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Structure of Fixed Point Sets of Condensing Maps in B_o Spaces with Applications to Differential Equations in Unbounded Domain

ABSTRACT. A class of "generalized" condensing maps is introduced. A theorem on the structure of fixed point set of such maps is obtained. Its applications to some boundary value problems for differential equations in unbounded domains are studied.

The purpose of this communication is to give an Aronszajn type theorem on the structure of solutions set of an equation in an abstract B_o space (written E) and show some of its applications to boundary value problems for differential equations studied in unbounded domains. The basic example is the finite dimensional Cauchy problem $x' = f(t,x)$, $x(t_0) = x_0$, where $f : [t_0, +\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous. Here, under some additional hypothesis, each solution extends on the interval $[t_0, +\infty)$ and the set of all solutions, treated as a subset of a B_o space $C([t_0, +\infty), \mathbb{R}^n)$, is a compact R_δ. Recall, that classical theorems of H. Knesser (1923) or N. Aronszajn ([1], 1942) give topological characterizations of the solutions set of the Cauchy problem treated as a subset of the Banach space $C([t_0, t_0 + a], \mathbb{R}^n)$ for some $a > 0$. Numerous results on the structure of solutions sets of equations appeared later, see for instance [20], [3], [17], [23], [24].

In the first section a class of "generalized" set-contractions and condensing maps is introduced and, in the second section, a degree theory for the corresponding class of vector fields is outlined. The proofs of propositions and theorems are ommitted, since they generally follow standard lines — compare [15], [16] and [8]. For a different treatment of condensing maps in general topological vector spaces see [18]. The proofs of propositions in the range (1)–(9) are similar to those in [8], (1)–(14). The only difference is that now one also has to verify that certain sets fall into certain classes $\mathcal{R}_q$, but this usually follows straight from the axiomatic definition of the classes $\mathcal{R}_Q$ given in (3). A complete treatment of the material included below is given in [9].

In the third section, Theorem (15) on the structure of fixed point set of a "generalized" condensing map is given. The theorem is a generalization of a theorem of W. V. Petryshyn [17]. We demonstrate its applications to some boundary value problems for differential equations studied in unbounded domains.

1. Generalized condensing maps in a $\mathbf{B}_o$ space. We assume that the topology of the space E is determined by a chosen countable family of seminorms Q. For a seminorm $q \in Q$, let $B_q = \{ u : \ q(u) < 1 \}$ denote the "unit ball" with respect to q and let $\mathcal{B}_q = \{ A \subset E : \sup q(A) < +\infty \}$ denote the family of all q-bounded subsets of E. Finally let $\mathcal{B} = \bigcap_{q \in Q} \mathcal{B}_q$ denote the family of all *bounded* subsets of E.

With each seminorm q we associate a function

$$\gamma_q : \mathcal{B}_q \to \mathbb{R}_+ \qquad (\mathbb{R}_+ \equiv [0, +\infty)),$$
$$\gamma_q(A) = \inf \{ \delta > 0 : \exists_{(\text{a finite set } S \subset E)} \ A \subset S + \delta B_q \}.$$

The family $\{ \gamma_q : \ q \in Q \}$, denoted γ_Q, is a version of the Hausdorff or "ball" measure of noncompactness. The functions γ_q satisfy properties well known for the Banach space case.

(1) Proposition. *For each seminorm $q \in Q$ the function γ_q satisfies the following properties:*

(a) *if $A \in \mathcal{B}_q$ i $B \subset A$, then $B \in \mathcal{B}_q$ and $\gamma_q(B) \le \gamma_q(A)$;*
(b) *if $A, B \in \mathcal{B}_q$, then $A \cup B \in \mathcal{B}_q$ and $\gamma_q(A \cup B) \le \max(\gamma_q(A), \gamma_q(B))$;*
(c) *if $A \in \mathcal{B}_q$, then $\overline{A} \in \mathcal{B}_q$ and $\gamma_q(\overline{A}) = \gamma_q(A)$;*
(d) *if $A \in \mathcal{B}_q$ i $\lambda \in \mathbb{R}$, then $\lambda A \in \mathcal{B}_q$ and $\gamma_q(\lambda A) = |\lambda| \gamma_q(A)$;*
(e) *if $A, B \in \mathcal{B}_q$, then $A + B \in \mathcal{B}_q$, and $\gamma_q(A + B) \le \gamma_q(A) + \gamma_q(B)$;*
(f) *if $A \in \mathcal{B}_q$, then $\operatorname{co} A \in \mathcal{B}_q$ and $\gamma_q(\operatorname{co} A) = \gamma_q(A)$;*
(g) $\gamma_q(B_q) = 1$.

(2) Proposition. *Let $A \in \mathcal{B}$. The set $\overline{A}$ is compact, iff $\gamma_q(A) = 0$ for each $q \in Q$.*

We assume that for each seminorm $q \in Q$ a family of „regular" sets $\mathcal{R}_q$ is distinguished, which satisfies the following

(3) Properties.

(a) $\mathcal{K} \subset \mathcal{R}_q \subset \mathcal{B}_q$, ($\mathcal{K}$ — the family of all compacta),

(b) if $A \in \mathcal{R}_q$ and $B \subset A$, then $B \in \mathcal{R}_q$,

(c) if $A, B \in \mathcal{R}_q$, then $A \cup B \in \mathcal{R}_q$,

(d) if $A \in \mathcal{R}_q$, then $\overline{A}$, $\operatorname{co} A \in \mathcal{R}_q$,

(e) if $A, B \in \mathcal{R}_q$ and $\lambda \in \mathbb{R}$, then $\lambda A, A + B \in \mathcal{R}_q$.

We use the following notation: $\mathcal{R}_Q = \{ \mathcal{R}_q : q \in Q \}$ and $\mathcal{R} = \bigcap_{q \in Q} \mathcal{R}_q$. We give some examples of families $\mathcal{R}_Q$.

(4) Examples.

(a) $\mathcal{R}_q = \mathcal{B}_q$.

(b) $\mathcal{R}_q = \{ A \subset E : \forall_{\varepsilon > 0} \exists_{(\text{a finite set } S \subset E)} \; A \subset S + \varepsilon B_q \}$.
In the case when E is a Banach space with the norm $\| \cdot \|$, the family $\mathcal{R}_{\|\cdot\|} = \{ A \subset E : \overline{A} \in \mathcal{K} \}$ is the family of all relatively compact sets.

(c) $\mathcal{R}_q = \{ A \in \mathcal{B}_q : \exists_{(\text{a countable set } S \subset E)} \forall_{\varepsilon > 0} \; A \subset S + \varepsilon B_q \}$.

(d) Let X be a Banach space and let $E = C(\mathbb{R}_+; X)$ be the B_o space of continuous maps $x : \mathbb{R}_+ \to X$ with the family of seminorms

$$Q = \{ q_T : T \in \mathbb{N} \}, \quad \text{where } \; q_T(x) = \sup\{ \|x(t)\| : t \in [0, T] \}.$$

We put

$$\mathcal{R}_{q_T} = \Big\{ A \in \mathcal{B}_{q_T} : \forall_{\varepsilon > 0} \exists_{\delta > 0} \forall_{x \in A} \forall_{t, t' \in [0, T]} \quad |t - t'| < \delta \; \Rightarrow \; \|x(t) - x(t')\| < \varepsilon \Big\}.$$

In the case of a Banach space $E = C([a, b]; X)$ with the maximum norm, the family $\mathcal{R}_{\|\cdot\|}$ is the family of all bounded sets of equicontinuous functions.

(5) Definition. Let $F : \overline{\Omega} \to E$, $\Omega \subset E$ an open set, be a continuous map.
F is $(Q, \mathcal{R}_Q)$-condensing iff

(a) $F(\overline{\Omega}) \in \mathcal{R}$ *and* $\forall_{q \in Q} \forall_{A \subset \overline{\Omega}, A \in \mathcal{R}_q} \quad \gamma_q(F(A)) < \gamma_q(A)$.

F is a $(Q, \mathcal{R}_Q)$-set-contraction iff

(b) $F(\overline{\Omega}) \in \mathcal{R}$ *and* $\forall_{q \in Q} \exists_{k_q < 1} \forall_{A \subset \overline{\Omega}, A \in \mathcal{R}_q} \quad \gamma_q(F(A)) \le k_q \gamma_q(A)$.

Similarly a homotopy $\Phi : \overline{\Omega} \times \mathbb{I} \to E$, $\mathbb{I} = [0,1]$, is $(Q, \mathcal{R}_Q)$-condensing or a $(Q, \mathcal{R}_Q)$-set-contraction iff it satisfies a condition which is obtained from (a) or (b) respectively, by substituting both $F(\overline{\Omega})$ and $F(A)$ by $\Phi(\overline{\Omega} \times \mathbb{I})$ and $\Phi(A \times \mathbb{I})$.

(6) Proposition. *If $F : \overline{\Omega} \to E$ is $(Q, \mathcal{R}_Q)$-condensing, then the map $I - F : \overline{\Omega} \to E$, I — the identity, is proper and closed. More generally: if a homotopy $\Phi : \overline{\Omega} \times \mathbb{I} \to E$ is $(Q, \mathcal{R}_Q)$-condensing, then for each compact $C \subset E$ the set $(I - \Phi)^{-1}(C) \equiv \{ x \in \overline{\Omega} : \exists_{t \in \mathbb{I}}\ x - \Phi(x,t) \in C \}$ is compact and for each closed $A \subset \overline{\Omega}$ the set*

$$(I - \Phi)(A \times \mathbb{I}) \equiv \{ x - \Phi(x,t) : x \in A, t \in \mathbb{I} \}$$

is closed.

(7) Proposition. *Let us assume that a continuous map $V : \overline{\Omega} \times E \to E$ such that $V(\overline{\Omega} \times E) \in \mathcal{R}$ satisfies the following two conditions*

(a) $\forall_{q \in Q}\ \forall_{y \in E}\ \forall_{A \subset \overline{\Omega}, A \in \mathcal{R}_q}\quad \gamma_q(V(A, y)) = 0;$

(b) $\forall_{q \in Q}\ \exists_{0 \le k_q < 1}\ \forall_{x \in \overline{\Omega}}\ \forall_{y_1, y_2 \in E}\quad q(V(x, y_1) - V(x, y_2)) \le k_q q(y_1 - y_2).$

Then the map $F : \overline{\Omega} \to E$ which is given by the formula $F(x) = V(x,x)$, is a $(Q, \mathcal{R}_Q)$-set-contraction.

2. Topological degree of a generalized condensing vector field.

In the class $\mathbb{S}(Q, \mathcal{R}_Q; \overline{\Omega})$ of vector fields $I - F$, where the map $F : \overline{\Omega} \to E$ is a $(Q, \mathcal{R}_Q)$-set-contraction, we shall define a topological degree, which is invariant with respect to the class of homotopies, denoted $\mathbb{H}S(Q, \mathcal{R}_Q; \overline{\Omega})$, of the form $I - \Phi$, where the homotopy Φ is a $(Q, \mathcal{R}_Q)$-set-contraction. We follow the method of R. D. Nussbaum [15].

We associate, with a given map $F : \overline{\Omega} \to E$, a decreasing sequence of closed and convex sets (or, starting from some index, empty sets) $K_0 = E$, $K_{n+1} = \overline{\mathrm{co}} F(\overline{\Omega} \cap K_n)$, $n = 0, 1, 2, \ldots$ and a closed and convex set (or an empty set) $K_\infty \equiv K_\infty(F, \overline{\Omega}) = \bigcap_{n=1}^{\infty} K_n$, which is invariant in the sense that $F(\overline{\Omega} \cap K_\infty)) \subset K_\infty$. With a homotopy $\Phi : \overline{\Omega} \times \mathbb{I} \to E$ we also associate a closed, convex and invariant set $K_\infty(\Phi, \overline{\Omega})$ by a similar construction, where the image $F(\overline{\Omega} \cap K_n)$ is substituted by $\Phi((\overline{\Omega} \cap K_n) \times \mathbb{I})$.

The set K_∞ constructed for a $(Q, \mathcal{R}_Q)$-set-contraction is compact, since for each $q \in Q$ we have $K_1 \in \mathcal{R}_q$ and for each $n > 1$,

$$\gamma_q(K_\infty) \le \gamma_q(K_n) \le k_q \cdot \gamma_q(K_{n-1}) \le \ldots \le k_q^{n-1} \cdot \gamma_q(K_1),$$

and hence $\gamma_q(K_\infty) = 0$.

(8) Definition. Let $I - F \in \mathbb{S}(Q, \mathcal{R}_Q; \overline{\Omega})$ and $x - F(x) \neq 0$ for $x \in \partial\Omega$. We define

$$\deg(I - F, \Omega, 0) = \begin{cases} \deg(I - rF, \Omega, 0), & \text{if } K_\infty \neq \emptyset, \\ 0, & \text{if } K_\infty = \emptyset, \end{cases}$$

where $r : E \to K_\infty$ is a retraction and $\deg(I - rF, \Omega, 0)$ is the degree of the compact vector field $I - rF$.

The degree $\deg(I - rF, \Omega, 0)$ does not depend on the choice of retraction and is equal to $\deg(I - F, \Omega, 0)$ whenever F is compact. Hence the above definition provides an extension of the topological degree from the class of compact vector fields onto the class $\mathbb{S}(Q, \mathcal{R}_Q; \overline{\Omega})$.

(9) Proposition. *We have the following properties:*

(a) $\deg(I, \Omega, 0) = \begin{cases} 1, & \text{if } 0 \in \Omega, \\ 0, & \text{if } 0 \notin \Omega, \end{cases}$

(b) *if* $f \in \mathbb{S}(Q, \mathcal{R}_Q; \overline{\Omega})$, $f(x) \neq 0$ *for each* $x \in \partial\Omega$ *and* $\deg(f, \Omega, 0) \neq 0$, *then* $f^{-1}(0) \neq \emptyset$,

(c) *if* $f \in \mathbb{S}(Q, \mathcal{R}_Q; \overline{\Omega})$, $\Omega_1, \Omega_2 \subset \Omega$, $\Omega_1 \cap \Omega_2 = \emptyset$ *and* $f(x) \neq 0$ *for each* $x \in \overline{\Omega} \setminus (\Omega_1 \cup \Omega_2)$, *then* $\deg(f, \Omega, 0) = \deg(f, \Omega_1, 0) + \deg(f, \Omega_2, 0)$,

(d) *if* $\phi \in \mathbb{H}S(Q, \mathcal{R}_Q; \overline{\Omega})$ *and* $\phi(x) \neq 0$ *for each* $x \in \partial\Omega$, *then* $\deg(\phi(\cdot, 0), \Omega, 0) = \deg(\phi(\cdot, 1), \Omega, 0)$.

The proofs of the above facts follow standard methods — see for instance [8]. It is also worth noticing that the linear homotopy $(x, t) \mapsto (1 - t) f_0(x) + t f_1(x)$ connecting two maps $f_0, f_1 \in \mathbb{S}(Q, \mathcal{R}_Q; \overline{\Omega})$ belongs to the class $\mathbb{H}S(Q, \mathcal{R}_Q; \overline{\Omega})$.

The following proposition allows the extension of the degree theory onto the class $\mathbb{S}'(Q, \mathcal{R}_Q; \overline{\Omega})$ of $(Q, \mathcal{R}_Q)$-condensing vector fields. The respective class of homotopies of the form $I - \Phi$, where the homotopy Φ is $(Q, \mathcal{R}_Q)$-condensing, is denoted by $\mathbb{H}S'(Q, \mathcal{R}_Q; \overline{\Omega})$.

(10) Proposition. *Let* $f \in \mathbb{S}'(Q, \mathcal{R}_Q; \overline{\Omega})$ *and* $f(x) \neq 0$ *for each* $x \in \partial\Omega$, *and let a convex and symmetric neighbourhood of zero* U *in the space* E *be chosen so that* $f(x) \notin U$ *for* $x \in \partial\Omega$. *Then the set*

$$U_f = \{ g \in \mathbb{S}(Q, \mathcal{R}_Q; \overline{\Omega}) : (f - g)(\overline{\Omega}) \subset U \}$$

is nonempty and the degree $\deg(g, \Omega, 0)$ *does not depend on* $g \in U_f$.

Moreover, if $\phi \in \mathbb{H}S'(Q, \mathcal{R}_Q; \overline{\Omega})$ and $\phi(x,t) \neq 0$ for each $x \in \partial\Omega$ and $t \in \mathbb{I}$, then the set

$$U_\phi = \left\{ \psi \in \mathbb{H}S(Q, \mathcal{R}_Q; \overline{\Omega}) : (\phi - \psi)(\overline{\Omega} \times \mathbb{I}) \subset U \right\}$$

is nonempty and the degree $\deg(\psi(\cdot,t), \Omega, 0)$ depends neither on $\psi \in U_f$ nor $t \in \mathbb{I}$, for any convex and symmetric neighbourhood of zero U such that $\phi(x,t) \notin U$ for $x \in \partial\Omega$ and $t \in \mathbb{I}$.

(11) Definition. Let $f \in \mathbb{S}'(Q, \mathcal{R}_Q; \overline{\Omega})$ and $f(x) \neq 0$ for each $x \in \partial\Omega$. Let a neighbourhood of zero U be chosen as in Proposition (10). We define

$$\deg(f, \Omega, 0) = \deg(g, \Omega, 0),$$

where the map g is arbitrary such that $g \in U_f$.

We have properties like those given in Proposition (9) — it is sufficient to substitute "$\mathbb{S}'$" instead of "$\mathbb{S}$".

We also have versions of theorems of Borsuk and invariance of domain.

(12) Theorem. *Let Ω be a convex and symmetric neighbourhood of zero in E and let a map $f \in \mathbb{S}'(Q, \mathcal{R}_Q; \overline{\Omega})$ be such that $f(x) \neq 0$ and $f(-x) = -f(x)$ for $x \in \partial\Omega$. Then $\deg(f, \Omega, 0) \equiv 1 (\mathrm{mod}\, 2)$.*

(13) Theorem. *Let Ω be a convex and symmetric neighbourhood of zero in E and let $f \in \mathbb{S}(Q, \mathcal{B}_Q; \overline{\Omega})$ be a one-to-one map such that $f(0) = 0$. Then $\deg(f, \Omega, 0) \equiv 1 (\mathrm{mod}\, 2)$.*

(14) Theorem. *If $f \in \mathbb{S}(Q, \mathcal{B}_Q; \Omega)$ is one-to-one, then the image $f(\Omega)$ is an open subset of the space E and f is a homeomorphism of Ω onto $f(\Omega)$.*

Let us note that Theorems (13) and (14) are stated in restricted generality — for the vector fields in $\mathbb{S}(Q, \mathcal{B}_Q; \overline{\Omega})$ instead of $\mathbb{S}'(Q, \mathcal{R}_Q; \overline{\Omega})$ (compare [16]).

3. Applications to structure of solutions sets of equations in B_0 spaces. Let (U_n) denote a decreasing sequence of convex and symmetric neighbourhoods of zero in E such that $\bigcap_{n=1}^{\infty} U_n = \{0\}$.

(15) Theorem. *Let $f \in \mathbb{S}'(Q, \mathcal{R}_Q; \overline{\Omega})$ be a map which satisfies conditions $f(x) \neq 0$ for $x \in \partial\Omega$ and $\deg(f, \Omega, 0) \neq 0$. Suppose that there exists a sequence of maps $f_n \in \mathbb{S}'(Q, \mathcal{R}_Q; \overline{\Omega})$ such that, for each n, $(f - f_n)(\overline{\Omega}) \subset U_n$ and the equation $f_n(x) = y$, for each $y \in \overline{U}_n$, has at most one solution.*

Then the set of all solutions of the equation $f(x) = 0$ is an R_δ (i.e. it is homeomorphic to an intersection of a decreasing sequence of compact absolute retracts).

Obviously, the condition $\deg(f, \Omega, 0) \neq 0$ is fullfilled in the case when $\Omega = E$ — the linear homotopy connecting the map f and the identity is admissible.

Now we present two examples of applications of the above theorem. Our goal here is to demonstrate applicability of the theory of sections 1 and 2 to certain classes of problems rather, than give truly new theorems. First of the examples deals with an ordinary first order differential equation in a Banach space. An existence theorem with hypothesis similar to that of Theorem (16) (but in a bounded interval and with a general Kamké function in (16c)) can be found in [12]. A series of theorems on the structure of sets of solutions (in bounded intervals) of differential and integral equations in Banach spaces can be found in papers of S. Szufla, e.g. [21], [22]. Recently some papers in which differential equations are studied in unbounded domain appeared, e.g. [6], [7], [19], [14], [5].

Let X denote a Banach space with a norm $\|\cdot\|$ and let μ be the Hausdorff (or "ball") measure of noncompactness associated with $\|\cdot\|$. Let us assume that the map $f : \mathbb{R}_+ \times X \to X$ is continuous and consider the following Cauchy problem

$$\text{(C)} \qquad \begin{cases} x'(t) = f(t, x(t)), & t \geq 0, \\ x(0) = 0. \end{cases}$$

The set of all solutions of the above problem is denoted by $\mathcal{S}$. We consider the B_o space $E = C(\mathbb{R}_+, X)$ of all continuous maps $x : \mathbb{R}_+ \to X$, with the family of seminorms

$$Q = \{ q_T : T > 0 \}, \quad q_T(x) = \sup\{ \|x(t)\| : t \in [0, T] \}.$$

Convergence in E is then equivalent to the uniform convergence on bounded subsets of $\mathbb{R}_+$.

(16) Theorem. *Let $b, c, k : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous functions. Let us assume that the following conditions hold:*

(a) *the map f is uniformly continuous on $[0, T] \times \{ x \in X : \|x\| \leq r \}$ for each $T, r > 0$,*

(b) $\|f(t,x)\| \le b(t)\|x\| + c(t)$ *for each* $x \in X$ *and* $t \in \mathbb{R}_+$;
(c) $\mu(f(t,A)) \le k(t)\mu(A)$ *for each bounded* $A \subset X$.
Then the set $\mathcal{S} \subset E$ *is an* R_δ.

We sketch the proof. From Gronwall inequality and (16b) it follows that if $x(\cdot)$ is a solution of (C), then for $t \ge 0$, $\|x(t)\| \le \alpha(t)$, where $\alpha(t) = \left(\int_0^t c(s)\,ds\right) e^{\int_0^t b(s)\,ds}$. Hence the right hand side f can be modified in such a way that all the conditions so far imposed on f will still hold, the set of solutions of the problem (C) will remain unchanged, but $f(t,x) = 0$ for $\|x\| > 1 + \alpha(t)$. For example it is sufficient to replace f by

$$\tilde{f}(t,x) = \theta\Big(\frac{\|x\|}{1+\alpha(t)}\Big) f(t,x),$$

where $\theta : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function such that $\theta(u) = 1$ for $|u| \le 1$, $0 \le \theta(u) \le 1$ for $1 < |u| < 2$ and $\theta(u) = 0$ for $|u| \ge 2$. Then the condition (16b) can be replaced by a stronger condition $\|f(t,x)\| \le a(t)$ with some continuous function $a(\cdot)$.

To complete the proof it is now sufficient to verify that the maps h, $h_n : E \to E$, $n \in \mathrm{N}$, given by formulae

$$h(x)(t) = x(t) - \int_0^t f(s, x(s)),$$

and

$$h_n(x)(t) = \begin{cases} x(t), & 0 \le t \le 1/n, \\ x(t) - \int_0^{t-(1/n)} f(s, u(s))\,ds, & t \ge 1/n, \end{cases}$$

fullfill the hypothesis of Theorem (15). The above maps belong to the class of vector fields $\mathbb{S}(\bar{Q}, \mathcal{R}_{\bar{Q}}; E)$ where

$$\bar{Q} = \{\, \bar{q}_T : T > 0 \,\}, \quad \bar{q}_T(x) = \sup\{\, e^{-\kappa \int_0^t k(s)\,ds} \|x(t)\| : t \in [0,T] \,\},$$

$\kappa > 1$ is arbitrarily chosen (the method of Bielecki is used here), and $\mathcal{R}_{\bar{q}_T}$ is defined as in Example (4d). The last choice is due to the fact that properties like

$$\mu\big(\textstyle\int_0^T A(s)\,ds\big) \le \displaystyle\int_0^T \mu(A(s))\,ds,$$

and $\gamma_T(A) = \sup\{\, \mu(A(t)) : t \in [0,T] \,\}$ hold for *equicontinuous* sets of functions $A \subset E$ (see [12]).

The second example is the Darboux problem for a hyperbolic equation. An existence theorem for a similar problem (but in a bounded domain and with continuous right side) was proved in [11]. Theorems on the structure of the set of solutions, in bounded domain, for various kinds of assumptions on the right side can be found in [13], [2], [4], [10]. Theorem (17), as stated below, was already proved in [8]. Here we wish to demonstrate that the theory of section 2 can be applied to this case.

Let $\Delta = \mathbb{R}_+ \times \mathbb{R}_+$ and $\Delta_T = [0,T] \times [0,T]$, $T > 0$. Let $f : \Delta \times \mathbb{R}^{4\nu} \to \mathbb{R}^\nu$ be a Carathéodory map, i.e. we assume that all its sections

$$f(x,y;\cdot,\cdot,\cdot,\cdot) : \mathbb{R}^{4\nu} \to \mathbb{R}^\nu, \qquad (x,y) \in \Delta$$

are continuous and all sections

$$f(\cdot,\cdot;u,r,s,t) : \Delta \to \mathbb{R}^\nu, \qquad (u,r,s,t) \in \mathbb{R}^{4\nu}$$

are Lebesgue measurable.

The Darboux problem is stated as follows:

$$(\mathrm{D}) \qquad \begin{cases} u_{xy} = f(x,y;u,u_x,u_y,u_{xy}) & \text{in } \Delta, \\ u(0,y) = g(y),\ u(x,0) = h(x) & \text{on } \partial\Delta, \end{cases}$$

where $g,h : \mathbb{R}_+ \to \mathbb{R}^\nu$ are given absolutely continuous maps which satisfy condition $g(0) = h(0)$. A solution of this problem is any absolutely continuous map $u : \Delta \to \mathbb{R}^\nu$ which satisfies the differential equation almost everywhere in Δ and the boundary condition for all $x, y \in \mathbb{R}_+$. The set of all solutions is denoted by $\mathcal{S}$.

We say that a measurable function $v : \Delta \to \mathbb{R}_+$ is *locally bounded (locally less then a, $a > 0$)*, if

$$ess\, sup_{(x,y)\in\Delta_T} \big|v(x,y)\big| < +\infty \qquad (\text{respectively: } \ldots < a)$$

for each $T > 0$.

In the following theorem we study the set of all solutions of the problem (D) as a subset of the B_o space of continuous maps $E = C(\Delta, \mathbb{R}^\nu)$ with the family of seminorms

$$Q = \{\, q_T : T > 0 \,\}, \qquad q_T(u) = \sup\Big\{\, |u(x,y)| : (x,y) \in \Delta_T \,\Big\}.$$

Besides, the space of locally integrable functions $E' = L^1(\Delta, \mathbb{R}^\nu)$ with the family of seminorms

$$P = \{\, p_T : T > 0 \,\}, \qquad p_T(u) = \int_0^T \int_0^T |u(x,y)|\, dx\, dy.$$

is useful.

(17) Theorem. *Let $b, c, K, M, N : \Delta \to \mathbb{R}_+$ be measurable locally bounded functions and let N be locally less then 1. We assume that the following two conditions are satisfied:*

(a) *for all $(x, y; u, r, s, t) \in \Delta \times \mathbb{R}^{4\nu}$,*

$$|f(x, y; u, r, s, t)| \leq b(x, y)|u| + c(x, y)$$

(b) *for all $(x, y; u, r_1, s_1, t_1), (x, y; u, r_2, s_2, t_2) \in \Delta \times \mathbb{R}^{4\nu}$,*

$$\begin{aligned}&|f(x, y; u, r_1, s_1, t_1) - f(x, y; u, r_2, s_2, t_2)|\\ &\qquad \leq K(x, y)|r_1 - r_2| + M(x, y)|s_1 - s_2| + N(x, y)|t_1 - t_2|\end{aligned}$$

Then the set of all solutions of the Darboux problem $\mathcal{S} \subset E$ is an R_δ.

We sketch the proof (we shall not be repeating some of the details which can be found in [8]). A suitable generalization of Gronwall inequality can be used to derive estimates on solutions of the problem (D) and further assumptions on f can be strengthened without loosing generality (as in the proof of (16)).

In particular we can assume that for all $(x, y; u, r, s, t) \in \Delta \times (\mathbb{R}^{4\nu})$,

$$\big|f(x, y; u, r, s, t)\big| \leq a(x, y),$$

where $a(\cdot, \cdot)$ is some locally integrable function.

Let $D_T = \{\, u \in E' : \ |u(x, y)| \leq a(x, y) \text{ almost everywhere in } \Delta_T \,\}$ and let us introduce the families of sets

$$\mathcal{R}_T = \Big\{ A \subset E' : \exists_{\text{a compact set } K \subset E'} \, \exists_{\mu > 0} \, A \subset K + \mu D_T \Big\}.$$

Then we consider the map

$h : E' \to E'$,

$$\begin{aligned}h(u)(x, y) = f\Big(x, y;\ & h(x) + g(y) - g(0) + \int_0^x \int_0^y u(\xi, \eta)\, d\xi\, d\eta,\\ & h'(x) + \int_0^y u(x, \eta)\, d\eta,\ g'(y) + \int_0^x u(\xi, y)\, d\xi, u(x, y)\Big).\end{aligned}$$

Using the method of Bielecki we find an equivalent family of seminorms $\tilde{P} = \{\, \tilde{p}_T \,\}$ such that if $\mathcal{R}_{\tilde{p}_T} = \mathcal{R}_T$, then $h \in \mathbb{S}(\tilde{P}, \mathcal{R}_{\tilde{P}}; E)$. The reason for the families $\mathcal{R}_{\tilde{P}}$ is that the map

$$S : E' \to E, \qquad S(v)(x, y) = \int_0^x \int_0^y v(\xi, \eta)\, d\xi\, d\eta,$$

is not completely continuous, but it sends sets from $\bigcap \mathcal{R}_{\tilde{P}}$ into compact sets (in [8] this difficulty is dealt with in a different way).

For the remaining elements of the proof (in particular the construction of suitable approximations, see [8]).

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