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Asymptotically Perturbed Norms of Classical Sequence Spaces with Applications to Fixed Point Theory

ABSTRACT. This article is a survey of results on Banach spaces containing asymptotically isometric copy of ℓ^1 and c_0, and their relationship to the fixed point property.

The concept of a Banach space containing an asymptotically isometric copy of ℓ^1 is a useful tool in identifying Banach spaces that fail the fixed point property. It was introduced by Dowling and Lennard in [DL] where it was initially used to prove that nonreflexive subspaces of $L^1[0,1]$ fail the fixed point property. Soon after, Dowling, Lennard and Turett [DLT1] used this notion to prove that every equivalent renorming of $\ell^1(\Gamma)$, for Γ uncountable, fails the fixed point property. In the same paper, the concept of a Banach space containing an asymptotically isometric copy of c_0 was introduced and it is proved that Banach spaces containing an asymptotically isometric copy of c_0 fail the fixed point property. Thus the concept of a Banach space containing an asymptotically isometric copy of c_0 is another tool in identifying Banach spaces failing the fixed point property.

This article is a survey of results on Banach spaces containing asymptotically isometric copy ℓ^1 and c_0, and their relationship to the fixed point property. We have divided this paper into five sections. In section 1 we define the concepts of a Banach space containing an asymptotically isometric copy of ℓ^1 or c_0. We prove that such Banach spaces fail the fixed point property. We also give alternative methods for recognizing asymptotically isometric copies of ℓ^1 and c_0. In section 2 we give examples of classes of Banach spaces containing asymptotically isometric copies of ℓ^1 and c_0 as well as examples of renormings of ℓ^1 (respectively, c_0) that fail to contain an asymptotically isometric copy of ℓ^1 (respectively, c_0). Section 3 deals with refinements of the James Distortion Theorems. The main result of this section is that Banach spaces containing c_0 fail the asymptotic fixed point property. Duality is the topic in section 4 and the main result is an asymptotically isometric analogue of a classical result of Bessaga and Pełczyński [BP]. Finally, in section 5 we consider the relationship between Banach spaces containing an asymptotically isometric copy of c_0 and Banach spaces with weak normal structure. We also prove that Banach spaces with the generalized Gossez-Lami Dozo property do not contain a copy of c_0.

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1. Asymptotically isometric copies of ℓ^1 and c_0. As a way of motivating the definitions of a Banach space containing asymptotically isometric copies of ℓ^1 and c_0, we begin by recalling James's Distortion Theorems.

Theorem 1.1. [J] *If a Banach space X contains an isomorphic copy of ℓ^1 and if $\varepsilon > 0$, then there exists a sequence $(x_n)_n$ in X so that*

$$(1-\varepsilon)\sum_{n=1}^{\infty}|a_n| \le \left\|\sum_{n=1}^{\infty} a_n x_n\right\| \le \sum_{n=1}^{\infty}|a_n|,$$

for all $(a_n)_n \in \ell^1$.

Theorem 1.2. [J] *If a Banach space X contains an isomorphic copy of c_0 and if $\varepsilon > 0$, then there exists a sequence $(x_n)_n$ in X so that*

$$(1-\varepsilon)\sup_n|a_n| \le \left\|\sum_{n=1}^{\infty} a_n x_n\right\| \le \sup_n|a_n|,$$

for all $(a_n)_n \in c_0$.

Definition 1.3. [DL] A Banach space X is said to contain an asymptotically isometric copy of ℓ^1 if there is a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and a sequence $(x_n)_n$ in X so that $\sum_{n=1}^{\infty}(1-\varepsilon_n)|a_n| \leq \left\|\sum_{n=1}^{\infty} a_n x_n\right\| \leq \sum_{n=1}^{\infty}|a_n|$, for all $(a_n)_n \in \ell^1$.

Definition 1.4. [DLT1] A Banach space X is said to contain an asymptotically isometric copy of c_0 if there is a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and a sequence $(x_n)_n$ in X so that $\sup_n(1-\varepsilon_n)|a_n| \leq \left\|\sum_{n=1}^{\infty} a_n x_n\right\| \leq \sup_n |a_n|$, for all $(a_n)_n \in c_0$.

The usefulness of these notions can be found in the next two results.

Theorem 1.5 [DL]. *If a Banach space X contains an asymptotically isometric copy of ℓ^1, then X fails the fixed point property for nonexpansive mappings on closed bounded convex subset of X.*

Proof. By assumption there is a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and a sequence $(x_n)_n$ in X so that $\sum_{n=1}^{\infty}(1-\varepsilon_n)|a_n| \leq \left\|\sum_{n=1}^{\infty} a_n x_n\right\| \leq \sum_{n=1}^{\infty}|a_n|$, for all $(a_n)_n \in \ell^1$. Let $(\lambda_n)_n$ be a strictly decreasing sequence in $(1,\infty)$ with $\lim_{n\to\infty} \lambda_n = 1$. By passing to subsequences if necessary we can assume that $\lambda_{n+1} < (1-\varepsilon_n)\lambda_n$.

Define $y_n = \lambda_n x_n$, for all $n \in \mathbb{N}$, and let $C = \overline{\text{co}}(\{y_n : n \in \mathbb{N}\})$, the closed convex hull of the sequence $(y_n)_n$. Clearly, C is a closed, bounded convex subset of X whose elements are of the form $z = \sum_{n=1}^{\infty} t_n y_n$, where $t_n \geq 0$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} t_n = 1$. Define a mapping $T : C \to C$ by $T(\sum_{n=1}^{\infty} t_n y_n) = \sum_{n=1}^{\infty} t_n y_{n+1}$, where $t_n \geq 0$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} t_n = 1$. It is easily seen that T has no fixed points in C. We will now show that T is nonexpansive (in fact, we will show that T is contractive).

Let $z = \sum_{n=1}^{\infty} t_n y_n$ and $w = \sum_{n=1}^{\infty} s_n y_n$ be elements of C with $z \neq w$. Then

$$\begin{aligned}
\|Tz - Tw\| &= \left\|\sum_{n=1}^{\infty}(t_n - s_n)y_{n+1}\right\| \leq \sum_{n=1}^{\infty}|t_n - s_n| \, \|y_{n+1}\| \\
&\leq \sum_{n=1}^{\infty}|t_n - s_n|\lambda_{n+1} < \sum_{n=1}^{\infty}|t_n - s_n|\lambda_n(1-\varepsilon_n) \\
&\leq \left\|\sum_{n=1}^{\infty}(t_n - s_n)\lambda_n x_n\right\| = \|z - w\|.
\end{aligned}$$

This completes the proof.

In this same manner we obtain

Theorem 1.6. [DLT1] *If a Banach space X contains an asymptotically isometric copy of c_0, then X fails the fixed point property for nonexpansive mappings on closed bounded convex subsets of X.*

Proof. By assumption there is a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and a sequence $(x_n)_n$ in X so that $\sup_n(1-\varepsilon_n)|a_n| \leq \left\| \sum_{n=1}^{\infty} a_n x_n \right\| \leq \sup_n |a_n|$, for all $(a_n)_n \in c_0$. Let $(\lambda_n)_n$ be a strictly decreasing sequence in $(1,\infty)$ converging to 1. By passing to subsequences if necessary we can assume that $\lambda_{n+1} < (1-\varepsilon_n)\lambda_n$.

Define $y_n = \lambda_n x_n$ for all $n \in \mathrm{N}$ and

$$C = \left\{ \sum_{n=1}^{\infty} t_n y_n : (t_n)_n \in c_0 \ , \ \ 0 \leq t_n \leq 1 \ \text{ for all } n \in \mathrm{N} \right\} .$$

C is clearly a closed bounded and convex subset of X. Define $T : C \to C$ by

$$T\left(\sum_{n=1}^{\infty} t_n y_n\right) = y_1 + \sum_{n=1}^{\infty} t_n y_{n+1} \ , \quad \text{for} \ \sum_{n=1}^{\infty} t_n y_n \in C \ .$$

It is easily seen that T has no fixed points in C. To see that T is nonexpansive (in fact, contractive), let $z = \sum_{n=1}^{\infty} t_n y_n$ and $w = \sum_{n=1}^{\infty} s_n y_n$ be elements of C with $z \neq w$.

Then

$$\begin{aligned}
\|Tz - Tw\| &= \left\| \sum_{n=1}^{\infty} (t_n - s_n) y_{n+1} \right\| = \left\| \sum_{n=1}^{\infty} (t_n - s_n)\lambda_{n+1} x_{n+1} \right\| \\
&\leq \sup_n |t_n - s_n| \lambda_{n+1} < \sup_n |t_n - s_n| \lambda_n (1-\varepsilon_n) \\
&\leq \left\| \sum_{n=1}^{\infty} (t_n - s_n)\lambda_n x_n \right\| = \|z - w\| \ .
\end{aligned}$$

This completes the proof.

The last results in this section give alternative methods of recognizing asymptotically isometric copies of ℓ^1 and c_0.

Theorem 1.7. *A Banach space X contains an asymptotically isometric copy of ℓ^1 if and only if there is a sequence $(x_n)_n$ in X such that*

(1) *there are constants $0 < m \leq M < \infty$ so that for all $(t_n)_n \in \ell^1$,*

$$m \sum_{n=1}^{\infty} |t_n| \leq \left\| \sum_{n=1}^{\infty} t_n x_n \right\| \leq M \sum_{n=1}^{\infty} |t_n| ,$$

and

(2) $\lim_{n\to\infty} \|x_n\| = m$.

Proof. Suppose that X contains an asymptotically isometric copy of ℓ^1. Then there is a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and a sequence $(y_n)_n$ in X so that

$$\sum_{n=1}^{\infty} (1-\varepsilon_n)|t_n| \leq \left\| \sum_{n=1}^{\infty} t_n y_n \right\| \leq \sum_{n=1}^{\infty} |t_n|$$

for all $(t_n)_n \in \ell^1$. Let $x_n = (1-\varepsilon_n)^{-1} y_n$ for each $n \in \mathbb{N}$. Then for all $(t_n)_n \in \ell^1$,

$$\sum_{n=1}^{\infty} |t_n| \leq \left\| \sum_{n=1}^{\infty} t_n x_n \right\| \leq \sum_{n=1}^{\infty} (1-\varepsilon_n)^{-1} |t_n| \leq (1-\varepsilon_1)^{-1} \sum_{n=1}^{\infty} |t_n| .$$

Also, since $1 \leq \|x_n\| \leq (1-\varepsilon_n)^{-1}$, $\lim\limits_{n\to\infty} \|x_n\| = 1$. Thus conditions (1) and (2) hold.

Conversely, suppose that conditions (1) and (2) hold. Fix a null sequence $(\varepsilon_n)_n$ in $(0,1)$. By scaling if necessary, we can assume that $m = 1$. Since $\lim\limits_{n\to\infty} \|x_n\| = m = 1$, and $\|x_n\| \geq m = 1$ for all $n \in \mathbb{N}$, by passing to subsequences, if necessary, we can assume that $1 \leq \|x_n\| \leq 1 + \varepsilon_n$ for all $n \in \mathbb{N}$. Define $y_n = (1+\varepsilon_n)^{-1} x_n$ for all $n \in \mathbb{N}$. Then since $\|y_n\| \leq 1$ we have

$$\left\| \sum_{n=1}^{\infty} t_n y_n \right\| \leq \sum_{n=1}^{\infty} |t_n| \quad \text{for all } (t_n)_n \in \ell^1 .$$

Also

$$\left\| \sum_{n=1}^{\infty} t_n y_n \right\| = \left\| \sum_{n=1}^{\infty} t_n (1+\varepsilon_n)^{-1} x_n \right\|$$
$$\geq \sum_{n=1}^{\infty} (1+\varepsilon_n)^{-1} |t_n| \geq \sum_{n=1}^{\infty} (1-\varepsilon_n)|t_n| .$$

Thus X contains an asymptotically isometric coy of ℓ^1.

An obvious consequence of Proposition 1.7 is the following

Corollary 1.8. *A Banach space X contains an asymptotically isometric copy of ℓ^1 if and only if there is a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and a sequence $(x_n)_n$ in X so that*

$$\sum_{n=k}^{\infty}|t_n| \le \Big\|\sum_{n=k}^{\infty} t_n x_n\Big\| \le (1+\varepsilon_k)\sum_{n=k}^{\infty}|t_n| \ ,$$

for all $(t_n)_n \in \ell^1$ and for all $k \in \mathbb{N}$.

We finish this section with analogous results for asymptotically isometric copies of c_0.

Theorem 1.9. *A Banach space X contains an asymptotically isometric copy of c_0 if and only if there exists a sequence $(x_n)_n$ in X and constants $0 < m < M < \infty$ such that, for all $(t_n)_n \in c_0$,*

$$m \sup_n |t_n| \le \Big\|\sum_{n=1}^{\infty} t_n x_n\Big\| \le M \sup_n |t_n|, \tag{1}$$

and

$$\lim_{n\to\infty} \|x_n\| = M. \tag{2}$$

Proof. Suppose that X contains an asymptotically isometric copy of c_0. Then there is a null sequence (ε_n) in $(0,1)$ and a sequence (x_n) in X such that $\sup_n(1-\varepsilon_n)|t_n| \le \left\|\sum_n t_n x_n\right\| \le \sup_n |t_n|$ for all $(t_n)_n \in c_0$. Let $m = \inf_n(1-\varepsilon_n)$. Then $0 < m < 1$ and for all $(t_n)_n \in c_0$,

$$m \sup_n |t_n| \le \Big\|\sum_n t_n x_n\Big\| \le \sup_n |t_n|.$$

Also, since $1-\varepsilon_n \le \|x_n\| \le 1$, we have $\lim_{n\to\infty} \|x_n\| = 1$.

Conversely, suppose that X contains a sequence $(x_n)_n$ satisfying conditions (1) and (2), and let $(\varepsilon_n)_n$ be a null sequence in $(0,1)$. By considering x_n/M rather than x_n, we can assume that $M = 1$. In particular, $m \le \|x_n\| \le 1$, for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} \|x_n\| = 1$. Hence, by passing to subsequences if necessary we can assume that $1-\varepsilon_n \le \|x_n\| \le 1$, for all $n \in \mathbb{N}$.

By passing to subsequences again, if necessary, we can assume that $\varepsilon_n < m/4$ for all $n \in \mathbb{N}$. Define $\delta_1 = m$ and $\delta_n = (4/m)\varepsilon_n$ for all $n \ge 2$.

Consider the expression $\|\sum_{k=1}^{n} t_k x_k\|$ for scalars $t_1, t_2, \ldots, t_n$. By assumption we have

$$m \max_{1\le k\le n} |t_k| \le \Big\|\sum_{k=1}^{n} t_k x_k\Big\| \le \max_{1\le k\le n} |t_k|.$$

By scaling we can assume that $\max\limits_{1\le k\le n} |t_k| = 1$. Thus we have

$$m \le \Big\|\sum_{k=1}^{n} t_k x_k\Big\| \le 1.$$

To show that X contains an asymptotically isometric copy of c_0 it suffices to show that $\max\limits_{1\le k\le n} (1-\delta_k)|t_k| \le \|\sum_{k=1}^{n} t_k x_k\| \le 1$. Since we already have the right-hand inequality, it remains only to show the left-hand inequality. First, note that if $|t_j| < m$, then

$$(1-\delta_j)|t_j| \le (1-\varepsilon_j)|t_j| < m \le \Big\|\sum_{k=1}^{n} t_k x_k\Big\|.$$

Secondly, if $|t_j| \ge m$, choose c_j with $c_j t_j = |t_j|$. By convexity we have

$$\begin{aligned}
1-\varepsilon_j &\le \|x_j\| \\
&\le \frac{1}{2}\Big\|x_j + \sum_{1\le k\le n, k\ne j} c_j t_k x_k\Big\| + \frac{1}{2}\Big\|x_j - \sum_{1\le k\le n, k\ne j} c_j t_k x_k\Big\| \\
&\le \frac{1}{2}\Big\|x_j + \sum_{1\le k\le n, k\ne j} c_j t_k x_k\Big\| + \frac{1}{2}.
\end{aligned}$$

Hence $\|x_j + \sum\limits_{1\le k\le n, k\ne j} c_j t_k x_k\| \ge 1 - 2\varepsilon_j$. By convexity again we have

$$\begin{aligned}
1-2\varepsilon_j &\le \Big\|x_j + \sum_{1\le k\le n, k\ne j} c_j t_k x_k\Big\| \\
&\le \frac{1}{2}\Big\|\sum_{k=1}^{n} c_j t_k x_k\Big\| + \frac{1}{2}\Big\|(2-c_j t_j)x_j + \sum_{1\le k\le n, k\ne j} c_j t_k x_k\Big\| \\
&\le \frac{1}{2}\Big\|\sum_{k=1}^{n} t_k x_k\Big\| + \frac{1}{2}(2-|t_j|).
\end{aligned}$$

Thus $\|\sum_{k=1}^{n} t_k x_k\| \geq |t_j| - 4\varepsilon_j$. Therefore, since $|t_j| \geq m$ we have

$$\Big\|\sum_{k=1}^{n} t_k x_k\Big\| \geq |t_j| - 4\varepsilon_j \geq |t_j| - (4/m)|t_j|\varepsilon_j = (1-\delta_j)|t_j|.$$

Putting the two pieces together we have $\|\sum_{j=1}^{n} t_j x_j\| \geq \max_{1\leq j\leq n}(1-\delta_j)|t_j|$ and so the proof is complete.

The following result follows immediately from Theorem 1.9.

Corollary 1.10. *A Banach space X contains an asymptotically isometric copy of c_0 if and only if there is a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and a sequence $(x_n)_n$ in X such that for all $(t_n)_n \in c_0$ and for all $k \in \mathbb{N}$*

$$(1-\varepsilon_k)\sup_{n\geq k}|t_n| \leq \Big\|\sum_{n\geq k} t_n x_n\Big\| \leq \sup_{n\geq k}|t_n|.$$

2. Spaces containing asymptotically isometric copies of ℓ^1. In Section 1, we saw that any Banach space containing an asymptotically isometric copy of ℓ^1 fails the fixed point property. In this section we will identify some Banach spaces containing asymptotically isometric copies of ℓ^1. Our first result was proved by Kadec and Pełczyński [KP], although they stated the result in a weaker form. The form of the Kadec-Pełczyński result we state here can be proved by analyzing the proof of the original result (see [Di]).

Theorem 2.1. *If X is a nonreflexive subspace of $(L^1[0,1], \|\cdot\|_1)$, then X contains an asymptotically isometric copy of ℓ^1. In particular, every nonreflexive subspace of $(L^1[0,1], \|\cdot\|_1)$ fails the fixed point property.*

Theorem 2.1, used in tandem with Maurey's Theorem [M], yields

Corollary 2.2. *Let X be a subspace of $(L^1[0,1], \|\cdot\|_1)$. Then X is reflexive if and only if X has the fixed point property.*

Our next result is of an isomorphic flavor.

Theorem 2.3. *If Γ is an uncountable set, then every renorming of $\ell^1(\Gamma)$ contains an asymptotically isometric copy of ℓ^1. Consequently, if Γ is uncountable, $\ell^1(\Gamma)$ cannot be renormed to have the fixed point property.*

Proof. Let e_γ be the element in $\ell^1(\Gamma)$ with $e_\gamma(\gamma) = 1$ and $e_\gamma(\alpha) = 0$ if $\alpha \neq \gamma$. Let $|||\cdot|||$ be an equivalent norm on $\ell^1(\Gamma)$. Then there exists constants $0 < m \leq M < \infty$ such that

$$m \sum_{\gamma \in F} |a_\gamma| \leq \left|\left|\left| \sum_{\gamma \in F} a_\gamma e_\gamma \right|\right|\right| \leq M \sum_{\gamma \in F} |a_\gamma| \ ,$$

for all finite subsets F of Γ and for all scalars $a_\gamma, \gamma \in F$.

Define $m_A = \inf\{|||\sum_{\gamma \in F} a_\gamma e_\gamma||| : \sum_{\gamma \in F} |a_\gamma| = 1, F$ is a finite subset of $A\}$, where A is an uncountable subset of Γ. Note that $m \leq m_A \leq M$ for all uncountable subsets of A of Γ and m_A increases as A decreases. Let $(A_\alpha)_{\alpha<\omega_1}$ be a decreasing chain of uncountable subsets of Γ with $\bigcap_{\alpha<\omega} A_\alpha = \emptyset$, where ω_1 is the first uncountable ordinal. Then $(m_{A_\alpha})_{\alpha<\omega_1}$ is a nondecreasing transfinite sequence of real numbers and hence eventually constant. Thus there exists α_0 such that if $\alpha \geq \alpha_0$, then $m_{A_\alpha} = m_{A_{\alpha_0}} \equiv m_0$.

Consider A_{α_0}. There exists $n_1 \in \mathbb{N}$, and real numbers a_j^1 and elements $\gamma_j^1 \in A_{\alpha_0}$ for $j = 1, ..., n_1$ such that $\sum_{j=1}^{n_1} |a_j^1| = 1$ and $m_0 \leq |||\sum_{j=1}^{n_1} a_j^1 e_{\gamma_j^1}||| \leq m_0 + 2^{-1}$. Since $\cap_{\alpha<\omega_1} A_\alpha = \emptyset$, there exists $\alpha_1 \geq \alpha_0$ so that $\gamma_j^1 \notin A_{\alpha_1}$, for $j = 1, ..., n_1$. Since $m_{A_{\alpha_1}} = m_0$, there exists $n_2 \in \mathbb{N}$, and real numbers a_j^2 and elements $\gamma_j^2 \in A_{\alpha_1}$ for $j = 1, ..., n_2$ such that $\sum_{j=1}^{n_2} |a_j^2| = 1$ and $m_0 \leq |||\sum_{j=1}^{n_2} a_j^2 e_{\gamma_j^2}||| \leq m_0 + 2^{-2}$. Continuing in this manner we obtain a block basic sequence (x_k) of (e_γ) where $x_k = \sum_{j=1}^{n_k} a_j^k e_{\gamma_j^k}$ and $m_0 \leq |||x_k||| \leq m_0 + 2^{-k}$.

Then for all scalars $a_1, ..., a_n$ we have

$$m_0 \sum_{k=1}^{n} |a_k| \leq \left|\left|\left| \sum_{k=1}^{n} a_k x_k \right|\right|\right| \leq M \sum_{k=1}^{n} |a_k|$$

and $\lim_{n\to\infty} |||x_k||| = m_0$. Hence, by Theorem 1.7, $(\ell^1(\Gamma), |||\cdot|||)$ contain an asymptotically isometric copy of ℓ^1.

We are now easily able to obtain some corollaries.

Corollary 2.4. *If X is a separable Banach containing a copy of ℓ^1, then X^* cannot be renormed to have the fixed point property. In particular, ℓ^∞ cannot be renormed to have the fixed point property.*

Proof. By a result of Pełczyński [Pe], if a separable Banach space X contains a copy of ℓ^1, then X^* contains a copy of $\ell^1(\Gamma)$, for some uncountable Γ. An application of Theorem 2.3 completes the proof.

Corollary 2.5. *If a Banach space X contains a complemented copy of ℓ^1, then X^* cannot be renormed to have the fixed point property.*

Proof. If X contains a complemented copy of ℓ^1, then X^* contains a copy of ℓ^∞. By Corollary 2.4, ℓ^∞ cannot be renormed to have the fixed point property, and hence X^* cannot be renormed to have the fixed point property.

There are other situations where asymptotically isometric copies of ℓ^1 appear. For example, if (Ω, Σ, μ) is a finite measure space that is not purely atomic, then the Orlicz space $L^\Phi(\mu)$, endowed with the Orlicz norm, contains an asymptotically isometric copy of ℓ^1 whenever $L^\Phi(\mu)$ is nonreflexive [DLT1]. Nonreflexive subspaces of the Lorentz function space $L_{w,1}(0,\infty)$ also contain asymptotically isometric copies of ℓ^1 [CDL]. Similarly, nonreflexive subspaces of the trace class C_1 contain asymptotically isometric copies of ℓ^1 as do nonreflexive subspaces of the predual $\mathcal{M}_*$ of a von Neumann algebra $\mathcal{M}$ with a faithful, normal, finite trace τ [DDDL]. However, not every Banach space containing a copy of ℓ^1 contains an asymptotically isometric copy of ℓ^1 as the following example illustrates.

Example 2.6. Define an equivalent norm on ℓ^1 by

$$|||x||| = \sup_n \gamma_n \sum_{k=n}^{\infty} |\xi_k|,$$

for all $x = (\xi_n) \in \ell^1$, where (γ_n) is a fixed sequence in $(0,1)$ that strictly increases to 1. We will show that $(\ell^1, |||\cdot|||)$ does not contain an asymptotically isometric copy of ℓ^1. Let us assume that $(\ell^1, |||\cdot|||)$ contains an asymptotically isometric copy of ℓ^1. Then there is a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and a $|||\cdot|||$ - normalized sequence $(x_n)_n$ in ℓ^1 so that

$$(*) \qquad \sum_{j=1}^{\infty}(1-\varepsilon_j)|t_j| \le \left|\left|\left| \sum_{j=1}^{\infty} t_j x_j \right|\right|\right| \le \sum_{j=1}^{\infty} |t_j|, \quad \text{for all } t = (t_j) \in \ell^1.$$

Without loss of generality we can assume that the sequence $(x_n)_n$ is disjointly supported, i.e., that the support of x_m is disjoint from the support of

x_n if $m \neq n$. This is so because, since the closed unit ball of ℓ^1 is weak-star sequentially compact with respect to the predual c_0, by passing to a subsequence, we may suppose that (x_n) converges weak-star (and hence pointwise with respect to the canonical basis (e_n) of ℓ^1) to some $y \in \ell^1$. By replacing (x_n) by the $|||\cdot|||$-normalization of the sequence $(\frac{x_{2n}-x_{2n-1}}{2})_n$, we may suppose that $y = 0$. By the proof of the Bessaga-Pełczyński Theorem [BP,Di], we can pass to a subsequence of $(x_n)_n$ which is essentially disjointly supported. Truncating this subsequence appropriately we obtain a disjointly supported sequence, which when normalized, satisfies $(*)$. Consequently, we can and do assume that $(x_n)_n$ is disjointly supported.

By passing to subsequences if necessary, we can also assume that $\varepsilon_n < \frac{1}{2n}$ for all $n \in \mathbb{N}$.

Let $(m(k))_{k=0}^{\infty}$ be a strictly increasing sequence in $\mathbb{N} \cup \{0\}$ with $m(0) = 0$ and $(\xi_j)_{j=1}^{\infty}$ a sequence of scalars such that, for each $k \in \mathbb{N}$,

$$x_k = \sum_{j=m(k-1)+1}^{m(k)} \xi_j e_j \,.$$

For each $N \in \mathbb{N}$, we have

$$\begin{aligned}
N + 1 - \varepsilon_1 - N\varepsilon_N &\le |||x_1 + Nx_N||| \\
&= \sup_{\substack{1\le j\le m(1)\\ m(N-1)+1\le i\le m(N)}} \left\{ \gamma_j \left(\sum_{k=j}^{m(1)} |\xi_k| + N \sum_{k=m(N-1)+1}^{m(N)} |\xi_k| \right),\ \gamma_i N \sum_{k=i}^{m(N)} |\xi_k| \right\} \\
&\le \sup_{\substack{1\le j\le m(1)\\ m(N-1)+1\le i\le m(N)}} \left\{ \gamma_j \sum_{k=j}^{m(1)} |\xi_k| + N\gamma_{m(1)} \sum_{k=m(N-1)+1}^{m(N)} |\xi_k|,\ N\gamma_i \sum_{k=i}^{m(N)} |\xi_k| \right\} \\
&= \sup_{\substack{1\le j\le m(1)\\ m(N-1)+1\le i\le m(N)}} \left\{ \gamma_j \sum_{k=j}^{m(1)} |\xi_k| + \frac{N\gamma_{m(1)}}{\gamma_{m(N-1)+1}} \left(\gamma_{m(N-1)+1} \sum_{j=m(N-1)+1}^{m(N)} |\xi_k| \right), \right. \\
&\qquad \left. N\gamma_i \sum_{k=i}^{m(N)} |\xi_k| \right\} \le \max\left\{ 1 + \frac{N\gamma_{m(1)}}{\gamma_{m(N-1)+1}}\ ,\ N \right\}
\end{aligned}$$

Thus $N + 1 - \varepsilon_1 - N\varepsilon_N \le \max\left\{1 + \frac{N\gamma_{m(1)}}{\gamma_{m(N-1)+1}}, N\right\}$ for all $N \in \mathbb{N}$. Since $\varepsilon_1 < 1/2$ and $N\varepsilon_N < 1/2$, we have $N + 1 - \varepsilon_1 - N\varepsilon_n > N$, and hence

$$N + 1 - \varepsilon_1 - N\varepsilon_N \le 1 + \frac{N\gamma_{m(1)}}{\gamma_{m(N-1)+1}} \quad \text{for all } N \in \mathbb{N}\,.$$

Therefore

$$1+\frac{1}{N}-\frac{\varepsilon_1}{N}-\varepsilon_N \le \frac{1}{N}+\frac{\gamma_{m(1)}}{\gamma_{m(N-1)+1}}$$

and letting $N \to \infty$ yields $1 \le \gamma_{m(1)}$. This is a contradiction since $\gamma_n < 1$ for all $n \in \mathbb{N}$.

Remark. This example and similar type examples of renormings of ℓ^1 which do not contain asymptotically isometric copies of ℓ^1 can be found in [DJLT]. Examples of renormings of c_0 which do not contain asymptotically isometric copies of c_0 can also be found in [DJLT].

We will now move to Banach spaces containing asymptotically isometric copies of c_0. Our first result is similar to Theorem 2.3.

Theorem 2.7 [DLT2]. *If Γ is an uncountable set, then every renorming of $c_0(\Gamma)$ contains an asymptotically isometric copy of c_0. Consequently, if Γ is uncountable, $c_0(\Gamma)$ cannot be renormed to have the fixed point property.*

Proof. Let e_γ be the element in $c_0(\Gamma)$ with $e_\gamma(\gamma) = 1$ and $e_\gamma(\alpha) = 0$ if $\alpha \neq \gamma$. Let $|||\cdot|||$ be an equivalent norm on $c_0(\Gamma)$. Hence there exists m, $M > 0$ such that

$$m \sup_{\gamma\in F} |a_\gamma| \le \left|\left|\left| \sum_{\gamma\in F} a_\gamma e_\gamma \right|\right|\right| \le M \sup_{\gamma\in F} |a_\gamma|,$$

for all finite subsets F of Γ and for all scalars $a_\gamma, \gamma \in F$.

Define $m_A \equiv \sup\{|||\sum_{\gamma\in F} a_\gamma e_\gamma||| : \max_{\gamma\in F} |a_\gamma| = 1, F \text{ is a finite subset of } A\}$, where A is an uncountable subset of Γ. Note that $m \le m_A \le M$ for all uncountable subsets A of Γ and m_A decreases as A decreases. Let $(A_\alpha)_{\alpha<\omega_1}$ be a decreasing chain of uncountable subsets of Γ with $\bigcap_{\alpha<\omega_1} A_\alpha = \emptyset$, where ω_1 is the first uncountable ordinal. Then $(m_{A_\alpha})_{\alpha<\omega_1}$ is a non-increasing transfinite sequence of real numbers and hence eventually constant. Thus there exists α_0 such that if $\alpha \ge \alpha_0$, then $m_{A_\alpha} = m_{A_{\alpha_0}} \equiv m_0$.

Consider A_{α_0}. There exist a natural number n_1, and real numbers a_j^1 and elements γ_j^1 in Γ for $j = 1, \cdots, n_1$ such that $\max_{1\le j\le n_1} |a_j^1| = 1$ and $m_0 - 2^{-1} \le |||\sum_{j=1}^{n_1} a_j^1 e_{\gamma_j^1}||| \le m_0$. Since $\bigcap_{\alpha<\omega_1} A_\alpha = \emptyset$, there exists $\alpha_1 \ge \alpha_0$ such that $\gamma_j^1 \notin A_{\alpha_1}$ for $j = 1, \cdots, n_1$. Since $m_{A_{\alpha_1}} = m_0$, there exist a natural number n_2, and real numbers a_j^2 and elements γ_j^2 in Γ for $j = 1, \cdots, n_2$ such that $\max_{1\le j\le n_2} |a_j^2| = 1$ and $m_0 - 2^{-2} \le |||\sum_{j=1}^{n_2} a_j^2 e_{\gamma_j^2}||| \le m_0$. Continue in this manner to obtain a block basic sequence (x_k) of (e_γ) where $x_k = \sum_{j=1}^{n_k} a_j^k e_{\gamma_j^k}$ and $m_0 - 2^{-k} \le |||x_k||| \le m_0$.

Then for any $(a_k)_k \in c_0$ we have,

$$\left|\left|\left|\sum_{k=1}^{\infty} a_k x_k\right|\right|\right| = \left|\left|\left|\sum_{k=1}^{\infty} a_k \left(\sum_{j=1}^{n_k} a_j^k e_{\gamma_j^k}\right)\right|\right|\right| = \left|\left|\left|\sum_{k=1}^{\infty}\sum_{j=1}^{n_k} a_k a_j^k e_{\gamma_j^k}\right|\right|\right|$$

$$\leq m_0 \sup |a_k a_j^k| = m_0 \sup |a_k|,$$

and

$$\left|\left|\left|\sum_{k=1}^{\infty} a_k x_k\right|\right|\right| = \left|\left|\left|\sum_{k=1}^{\infty}\sum_{j=1}^{n_k} a_k a_j^k e_{\gamma_j^k}\right|\right|\right| \geq m \sup |a_k a_j^k| = m \sup |a_k|,$$

and $\lim_{n\to\infty} |||x_k||| = m_0$. Hence by Theorem 1.9, $(c_0(\Gamma), |||\cdot|||)$ contains an asymptotically isometric copy of c_0.

Remark. Even though the proof of Theorem 2.7 is quite similar to the proof of Theorem 2.3, we decided to include it because of the applications of Theorem 2.7 that will appear in section 5.

Another place where asymptotically isometric copies of c_0 appear naturally is in subspaces of $(c_0, \|\cdot\|_\infty)$.

Theorem 2.8. *If Y is a closed infinite dimensional subspace of $(c_0, \|\cdot\|_\infty)$, then Y contains an asymptotically isometric copy of c_0 and hence Y fails the fixed point property.*

Proof. Let $(e_n)_n$ denote standard unit vector basis in c_0. Let Z_n denote the closed linear span of $(e_j)_{j\geq n}$ in $(c_0, \|\cdot\|_\infty)$. Since Y is an infinite dimensional subspace of c_0, $Y \cap Z_n \neq \{0\}$ for all $n \in \mathbb{N}$. Thus we can choose $x_1 \in Y \cap Z_1$ with $\|x_1\|_\infty = 1$. We can write $x_1 = \sum_{n=1}^{\infty} \alpha_n^1 e_n$, where $\alpha_n^1 \in \mathbb{R}$ for all $n \in \mathbb{N}$. Let $n_0 = 1$. Choose $n_1 > 1$ so that $\sup_{n\geq n_1} |\alpha_n^1| < 2^{-2}$. Since $Y \cap Z_{n_1} \neq \{0\}$, choose $x_2 \in Y \cap Z_{n_1}$ with $\|x_2\|_\infty = 1$. We can write $x_2 = \sum_{n=n_1}^{\infty} \alpha_n^2 e_n$, where $\alpha_n^2 \in \mathbb{R}$ for all $n \in \mathbb{N}$. Choose $n_2 > n_1$ so that $\sup_{n\geq n_2} |\alpha_n^i| < 2^{-1}\cdot 2^{-3}$ for $i = 1, 2$. Since $Y \cap Z_{n_2} \neq \{0\}$, choose $x_3 \in Y \cap Z_{n_2}$ with $\|x_3\|_\infty = 1$. We can write $x_3 = \sum_{n=n_2}^{\infty} \alpha_n^3 e_n$, where $\alpha_n^3 \in \mathbb{R}$ for all $n \in \mathbb{N}$. Continuing inductively in this manner we obtain a strictly increasing sequence $(n_k)_k$ in $\mathbb{N}$ and a sequence $(x_k)_k$ in Y with $x_k = \sum_{n=n_{k-1}}^{\infty} \alpha_n^k e_n$, where $\sup_{n\geq n_{k-1}} |\alpha_n^i| < (k-1)^{-1}\cdot 2^{-k-1}$ for all $1 \leq i \leq k-1$ with $k \geq 2$. Fix $(t_k)_{k=1}^{\infty} \in c_0$ and consider the element $z = \sum_{k=1}^{\infty} t_k x_k \in Y$. Then

$$z = \sum_{k=1}^{\infty} t_k \sum_{n=n_{k-1}}^{\infty} \alpha_n^k e_n = \sum_{k=1}^{\infty}\sum_{n=n_{k-1}}^{n_k-1} \left(\sum_{i=1}^{k} t_i \alpha_n^i\right) e_n .$$

Also, since $z \in c_0$, we can write $z = \sum_{n=1}^{\infty} z_n e_n$. Hence for $n_{k-1} \leq n \leq n_k - 1$, $z_n = \sum_{i=1}^{k} t_i \alpha_n^i$. Let $b_k = \sum_{n=n_{k-1}}^{n_k-1} \alpha_n^k e_n$. Note that $\|b_k\|_\infty = 1$ for all $k \in \mathbb{N}$. Thus, for each $k \in \mathbb{N}$, we obtain

$$\begin{aligned}
\Big\| \sum_{n=n_{k-1}}^{n_k-1} z_n e_n - t_k b_k \Big\|_\infty &= \Big\| \sum_{n=n_{k-1}}^{n_k-1} \Big(\sum_{i=1}^{k} t_i \alpha_n^i \Big) e_n - t_k \sum_{n=n_{k-1}}^{n_k-1} \alpha_n^k e_n \Big\|_\infty \\
&= \Big\| \sum_{n=n_{k-1}}^{n_k-1} \Big(\sum_{i=1}^{k-1} t_i \alpha_n^i \Big) e_n \Big\|_\infty \\
&= \max_{n_{k-1} \leq n \leq n_k - 1} \Big| \sum_{i=1}^{k-1} t_i \alpha_n^i \Big| \\
&\leq (k-1)\|t\|_\infty \Big(\max_{\substack{n_{k-1} \leq n \leq n_k - 1 \\ 1 \leq i \leq k-1}} |\alpha_n^i| \Big) \\
&\leq (k-1)\|t\|_\infty \Big(\frac{1}{2^{k+1}(k-1)} \Big) = 2^{-k-1}\|t\|_\infty .
\end{aligned}$$

Also for $n_{k-1} \leq n \leq n_k - 1$, we have

$$\begin{aligned}
|z_n| = \Big| \sum_{i=1}^{k} t_i \alpha_n^i \Big| &\geq |t_k \alpha_n^k| - \Big| \sum_{i=1}^{k-1} t_i \alpha_n^i \Big| \\
&\geq |t_k||\alpha_n^k| - (k-1)\|t\|_\infty \max_{1 \leq i \leq k-1} |\alpha_n^i| \\
&\geq |t_k||\alpha_n^k| - (k-1)\|t\|_\infty \Big(\frac{1}{(k-1)2^{k+1}} \Big) \\
&= |t_k||\alpha_n^k| - 2^{-k-1}\|t\|_\infty .
\end{aligned}$$

Therefore, $\max_{n_{k-1} \leq n \leq n_k - 1} |z_n| \geq |t_k| - 2^{-k-1}\|t\|_\infty$, and so

$$\|z\|_\infty = \sup_{k \geq 1} \Big(\max_{n_{k-1} \leq n \leq n_k - 1} |z_n| \Big) \geq \sup_{k \geq 1} \big(|t_k| - 2^{-k-1}\|t\|_\infty \big) \geq 2^{-1}\|t\|_\infty .$$

Thus

$$\Big\| \sum_{n=n_{k-1}}^{n_k-1} z_n e_n - t_k b_k \Big\|_\infty \leq 2^{-k-1}\|t\|_\infty \leq 2^{-k}\|z\|_\infty$$

and therefore

$$\Big| \Big\| \sum_{n=n_{k-1}}^{n_k-1} z_n e_n \Big\|_\infty - |t_k| \Big| \leq 2^{-k}\|z\|_\infty .$$

In particular, for each $k \in \mathbb{N}$ we have $\|\sum_{n=n_{k-1}}^{n_k-1} z_n e_n\|_\infty - |t_k| \le 2^{-k}\|z\|_\infty$. Choose $k_0 \in \mathbb{N}$ so that $\|\sum_{n=n_{k_0-1}}^{n_{k_0}-1} z_n e_n\|_\infty = \|z\|_\infty$. Then $\|z\|_\infty - |t_{k_0}| \le 2^{-k_0}\|z\|_\infty$. Consequently,

$$\|z\|_\infty \le \frac{1}{1-2^{-k_0}}|t_{k_0}| \le \max_{k\ge 1}\left(\frac{1}{1-2^{-k}}\right)|t_k|.$$

We also have that for all $k \in \mathbb{N}$, $|t_k| - \|\sum_{n=n_{k-1}}^{n_k-1} z_n e_n\|_\infty \le 2^{-k}\|z\|_\infty$, and hence $|t_k| \le \|\sum_{n=n_{k-1}}^{n_k-1} z_n e_n\|_\infty + 2^{-k}\|z\|_\infty \le \|z\|_\infty + 2^{-k}\|z\|_\infty$. Therefore $\sup_{k\ge 1}\left(\frac{1}{1+2^{-k}}\right)|t_k| \le \|z\|_\infty$. Putting the above inequalities together we get that for all $(t_k)_{k=1}^\infty \in c_0$

$$\sup_{k\ge 1}\left(\frac{1}{1+2^{-k}}\right)|t_k| \le \|\sum_{k=1}^\infty t_k x_k\|_\infty \le \sup_{k\ge 1}\left(\frac{1}{1-2^{-k}}\right)|t_k|.$$

This means that $(x_k)_k$ spans an asymptotically isometric copy of c_0 in Y.

Remarks.

(1) The real Banach space $(c_0, \|\cdot\|_\infty)$ contains an infinite dimensional subspace that does not contain an isometric copy of c_0. Examples of such subspaces were independently constructed by D. Alspach [A] and E. Behrends [B]. Thus Theorem 2.8 can be considered as the optimal result of this type. We will outline Alspach's example below.

(2) We could modify the proof of Theorem 2.8 to show that every infinite dimensional subspace of $(\ell^1, \|\cdot\|_1)$ contains an asymptotically isometric copy of ℓ^1. However, since every infinite dimensional subspace of $(\ell^1, \|\cdot\|_1)$ is isometric to a nonreflexive subspace of $(L^1[0,1], \|\cdot\|_1)$, the result is already known from Theorem 2.1. Again this result can be considered as the optimal result of this type because Fonf and Kadec [FK] has constructed an infinite dimensional subspace of $(\ell^1, \|\cdot\|_1)$ which is strictly convex (and hence does not contain an isometric copy of ℓ^1).

Example 2.9 [A]. The space $(c_0, \|\cdot\|_\infty)$ contains a subspace that is isometric to $(\sum_n \ell_n^1)_{c_0}$, where ℓ_n^1 is n-dimensional ℓ^1. Let $(e_{n,k})_{k=1}^{n}{}_{,n=1}^{\infty}$ be the standard basis for $(\sum_n \ell_n^1)_{c_0}$. Thus $\|\sum a_{n,k} e_{n,k}\| = \sup_n \sum_{k=1}^n |a_{n,k}|$. For each $n \in \mathbb{N}$, define $f_n = e_{n,n} + \sum_{k>n} \frac{1}{8k} e_{k,n}$. Then, for $(a_n)_{n=1}^\infty \in c_0$, we have

$$\left\|\sum_{n=1}^\infty a_n f_n\right\| = \sup_n\left(|a_n| + \sum_{k<n}\frac{1}{8n}|a_k|\right).$$

Therefore,

$$\sup_n |a_n| \le \left\| \sum_{n=1}^{\infty} a_n f_n \right\| \le \sup_n \left(1+\frac{1}{8}\right)|a_n| .$$

Let $X = \overline{span}\{f_n\}_{n=1}^{\infty}$. We will show that X does not contain an isometric copy of c_0. To do this we will show that X does not contain an isometric copy of ℓ_2^∞, two-dimensional ℓ^∞. Suppose, to obtain a contradiction, that X does contain an isometric copy of ℓ_2^∞. Then there are elements x and y in X with $\|x\| = \|y\| = \|x+y\| = \|x-y\| = 1$.

Let $x = \sum_n a_n f_n$ and $y = \sum_n b_n f_n$. We may assume that there are indices n_0 and n_1 with $n_0 \le n_1$ such that

$$\|x\| = |a_{n_0}| + \sum_{k<n_0} \frac{1}{8n_0}|a_k| \quad \text{and} \quad \|y\| = |b_{n_1}| + \sum_{k<n_1} \frac{1}{8n_1}|b_k| .$$

Then $7/8 \le |a_{n_0}| \le 1$ and $7/8 \le |b_{n_1}| \le 1$.

Note that

$$|a_{n_0} - b_{n_0}| + \sum_{k<n_0} \frac{1}{8n_0}|a_k - b_k| \le \|x-y\| = 1, \text{ and}$$

$$|a_{n_0} + b_{n_0}| + \sum_{k<n_0} \frac{1}{8n_0}|a_k + b_k| \le \|x+y\| = 1 \text{ , so}$$

$$\begin{aligned} 2 &= 2\left(|a_{n_0}| + \sum_{k<n_0} \frac{1}{8n_0}|a_k|\right) \\ &\le 2\left(\max\{|a_{n_0}|, |b_{n_0}|\} + \sum_{k<n_0} \frac{1}{8n_0}\max\{|a_k|, |b_k|\}\right) \\ &\le |a_{n_0} - b_{n_0}| + |a_{n_0} + b_{n_0}| + \sum_{k<n_0} \frac{1}{8n_0}(|a_k - b_k| + |a_k + b_k|) \\ &\le 2 . \end{aligned}$$

Therefore $|b_k| \le |a_k|$ for all $k \le n_0$. Similarly, we can show that $|a_k| \le |b_k|$ for all $k \le n_1$. In particular, we have $|b_{n_0}| = |a_{n_0}| \ge 7/8$. Hence $|a_{n_0} + \varepsilon b_{n_0}| \ge 7/4$ for $\varepsilon = 1$ or $\varepsilon = -1$. For this ε, we have $1 = \|x + \varepsilon y\| \ge \sup_n |a_n + \varepsilon b_n| \ge 7/4$ and this is a contradiction. Hence X does not contain an isometric copy of ℓ_2^∞.

Our last example is of a renorming of ℓ^∞ which fails to contain an asymptotically isometric copy of c_0. We will give another verification of this result in section 5.

Example 2.10. $(\ell^\infty, |||\cdot|||)$ does not contain an asymptotically isometric copy of c_0, where the $|||\cdot|||$ norm is defined on ℓ^∞ as follows:

$$\text{if } x = (\xi_j)_{j=1}^\infty \in \ell^\infty \text{ then } |||x||| = \sup_j |\xi_j| + \sum_{j=1}^\infty 2^{-j}|\xi_j|.$$

Note that $|||\cdot|||$ is an equivalent norm on ℓ^∞ and $\|x\|_\infty \le |||x||| \le 2\|x\|_\infty$ for all $x \in \ell^\infty$.

Proof. Suppose that $(\ell^\infty, |||\cdot|||)$ does contain an asymptotically isometric copy of c_0. That is, there is a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and a sequence $(x_n)_n$ in ℓ^∞ such that

$$\max_{1\le j\le n}(1-\varepsilon_j)|t_j| \le \left|\left|\left|\sum_{j=1}^n t_j x_j\right|\right|\right| \le \max_{1\le j\le n}|t_j|,$$

for all scalars $t_1, t_2, \ldots, t_n$ and for all $n \in \mathbb{N}$.

Without loss of generality, we can assume that the sequence $(x_n)_n$ converges pointwise to 0. For each $n \in \mathbb{N}$ let $x_n = (\xi_j^n)_{j=1}^\infty$. Since $|||x_1||| \ge 1-\varepsilon_1 > 0$, there exists $j \in \mathbb{N}$ such that $\xi_j^1 \ne 0$. Let $k = \min\{j : \xi_j^1 \ne 0\}$ and $\alpha = \frac{1}{3\cdot 2^k}|\xi_k^1|$. Choose $N_1 \ge k$ so that $\sum_{j=N_1+1}^\infty 2^{-j} < \alpha/4$. Choose $N_2 \in \mathbb{N}$ so that $\varepsilon_n < \alpha$, for all $n \ge N_2$. Since $(x_n)_n$ converges pointwise to 0, choose $N \ge N_2$ such that $|\xi_j^n| < \alpha/4$ for $j = 1, 2, \ldots, N_1$ and for all $n \ge N$. Hence, for each $n \ge N$, we have the following

$$\begin{aligned}
\|x_n\|_\infty \le |||x_n||| &= \|x_n\|_\infty + \sum_{j=1}^\infty 2^{-j}|\xi_j^n| \\
&= \|x_n\|_\infty + \sum_{j=1}^{N_1} 2^{-j}|\xi_j^n| + \sum_{j=N_1+1}^\infty 2^{-j}|\xi_j^n| \\
&\le \|x_n\|_\infty + \sum_{j=1}^{N_1} 2^{-j}\frac{\alpha}{4} + \sum_{j=N_1+1}^\infty 2^{-j} \le \|x_n\|_\infty + \frac{\alpha}{2}
\end{aligned}$$

By convexity of $\|\cdot\|_\infty$, we have $\|x_n\|_\infty \le \frac{1}{2}(\|x_1+x_n\|_\infty + \|x_1-x_n\|_\infty)$, so either $\|x_1+x_n\|_\infty \ge \|x_n\|_\infty$ or $\|x_1-x_n\|_\infty \ge \|x_n\|_\infty$. If $\|x_1+x_n\|_\infty \ge \|x_n\|_\infty$, then we have

$$\begin{aligned}
1 \ge |||x_1+x_n||| &= \|x_1+x_n\|_\infty + \sum_{j=1}^\infty 2^{-j}|\xi_j^1+\xi_j^n| \\
&\ge \|x_n\|_\infty + 2^{-k}|\xi_k^1+\xi_k^n| \ge |||x_n||| - \frac{\alpha}{2} + 2^{-k}(|\xi_k^1| - |\xi_k^n|) \\
&\ge |||x_n||| - \frac{\alpha}{2} + 2^{-k}(|\xi_k^1| - \frac{\alpha}{4}) \ge |||x_n||| - \alpha + 2^{-k}|\xi_k^1| \\
&\ge 1 - \varepsilon_n - \alpha + 2^{-k}|\xi_k^1| \ge 1 - \alpha - \alpha + 2^{-k}|\xi_k^1| = 1 + \alpha,
\end{aligned}$$

which is clearly impossible. Similarly we arrive at a contradiction if we assume that $\|x_1 - x_n\|_\infty \geq \|x_n\|_\infty$. This completes the proof.

3. Refinements of James's Distortion Theorems. We saw in section 1 that the concepts of asymptotically isometric copies of ℓ^1 and c_0 arise naturally when one considers the James's Distortion Theorems. In this section we will consider some refinements of James's Theorems.

Our first result is the isomorphic analogue of Theorem 1.7.

Theorem 3.1. *A Banach space X contains an isomorphic copy of ℓ^1 if and only if there is a sequence $(x_n)_n$ in X such that*

(1) *there are constants $0 < m \leq M < \infty$ so that for all $(t_n)_n \in \ell^1$,*

$$m \sum_{n=1}^{\infty} |t_n| \leq \left\| \sum_{n=1}^{\infty} t_n x_n \right\| \leq M \sum_{n=1}^{\infty} |t_n|,$$

and

(2) $\lim_{n\to\infty} \|x_n\| = M$.

The proof of Theorem 3.1 follows directly from the proof of the James's Distortion Theorem for ℓ^1 [J]. A direct consequence is the following.

Corollary 3.2. *A Banach space X contains an isomorphic copy of ℓ^1 if and only if there is a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and a sequence $(x_n)_n$ in X so that*

$$(1-\varepsilon_k) \sum_{n=k}^{\infty} |t_n| \leq \left\| \sum_{n=k}^{\infty} t_n x_n \right\| \leq \sum_{n=k}^{\infty} |t_n|,$$

for all $(t_n)_n \in \ell^1$ and for all $k \in \mathbb{N}$.

The corresponding results for c_0 do hold, but they do not follow so easily from the James's Distortion Theorem for c_0. We begin with a simple consequence of James's Theorem for c_0, which can easily be obtained by modifying James's original proof.

Proposition 3.3. *A Banach space X contains an isomorphic copy of c_0 if and only if there is a decreasing null sequence $(\varepsilon_n)_n$ in $(0,1)$ and a sequence $(x_n)_n$ in X so that*

$$(1-\varepsilon_k) \sup_{n\geq k} |t_n| \leq \left\| \textstyle\sum_{n=1}^{\infty} t_n x_n \right\| \leq (1+\varepsilon_k) \sup_{n\geq k} |t_n|,$$

for all $(t_n)_n \in c_0$ and for all $k \in \mathbb{N}$.

We will improve the left-hand estimate in Proposition 3.3 using the following result from [CGJ].

Lemma 3.4. *Let $(x_n)_n$ be a basic sequence in an infinite dimensional Banach space X. Then there is a block basic sequence $(y_n)_n$ of (x_n) and a sequence of functionals $(y_n^*)_n$ in X^* which form a unit biorthogonal system of X. That is, for each $n \in \mathbb{N}$, $\|y_n\| = \|y_n^*\| = y_n^*(y_n) = 1$ and $y_n^*(y_m) = 0$ for all $m \neq n$.*

Theorem 3.5. *A Banach space X contains an isomorphic copy of c_0 if and only if there is a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and a sequence $(x_n)_n$ in X so that*

$$\sup_{n\geq k} |t_n| \leq \left\| \sum_{n=k}^{\infty} t_n x_n \right\| \leq (1+\varepsilon_k) \sup_{n\geq k} |t_n| ,$$

for all $(t_n)_n \in c_0$ and for all $k \in \mathbb{N}$.

Proof. Suppose that X contains an isomorphic copy of c_0. Then by Proposition 3.3, there is a decreasing null sequence $(\delta_n)_n$ in $(0,1)$ and a sequence $(x_n)_n$ in X so that

$$(1-\delta_k) \sup_{n\geq k} |t_n| \leq \left\| \sum_{n=k}^{\infty} t_n x_n \right\| \leq (1+\delta_n) \sup_{n\geq k} |t_n|,$$

for all $(t_n)_n \in c_0$ and for all $k \in \mathbb{N}$.

Since $(x_n)_n$ is a basic sequence in X, there is a block basic sequence $(y_n)_n$ of $(x_n)_n$ and a sequence of functionals $(y_n^*)_n$ in X^* which form a unit biorthogonal system of X, by Lemma 3.4. Thus there is a strictly increasing sequence of integers $(k_n)_{n=0}^{\infty}$, with $k_0 = 0$, and scalars t_j^n, where $k_{n-1} < j \leq k_n$ and $n \in \mathbb{N}$, so that $y_n = \sum_{j=k_{n-1}+1}^{k_n} t_j^n x_j$. Since $\|y_n\| = 1$, we have $(1-\delta_{k_{n-1}+1}) \max_{k_{n-1}+1\leq j\leq k_n} |t_j^n| \leq 1$, so $|t_j^n| \leq (1-\delta_{k_{n-1}+1})^{-1}$, for all $k_{n-1}+1 \leq j \leq k_n$ and for all $n \in \mathbb{N}$.

Let $(a_n)_n \in c_0$ and $\ell \in \mathbb{N}$. Then for each $m \geq \ell$,

$$\left\| \sum_{n=\ell}^{\infty} a_n y_n \right\| \geq \left| y_m^* \left(\sum_{n=\ell}^{\infty} a_n y_n \right) \right| = |a_m| .$$

Hence $\| \sum_{n=\ell}^{\infty} a_n y_n \| \geq \sup_{m\geq \ell} |a_m|$. Also

$$\left\| \sum_{n=\ell}^{\infty} a_n y_n \right\| = \left\| \sum_{n=\ell}^{\infty} a_n \sum_{j=k_{n-1}+1}^{k_n} t_j^n x_j \right\| \leq \left(1+\delta_{k_{\ell-1}+1}\right) \sup_{\substack{k_{n-1}+1\leq j\leq k_n \\ n\geq \ell}} |a_n t_j^n|$$

$$\leq \left(1+\delta_{k_{\ell-1}+1}\right)\left(1-\delta_{k_{\ell-1}+1}\right)^{-1} \sup_{n\geq \ell} |a_n| = (1+\varepsilon_\ell) \sup_{n\geq \ell} |a_n|,$$

where $\varepsilon_\ell = \left(1+\delta_{k_{\ell-1}+1}\right)\left(1-\delta_{k_{\ell-1}+1}\right)^{-1} - 1$.

Since $\delta_n \to 0$ as $n \to \infty$, $\varepsilon_\ell \to 0$ as $\ell \to \infty$, and hence the proof is complete.

An obvious consequence of Theorem 3.5 is the following result which is similar to Theorem 3.1.

Corollary 3.6. *A Banach space X contains an isomorphic copy of c_0 if and only if there is a sequence $(x_n)_n$ in X such that*

(1) *there are constants $0 < m \le M < \infty$ so that for all $(t_n)_n \in c_0$,*

$$m \sup_n |t_n| \le \left\| \sum_{n=1}^{\infty} t_n x_n \right\| \le M \sup_n |t_n|,$$

and

(2) $\lim_{n\to\infty} \|x_n\| = m.$

Our last result in this section returns us to fixed point theory. It is proved in [DLT3].

Theorem 3.7. *If a Banach space X contains an isomorphic copy of c_0, then X fails the fixed point property for asymptotically nonexpansive mappings on closed bounded convex subsets of X. (A mapping $T : C \to C$ is said to be asymptotically nonexpansive if $\|T^n x - T^n y\| \le k_n\|x - y\|$ for all $x, y \in C$ and for all $n \in \mathbb{N}$, where $(k_n)_n$ is a sequence of real numbers converging to 1.)*

Proof. If X contains an isomorphic copy of c_0, then, by Theorem 3.5, there is a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and a sequence $(x_n)_n$ in X so that

$$\sup_{n\ge k} |t_n| \le \left\| \sum_{n=k}^{\infty} t_n x_n \right\| \le (1+\varepsilon_k) \sup_{n\ge k} |t_n| \ ,$$

for all $(t_n)_n \in c_0$ and for all $n \in \mathbb{N}$.

Define $C = \{\sum_{n=1}^{\infty} t_n x_n : 0 \le t_n \le 1 \text{ and } (t_n)_n \in c_0\}$. Clearly, C is a closed bounded convex subset of X. Define $T : C \to C$ by $T(\sum_{n=1}^{\infty} t_n x_n) = x_1 + \sum_{n=1}^{\infty} t_n x_{n+1}$, for all $\sum_{n=1}^{\infty} t_n x_n \in C$. It is easily seen that T has no fixed points in C. Let $z = \sum_{n=1}^{\infty} t_n x_n$ and $w = \sum_{n=1}^{\infty} s_n x_n$ be elements of C and let $k \in \mathbb{N}$. Then

$$\begin{aligned}\|T^k z - T^k w\| &= \left\| \sum_{n=1}^{\infty} (t_n - s_n) x_{n+k} \right\| \le (1+\varepsilon_{k+1}) \sup_n |t_n - s_n| \\ &\le (1+\varepsilon_{k+1}) \left\| \sum_{n=1}^{\infty} (t_n - s_n) x_n \right\| = (1+\varepsilon_{k+1})\|z - w\|.\end{aligned}$$

Since $1+\varepsilon_{k+1} \to 1$ as $k \to \infty$, this shows that T is an asymptotically nonexpansive mapping on C. This completes the proof.

Remark. It is unknown where the corresponding result for ℓ^1 holds; that is, if a Banach space X contains an isomorphic copy of ℓ^1, does X fail the fixed point property for asymptotically nonexpansive mappings on closed bounded convex subsets of X?

4. Duality results. In this section we will consider the behaviour of containment of asymptotically isometric copies of c_0 and ℓ^1 with respect to duality. Our first result appears in [DJLT].

Theorem 4.1. *Let $(X, \|\cdot\|)$ be a Banach space that contains an asymptotically isometric copy of c_0. Then X^*, with the dual norm, contains an asymptotically isometric copy of ℓ^1.*

Proof. Since X contains an asymptotically isometric copy of c_0, there is a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and a sequence $(x_n)_n$ in X so that

$$\max_{1\le n\le N}(1-\varepsilon_n)|a_n| \le \Big\|\sum_{n=1}^{N} a_n x_n\Big\| \le \max_{1\le n\le N}|a_n|\,,$$

for all scalars $a_1, \ldots, a_N$ and all $N \in \mathbb{N}$. Let (x_n^*) be the Hahn-Banach extensions to elements of X^* of the linear functionals on the span of (x_n) that are biorthogonal to (x_n).

Fix $m \in \mathbb{N}$. Then, for all vectors x of the form $\sum_{n=1}^{N} a_n x_n$, with $N \ge m$, we have

$$\begin{aligned} |x_m^*(x)| &= |a_m| = (1-\varepsilon_m)^{-1}(1-\varepsilon_m)|a_m| \\ &\le (1-\varepsilon_m)^{-1}\max_{1\le n\le N}(1-\varepsilon_n)|a_n| \le (1-\varepsilon_m)^{-1}\|x\|. \end{aligned}$$

Therefore, $\|x_m^*\| \le (1-\varepsilon_m)^{-1}$. Define $y_n^* = x_n^*\|x_n^*\|^{-1}$ for each $n \in \mathbb{N}$. Fix scalars $a_1, a_2, \ldots, a_N$ and let $b_n = \operatorname{sign} a_n$ for all $1 \le n \le N$. Then since $\|\sum_{n=1}^{N} b_n x_n\| \le \max_{1\le n\le N}|b_n| = 1$, we have

$$\begin{aligned} \sum_{n=1}^{N}|a_n| \ge \Big\|\sum_{n=1}^{N} a_n y_n^*\Big\| &\ge \left(\sum_{n=1}^{N} a_n y_n^*\right)\left(\sum_{n=1}^{N} b_n x_n\right) \\ &= \sum_{n=1}^{N}\|x_n^*\|^{-1}|a_n| \ge \sum_{n=1}^{N}(1-\varepsilon_n)|a_n|\,. \end{aligned}$$

Thus X^* contains an asymptotically isometric copy of ℓ^1.

Remark. The converse of Theorem 4.1 does not hold and this is easily seen by considering $X = \ell^1$ with its canonical norm. Then X does not contain an asymptotically isometric copy of c_0, but $X^* = \ell^\infty$, with its canonical norm, and X^* contains an isometric copy of ℓ^1.

Our next result can be considered to be the asymptotically isometric version of a result of Bessaga and Pełczyński [BP]. The proof of this result can be found in [DR].

Theorem 4.2. *For a Banach space X, the following conditions are equivalent:*

(a) *X^* contains an asymptotically isometric copy of c_0.*
(b) *X contains a complemented asymptotically isometric copy of ℓ^1.*
(c) *X^* contains an asymptotically isometric copy of ℓ^∞.*

Before we get to the proof of Theorem 4.2, we need the following definition and some results.

Definition 4.3. A Banach space X is said to contain an asymptotically isometric copy of ℓ^∞ if there is a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and a bounded linear mapping $T : \ell^\infty \to X$ so that

$$\sup_n (1-\varepsilon_n)|t_n| \le \|T((t_n)_n)\| \le \sup_n |t_n|,$$

for all $(t_n)_n \in \ell^\infty$.

Remark. The definition of a Banach space containing an asymptotically isometric copy of ℓ^∞ is completely analogous to the definition of a Banach space containing an asymptotically isometric copy of c_0 or ℓ^1. However, because ℓ^∞ does not have a basis, we have cloaked the definition in terms of an operator rather than a basis.

By modifying the proof of Theorem 1.9, we obtain

Proposition 4.4. *A Banach space X contains an asymptotically isometric copy of ℓ^∞ if and only if there is a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and an operator $T : \ell^\infty \to X$ so that*

$$(1-\varepsilon_k)\sup_{n\ge k} |t_n| \le \|T((t_n)_{n\ge k})\| \le \sup_{n\ge k} |t_n|$$

for all $(t_n)_n \in \ell^\infty$ and for all $k \in \mathbb{N}$.

The following three results will also be needed in the proof of Theorem 4.2.

Theorem 4.5. [JR] *Let X be a separable infinite dimensional Banach space. If $(x_n^*)_n$ is a weak* null normalized sequence in X^*, then $(x_n^*)_n$ has a subsequence $(y_n^*)_n$ which is a weak* basic sequence.*

Theorem 4.6. [Me] *Let $(x_n)_n$ be a bounded sequence in a Banach space X and suppose that there is a weakly unconditionally Cauchy series $\sum_n x_n^*$ in X^* so that the sequence $(x_n^*(x_n))_n$ does not converge to 0. Then $(x_n)_n$ has a subsequence $(y_n)_n$ which is equivalent to the unit vector basis of ℓ^1 and the closed linear span of $(y_n)_n$ is complemented in X.*

Theorem 4.7. [HM] *Let X be a Banach space and let X_0 be a separable subspace of X. Then there exists a separable subspace Z of X which contains X_0, and an isometric embedding $J : Z^* \to X^*$ such that $(J(z^*))(z) = z^*(z)$ for all $z \in Z$ and $z^* \in Z^*$.*

Proof of Theorem 4.2. The implication (c) implies (a) is trivial.
For (b) implies (c), let Y be a complemented subspace of X which is an asymptotically isometric copy of ℓ^1. By the statement and proof of Corollary 1.8, there is a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and a basis $(x_n)_n$ in Y so that

$$\sum_{n=k}^{\infty} |t_n| \leq \left\| \sum_{n=k}^{\infty} t_n x_n \right\| \leq (1+\varepsilon_k) \sum_{n=k} |t_n|$$

for all $(t_n)_n \in \ell^1$ and for all $k \in \mathbf{N}$. Define $T : Y \to \ell^1$ by $T(\sum_n t_n x_n) = (t_n)_n$, for all $(t_n)_n \in \ell^1$. Note that T is a bounded linear mapping from Y onto ℓ^1 with $||T|| \leq 1$. Hence $T^* : \ell^\infty \to Y^*$ is a bounded linear mapping with $||T^*|| \leq 1$.

Let $(a_n)_n \in \ell^\infty$ and let $k \in \mathbf{N}$. Since $||T^*|| \leq 1$, $||T^*((a_n)_{n\geq k})|| \leq \sup_{n\geq k} |a_n|$. Also

$$||T^*((a_n)_{n\geq k})|| = \sup\Big\{|[T^*((a_n)_{n\geq k})](x)| : x \in Y, ||x|| \leq 1\Big\}$$

$$= \sup\Big\{|[(a_n)_{n\geq k}](Tx)| : x \in Y,\ ||x|| \leq 1\Big\}$$

$$= \sup\Big\{|[(a_n)_{n\geq k}](T(\sum_{j=1}^{\infty} \beta_j x_j)| : \sum_{j=1}^{\infty} \beta_j x_j \in Y, ||\sum_{j=1}^{\infty} \beta_j x_j|| \leq 1\Big\}$$

$$= \sup\Big\{|\sum_{n=k}^{\infty} a_n \beta_n| : \sum_{j=1}^{\infty} \beta_j x_j \in Y, ||\sum_{j=1}^{\infty} \beta_j x_j|| \leq 1\Big\}$$

$$\geq \sup\Big\{\Big|\sum_{n=k}^{\infty} a_n\beta_n\Big| : \sum_{j=k}^{\infty}\beta_j x_j \in Y, \Big\|\sum_{j=1}^{\infty}\beta_j x_j\Big\| \leq 1\Big\}$$

$$\geq \sup\Big\{\Big|\sum_{n=k}^{\infty} a_n\beta_n\Big| : \sum_{j=k}^{\infty}|\beta_j| \leq \frac{1}{1+\varepsilon_k}\Big\} = \frac{1}{1+\varepsilon_k}\sup_{n\geq k}|a_n| .$$

Hence by Proposition 4.4, Y^* contains an asymptotically isometric copy of ℓ^∞. Since Y is complemented in X, Y^* is isometric to a subspace of X^* and thus X^* contains an asymptotically isometric copy of ℓ^∞.

For the proof of (a) implies (b) we will first assume that X is separable. We now assume that X is a separable Banach space and X^* contains an asymptotically isometric copy of c_0. Then there is a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and a sequence (x_n^*) in X^* so that

$$\sup_n(1-\varepsilon_n)|t_n| \leq \Big\|\sum_n t_n x_n^*\Big\| \leq \sup_n|t_n| ,$$

for all $(t_n)_n \in c_0$.

Since $(x_n^*)_n$ is a basic sequence in X^*, there is a block basic sequence $(y_n^*)_n$ of $(x_n^*)_n$ and a sequence of functionals (y_n^{**}) in X^{**} which form a unit biorthogonal system of X, by Lemma 3.4. Hence there is a strictly increasing sequence of integers, $(k_n)_{n=0}^{\infty}$ with $k_0 = 0$, and scalars, $\alpha_j^{(n)}$, where $k_{n-1}+1 \leq j \leq k_n$ and $n \in \mathbb{N}$, so that

$$y_n^* = \sum_{j=k_{n-1}+1}^{k_n} \alpha_j^{(n)} x_j^* .$$

Since $\|y_n^*\| = 1$, we get

$$\max_{k_{n-1}+1\leq j\leq k_n}(1-\varepsilon_j)|\alpha_j^{(n)}| \leq 1 \leq \max_{k_{n-1}+1\leq j\leq k_n}|\alpha_j^{(n)}|.$$

In particular, $|\alpha_j^{(n)}| \leq (1-\varepsilon_j)^{-1} \leq [1-\min_{k_{n-1}+1\leq j\leq k_n}\varepsilon_j]^{-1}$ for all $k_{n-1}+1 \leq j \leq k_n$ and for all $n \in \mathbb{N}$. Define $\delta_n = 1-\min_{k_{n-1}+1\leq j\leq k_n}\varepsilon_j$, and let $z_n^* = \delta_n y_n^*$ for each $n \in \mathbb{N}$. Then, for each $(a_n)_n \in c_0$, we have

$$\Big\|\sum_n a_n z_n^*\Big\| = \Big\|\sum_n a_n\delta_n\Big(\sum_{k_{n-1}+1\leq j\leq k_n}\alpha_j^{(n)}x_j^*\Big)\Big\|$$

$$= \Big\|\sum_n \sum_{k_{n-1}+1\leq j\leq k_n} a_n\delta_n\alpha_j^{(n)}x_j^*\Big\|$$

$$\leq \sup_{\substack{k_{n-1}+1\leq j\leq k_n\\ n\in\mathbb{N}}}\Big|a_n\delta_n\alpha_j^{(n)}\Big| \leq \sup_{n\in\mathbb{N}}|a_n| .$$

Now, for scalars $\beta_1, \beta_2, \ldots, \beta_k$, we will consider the expression $\| \sum_{n=1}^{k} \beta_n y_n^{**} \|$. Note that since $\|y_n^{**}\| = 1$ for all $n \in \mathbb{N}$, $\| \sum_{n=1}^{k} \beta_n y_n^{**} \| \leq \sum_{n=1}^{k} |\beta_n|$. Define $x^* = \sum_{n=1}^{k} (\operatorname{sgn}\beta_n) z_n^*$. From our calculations above, we see that $\|x^*\| \leq 1$. Also, since the pair of sequences (y_n^*), $(y_n^{**})_n$ form a unit biorthogonal system of X^*, we have

$$\Big\| \sum_{n=1}^{k} \beta_n y_n^{**} \Big\| \geq \Big| \Big(\sum_{n=1}^{k} \beta_n y_n^{**} \Big) x^* \Big| = \Big| \Big(\sum_{n=1}^{k} \beta_n y_n^{**} \Big) \Big(\sum_{n=1}^{k} (\operatorname{sgn}\beta_n) \delta_n y_n^* \Big) \Big|$$

$$= \sum_{n=1}^{k} \delta_n |\beta_n| \,.$$

Thus $\sum_{n=1}^{k} \delta_n |\beta_n| \leq \Big\| \sum_{n=1}^{k} \beta_n y_n^{**} \Big\| \leq \sum_{n=1}^{k} |\beta_n|$, and therefore we have $\sum_n \delta_n |\beta_n| \leq \| \sum_n \beta_n y_n^{**} \| \leq \sum_n |\beta_n|$, for all $(\beta_n)_n \in \ell^1$.

Since $(x_n^*)_n$ is a sequence in X^* which is equivalent to the unit vector basis of c_0, and since $(y_n^*)_n$ is a block basis of $(x_n^*)_n, (y_n^*)_n$ is equivalent to the unit vector basis of c_0. In particular, $(y_n^*)_n$ is a weak* null sequence in X^*. Thus since X is separable and $(y_n^*)_n$ is a weak* null normalized sequence in X^*, $(y_n^*)_n$ has a subsequence (which we will again denote by $(y_n^*)_n$) which is weak* basic, by Theorem 4.5. By the construction of this sequence (see the proof of Theorem 4.5 [LT; pages 11-12]), there is a bounded linear operator $T : X \to (\overline{span}\{y_n^*\}_{n=1}^{\infty})^*$, defined by $(Tx)(y^*) = y^*(x)$, for all $y^* \in \overline{span}\{y_n^*\}_{n=1}^{\infty}$ and for all $x \in X$. Moreover, this operator has the property that for each $\varepsilon > 0$ and for each $y^{**} \in span\{y_n^{**}\}_{n=1}^{\infty}$ of norm 1, there exists an $x \in X$ with $\|x\| = 1$ and $\|Tx - y^{**}\| < \varepsilon$. Hence we have that for each $n \in \mathbb{N}$, there exists $x_n \in X$ with $\|x_n\| = 1$ so that $\|Tx_n - y_n^{**}\| < 2^{-n}$.

For scalars $\beta_1, \beta_2, ..., \beta_k$, we have $\| \sum_{n=1}^{k} \beta_n x_n \| \leq \sum_{n=1}^{k} |\beta_n|$, since $\|x_n\| = 1$ for each $n \in \mathbb{N}$. If we define $x^* = \sum_{n=1}^{k} (\operatorname{sgn}\beta_n) z_n^*$, then $\|x^*\| \leq 1$ so by an earlier computation we have

$$\sum_{n=1}^{k} \delta_n |\beta_n| = \Big| \Big(\sum_{n=1}^{k} \beta_n y_n^{**} \Big) (x^*) \Big|$$

$$\leq \Big| \Big(\sum_{n=1}^{k} \beta_n T x_n \Big) (x^*) \Big| + \Big| \Big(\sum_{n=1}^{k} \beta_n \left(y_n^{**} - T x_n \right) \Big) (x^*) \Big|$$

$$= \left|x^*\Big(\sum_{n=1}^{k}\beta_n x_n\Big)\right| + \left|\sum_{n=1}^{k}\beta_n\,(y_n^{**} - Tx_n)\,(x^*)\right|$$
$$\le \|x^*\|\Big\|\sum_{n=1}^{k}\beta_n x_n\Big\| + \sum_{n=1}^{k}|\beta_n|\,\|y_n^{**} - Tx_n\|\,\|x^*\|$$
$$\le \Big\|\sum_{n=1}^{k}\beta_n x_n\Big\| + \sum_{n=1}^{k}|\beta_n|2^{-n}.$$

Hence $\|\sum_{n=1}^{k}\beta_n x_n\| \ge \sum_{n=1}^{k}(\delta_n - 2^{-n})\,|\beta_n|$. Thus we have shown that $\sum_n(\delta_n - 2^{-n})|\beta_n| \le \|\sum_n \beta_n x_n\| \le \sum_n |\beta_n|$ for all $(\beta_n)_n \in \ell^1$, and since $\delta_n - 2^{-n} \to 1$ as $n \to \infty$, we have shown that $(x_n)_n$ is an asymptotically isometric copy of ℓ^1.

Note that for each $n \in \mathrm{N}$,

$$(Tx_n - y_n^{**})(y_n^*) = y_n^*(x_n) - y_n^{**}(y_n^*) = y_n^*(x_n) - 1.$$

Thus, $|y_n^*(x_n)| \ge 1 - |(Tx_n - y_n^{**})(y_n^*)|$ for all $n \in \mathrm{N}$, and since

$$|(Tx_n - y_n^{**})(y_n^*)| \le \|Tx_n - y_n^{**}\|\,\|y_n^*\| = \|Tx_n - y_n^{**}\| < 2^{-n},$$

$(y_n^*(x_n))_n$ does not converge to 0. Also, since $(y_n^*)_n$ is equivalent to the unit vector basis of c_0, $\sum_n y_n^*$ is a weakly unconditionally Cauchy sequence in X^*. Hence, by Theorem 4.6, $(x_n)_n$ has a subsequence $(y_n)_n$ which is equivalent to the unit vector basis of ℓ^1 and the closed linear span of $(y_n)_n$ is complemented in X. However, since $(x_n)_n$ is an asymptotically isometric copy of ℓ^1, $(y_n)_n$ is also an asymptotically isometric copy of ℓ^1. This completes the proof for the case when X is a separable Banach space.

For the general case, let X be a Banach space such that X^* contains an asymptotically isometric copy of c_0. Then there is a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and a sequence $(x_n^*)_n$ in X^* so that

$$\sup_n(1-\varepsilon_n)|t_n| \le \Big\|\sum_n t_n x_n^*\Big\| \le \sup_n |t_n|\,,$$

for all $(t_n)_n \in c_0$.

Let $Z = \overline{span}\{x_n^*\}_{n=1}^{\infty}$. Then Z is a separable subspace of X^*. Let $\{z_n\}_{n=1}^{\infty}$ be a countable dense subset of the unit ball of Z. For each $n \in \mathrm{N}$ choose a sequence $(x_{n,k})_k$ in the unit ball of X so that $\|z_n\| = \lim_{k\to\infty} z_n(x_{n,k})$. Now we define $Y = \overline{span}\{x_{n,k}\}_{n,k=1}^{\infty}$. Then Y is a separable subspace of

X. By Theorem 4.7, there is a separable subspace Y_1 of X which contains Y and there is an operator $J : Y_1^* \to X^*$ satisfying $(Jy^*)(y) = y^*(y)$ for each $y^* \in Y_1^*$ and $y \in Y_1$. In particular, if $R : X^* \to Y_1^*$, is the natural restriction mapping, then $RJ = id_{Y_1^*}$. Therefore, for each $n \in \mathbb{N}$

$$\begin{aligned} \|z_n\| \geq \|Rz_n\| &= \sup\{|(Rz_n)(y)| : y \in Y_1, \|y\| \leq 1\} \\ &= \sup\{|z_n(y)| : y \in Y_1, \|y\| \leq 1\} \\ &\geq \sup\{|z_n(x_{n,k})| : k \in \mathbb{N}\} = \|z_n\|. \end{aligned}$$

Thus $\|Rz_n\| = \|z_n\|$ for each $n \in \mathbb{N}$, so the restriction of R to Z is an isometry. Thus Y_1^* contains an asymptotically isometric copy of c_0. Since Y_1 is separable, the first part of the proof says that Y_1 contains a complemented asymptotically isometric copy of ℓ^1. A close analysis of the proof reveals that this complemented asymptotically isometric copy of ℓ^1 in Y_1 is in fact a complemented asymptotically isometric copy of ℓ^1 in X. This concludes the proof.

Remark. In Theorem 4.2 if we replace the phrase "asymptotically isometric" by "isomorphic", then we have the statement of the Bessaga-Pełczyński Theorem [BP]. However, it is interesting to note that if we replace the phrase "asymptotically isometric" by "isometric", then the result is no longer true. More precisely, the implications (a) implies (b), and (c) implies (b) do not hold if "asymptotically isometric" is replaced by the word "isometric". This can be seen by considering $X = \ell^1(\Gamma)$, where Γ is an uncountable set, and equip X with a strictly convex norm $|||\cdot|||$. Then $(X, |||\cdot|||)$ does not contain an isometric copy of ℓ^1. On the other hand, $(X, |||\cdot|||)^*$ is isomorphic to $\ell^\infty(\Gamma)$. Hence, since Γ is uncountable, $(X, |||\cdot|||)^*$ contains an isometric copy of ℓ^∞ (and so contains an isometric copy of c_0) [Pa].

5. Asymptotically isometric copy of c_0 and renormings of Banach spaces. In this section we will investigate the relationship between Banach spaces containing an asymptotically isometric copy of c_0 and weak normal structure. We also consider the relationship between Banach spaces containing an isomorphic copy of c_0 and other well known geometric properties of Banach spaces, such as property (P), property asymptotic (P), property (WO) and the generalized Gossez-Lami Dozo property (GGLD). We refer the reader to the paper of Sims [S] for more details on these properties. All of the results in this section appear in [Do].

Theorem 5.1. *If X is a Banach space with weak normal structure, then X does not contain an asymptotically isometric copy of c_0.*

Proof. Suppose that X contains an asymptotically isometric copy of c_0. Then there is null sequence $(\varepsilon_n)_n$ in $(0,1)$ and a sequence $(x_n)_n$ in X such that $\sup_n(1-\varepsilon_n)|t_n| \leq \| \sum_{n=1}^{\infty} t_n x_n \| \leq \sup_n |t_n|$, for all $(t_n)_n \in c_0$.

Since the sequence $(x_n)_n$ is equivalent to the unit vector basis of c_0 and since the unit vector basis of c_0 is weakly null, $(x_n)_n$ is weakly null. Hence the set $\{x_n : n \in \mathrm{N}\}$ is a relatively weakly compact set in X and so by the Krein-Šmulian Theorem, $K = \overline{\mathrm{co}}(\{x_n : n \in \mathrm{N}\})$, the closed convex hull of $\{x_n : n \in \mathrm{N}\}$, is a weakly compact convex subset of X.

Since $\|x_n - x_m\| \leq 1$ for all $n, m \in \mathrm{N}$, diam $K \leq 1$. Since $\|x_n - x_m\| \geq 1 - \varepsilon_n$ for all $n, m \in \mathrm{N}$, with $n > m$, and since $(\varepsilon_n)_n$ is a null sequence in $(0,1)$, diam $K \leq 1$. Thus diam $K = 1$.

Consider an element $x \in \mathrm{co}(\{x_n : n \in \mathrm{N}\})$. Then $x = \sum_{j=1}^{N} t_j x_j$, where $t_j \geq 0$ for all $1 \leq j \leq N$ and $\sum_{j=1}^{N} t_j = 1$.

For each $n > N$, $\|x - x_n\| = \| \sum_{j=1}^{N} t_j x_j - x_n\| \geq 1 - \varepsilon_n$.

Since $(\varepsilon_n)_n$ is a null sequence, this implies that $\sup\{\|x - y\| : y \in K\} \geq 1$. Since diam $K = 1$, we get that $\sup\{\|x - y\| : y \in K\} = 1$. Hence $\sup\{\|x - y\| : y \in K\} = \mathrm{diam}\, K$ for all $x \in K$ and so K fails to have normal structure. Therefore X fails weak normal structure and this completes the proof.

We now obtain some easy consequences of Theorem 5.1. The first corollary follows immediately from Theorem 2.7, and the second corollary is immediate from Theorem 2.8.

Corollary 5.2. *If X is an infinite dimensional subspace of $(c_0, \|\cdot\|_\infty)$, then X fails to have weak normal structure.*

Corollary 5.3. *Every equivalent renorming of $c_0(\Gamma)$, for Γ uncountable, fails weak normal structure.*

Remark. It is well known that if a Banach space X is uniformly convex in every direction (UCED), then X has weak normal structure [Z]. Day, James and Swaminathan [DJS] proved that $c_0(\Gamma)$ does not admit an equivalent UCED norm if Γ is uncountable. Corollary 5.2 is therefore an improvement of this result. Corollary 5.2 was also proved by Landes [L1, L2] using different techniques.

In [DJS] and in [Z], it is proved that every separable Banach space can be equivalently renormed to be UCED, and thus can be renormed to have weak normal structure. They also show that ℓ^∞ can be equivalently renormed to be UCED, and hence to have weak normal structure. Combining these results with Theorem 5.1 we get the following

Corollary 5.4. *Every separable Banach space can be equivalently renormed so as not to contain an asymptotically isometric copy of c_0. Also, ℓ^∞ can be equivalently renormed so as not to contain an asymptotically isometric copy of c_0.*

Remark. Separability is necessary in Corollary 5.2, because every equivalent renorming of $c_0(\Gamma)$ contains an asymptotically isometric copy of c_0, if Γ is uncountable, by Theorem 2.7.

Two Banach space properties that imply weak normal structure are property (P) of Tan and Xu [TX] and (WO) of Tingley [T]. Sims and Smyth [SS] proved that these properties are equivalent. By Theorem 5.1, every Banach space with property (P) (or equivalently, (WO)) will not contain an asymptotically isometric copy of c_0. Two related properties are the generalized Gossez-Lami Dozo property (GGLD) of Jiménéz-Melado [JM], and asymptotic (P) of Sims and Smyth [SS] given below.

Definition 5.5. Let X be a Banach space.

(a) X is said to have the generalized Gossez-Lami Dozo property (GGLD) if whenever $(x_n)_n$ is a weakly null sequence in X which is not norm null then

$$\liminf_n \|x_n\| < \limsup_n \limsup_m \|x_n - x_m\|.$$

(b) X is said to have property asymptotic (P) if whenever $(x_n)_n$ is a weakly null sequence in X which is not norm null then

$$\liminf_n \|x_n\| < \mathrm{diam}_a\{x_n\},$$

where $\mathrm{diam}_a\{x_n\} = \lim_n \mathrm{diam}\{x_k : k \geq n\}$ is the asymptotic diameter of the sequence $(x_n)_n$.

Sims and Smyth [SS] have also proved that GGLD and property asymptotic (P) are equivalent. While GGLD implies (WO), Jiménéz-Melado [JM] has shown that they are not equivalent by constructing an equivalent norm, $|||\cdot|||$, on c_0, so that $(c_0, |||\cdot|||)$ has (WO), but $(c_0, |||\cdot|||)$ fails GGLD. Our next result shows the relationship between GGLD and c_0.

Theorem 5.6. *If X is a Banach space with GGLD, then X does not contain an isomorphic copy of c_0.*

Proof. Suppose that X contains an isomorphic copy of c_0. By Theorem 3.5, there is a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and a sequence $(x_n)_n$ in X so that

$$(*) \qquad \sup_{k\geq n} |t_k| \leq \| \sum_{k=n}^{\infty} t_k x_k \| \leq (1+\varepsilon_n) \sup_{k\geq n} |t_k|,$$

for all $(t_k)_k \in c_0$ and for all $n \in \mathbb{N}$. By passing to subsequences if necessary we can and do assume that the sequence $(\varepsilon_n)_n$ is decreasing. Since $(x_n)_n$ is equivalent to the unit vector basis of c_0, $(x_n)_n$ is weakly null. Clearly from $(*)$, $\lim_n \|x_n\| = 1$. Also, if $k > n$ then $1 \leq \|x_n - x_k\| \leq 1 + \varepsilon_n$. Hence $1 \leq \operatorname{diam}\{x_k : k \geq n\} \leq 1 + \varepsilon_n$ for all $n \in \mathbb{N}$. Therefore $\operatorname{diam}_a\{x_n\} = 1$, so X fails to have property asymptotic (P), and thus fails to have the GGLD. This completes the proof.

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