subset R^n$ with the nonempty interior and a vector $a \in R^n, a \neq 0$,

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we write $C(a) = \{x \in C :< a, x >= \max_{t \in C} < a, t >\}$; $C(a)$ is a face of C perpendicular to the vector a.

Two points of C say x and y, are said to be opposite if for some $a \in R^n$ $x \in C(a)$ and $y \in C(-a)$. In this case we may say that the two faces $C(a)$ and $C(-a)$ are opposite.

In [Id] we proved the following generalization of the Borsuk-Ulam theorem:

Theorem 1. *Let $C \subset R^{n+1}$ be a convex compact set with the nonempty interior. For every admissible map $\varphi : \partial C \to \mathcal{P}(R^n)$, there exist two opposite points of C say x and y, such that $\varphi(x) \cap \varphi(y) \neq \emptyset$.*

□

Also in [Id] we formulated the following

Conjecture. *For natural numbers n, p and $q = (p-1)(n+1)$ let $C \subset R^q$ be a convex compact set with the nonempty interior. For every admissible map $\varphi : \partial C \to \mathcal{P}(R^n)$ there exist p pairwise disjoint faces of $C : C_1, \ldots, C_p$ such that $\varphi(C_1) \cap \ldots \cap \varphi(C_p) \neq \emptyset$.*

□

Theorem 1 is the case $p = 2$ of this conjecture. Some special cases of this conjecture are also true. Tverberg [T] proved it for C - the unit simplex and φ - the linear single-value map and Bárány, Shlosman and Szücs [BSS] proved the following

Theorem 2 (Theorem in [BSS]). *Suppose p is prime, $n \geq 1$, $q = (p-1)(n+1)$ and $f : \Delta^q \to R^n$ is a continuous function. Then there exist p pairwise disjoint faces $\Delta^{t_1}, \ldots, \Delta^{t_p}$ of Δ^q such that the set $f(\Delta^{t_1}) \cap \ldots \cap f(\Delta^{t_p})$ is nonempty.*

□

A generalization of this theorem was made a few years ago by Sarkaria [Sa] and applied to prove the existence of skeletons nonembeddable in some Euclidean spaces.

In this paper we will prove two theorems which approach our conjecture.

The conjecture is true for C - the simplex and for $n = 1$. In this case we have the following:

Theorem 3. *For a natural p and an admissible map $\varphi : \partial \Delta^{2(p-1)} \to R$ there exist p pairwise disjoint faces (subsimplexes) of $\Delta^{2(p-1)} : C_1, \ldots, C_p$ such that $\varphi(C_1) \cap \ldots \cap \varphi(C_p) \neq \emptyset$.*

Proof. Let V denote the set of vertices of $\Delta^{2(p-1)}$. Now, take $x_v \in \varphi(v)$ for each $v \in V$ and define a linear function $\hat{f} : \partial\Delta^{2(p-1)} \to R$ by $\hat{f}(v) = x_v (v \in V)$ and then extend linearly. By Tverberg's theorem there exist p pairwise disjoint faces $C_1, \dots, C_p$ of $\Delta^{2(p-1)}$ such that $\hat{f}(C_1) \cap \dots \cap \hat{f}(C_p) \neq \emptyset$. These faces satisfy also our theorem because $\varphi(C_i) \supset \hat{f}(C_i)$ for each $i \in \{1, \dots, p\}$.

□

The conjecture is also true in the case p is a prime number and all the faces of the set C are points except for a finite number of pairwise disjoint faces.

To prove our theorem we recall a special case of the Izydorek theorem:

Theorem 4 (Theorem 1. 3 in [Iz]). *Let a cyclic group Z_p of a prime order act freely on a sphere S^n $(p > 2)$. If $\varphi : S^n \to R^d$ is an admissible map, then the covering dimension of the set*

$$A_\varphi = \{x \in S^n : \varphi(x) \cap \varphi(gx) \cap \dots \cap \varphi(g^{p-1}x) \neq \emptyset\}$$

is not less than $n - (p-1)d$ (g is a fixed generator of G and n, d are natural numbers).

□

Theorem 5. *For natural numbers n, p and $q = (p-1)(n+1)$, p prime, let $C \subset R^q$ be a convex compact set with the nonempty interior and let all the faces of C be points except for a finite number of pairwise disjoint faces. Then, for every admissible map $\varphi : \partial C \to \mathcal{P}(R^n)$ there exist p pairwise disjoint faces $C_1, \dots, C_p \in \partial C$ such that $\varphi(C_1) \cap \dots \cap \varphi(C_p) \neq \emptyset$.*

Proof. By Theorem 1 we may assume $p > 2$. Denote by I the set $\{1, \dots, p\}$. There is an u.s.c. map ψ from S^{q-1} onto ∂C which define one to one correspondence between points in S^{q-1} and faces in C. Let $x = (x_1, \dots, x_p) \in S^{p(n+1)-1}$ and $x_i \in R^{n+1}$ for $i \in I$.

Now, consider the sphere $\hat{S}^{q-1} := \{x \in S^{p(n+1)-1} : \Sigma_{i \in I} x_i = 0\}$, which is homeomorfic to S^{q-1}. Observe that there is a natural action Z_p on $\hat{S}^{q-1}$, which acts freely, defined by $g(x_1, \cdots, x_p) = (x_2, \dots, x_p, x_1)$.

The function $\sigma = \varphi\psi : S^{q-1} \to \mathcal{P}(R^n)$ is an admissible map. Because $q - 1 = (p-1)(n+1) - 1 \geq (p-1)n$ for $p \geq 2$, then by Theorem 4 there exist p pairwise different points in S^{q-1}: $x, g(x), \dots, g^{p-1}(x)$ such that $\sigma(x) \cap \sigma(g(x)) \cap \dots \cap \sigma(g^{p-1}(x)) \neq \emptyset$. And we can take $C_i = \psi(g^{i-1}(x))$ for $i \in I$.

□

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Institute of Computer Science
Polish Academy of Sciences
Ordona 21, 01-237 Warszawa, Poland
e-mail: adidzik@ipipan.waw.pl

received October 21, 1997

Faculty of Mathematics and Informatics
Nicolas Copernicus University
Chopina 12-18, 87-100 Toruń, Poland
e-mail: miklasze@mat.uni.torun.pl