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An Example of a Contractive Mapping Without Fixed Point

ABSTRACT. We answer an open question regarding the existence of a fixed point in a complete metric space X for a continuous self map $T : X \to X$ which satisfies the inequality $d(Tx,Ty) < \max\{d(x,Ty), d(y,Tx)\}$ for distinct x and y in X, and whose sequence of iterates, $\{T^n(x_0)\}_{n=0}^{\infty}$ for some x_0 in X, has a cluster point. This contractive condition appeared (under number ⟨13⟩) in a classification suggested by B. Rhoades in [3]. It was an open question whether any such map necessarily has a fixed point.

We will construct an example of a pair (X, T) which lacks a fixed point. Our example also answers open questions regarding the contractive conditions ⟨38⟩, ⟨88⟩, ⟨17⟩, ⟨42⟩ and ⟨92⟩ of the classification in [3].

1. Introduction. B. Rhoades selected and partially ordered in [3] five groups of contractive conditions. For many of these conditions he either established fixed point theorems or arranged examples of maps, which do not possess any fixed point. Still there were questions which remained open and later on J. Kincses and V. Totik in [2], answered some of them. They considered continuous self maps of complete metric spaces satisfying one of these contractive conditions and either no additional assumption or one of the following additional assumptions, imposed either on the space or on the map:

(A1) the sequence of iterates $\{T^n(x_0)\}_{n=1}^{\infty}$ for some x_0 in X has a cluster point in X;

(A2) the metric space X is compact.

After their contribution there still remained nine open questions regarding the existence of a fixed point for different contractive definitions from the classification of B. Rhoades. In this paper we present answers to six of them which involve the following contractive conditions:

⟨13⟩ For all distinct $x,y \in X$,

$$d(Tx,Ty) < \max\{d(x,Ty), d(y,Tx)\};$$

⟨38⟩ There exists a natural number p, such that for all distinct $x,y \in X$

$$d(T^p x, T^p y) < \max\{d(x,T^p y), d(y,T^p x)\};$$

⟨88⟩ For each $x \in X$ there exists a natural number $p(x)$ such that for all $y \in X$,

$$d\left(T^{p(x)}x, T^{p(x)}y\right) < \max\left\{d\left(x, T^{p(x)}y\right), d\left(y, T^{p(x)}x\right)\right\};$$

⟨17⟩ For all distinct $x,y \in X$,

$$d(Tx,Ty) < \max\{d(x,Ty), d(y,Tx), d(x,y)\};$$

⟨42⟩ There exists a natural number p, such that for all distinct $x,y \in X$

$$d(T^p x, T^p y) < \max\{d(x,T^p y), d(y,T^p x), d(x,y)\};$$

⟨92⟩ For each $x \in X$ there exists a natural number $p(x)$ such that

$$d\left(T^{p(x)}x, T^{p(x)}y\right) < \max\left\{d\left(x, T^{p(x)}y\right), d\left(y, T^{p(x)}x\right), d(x,y)\right\}$$

for all $y \in X$.

We address the question of *whether for any continuous self map $T : X \to X$ satisfying one of these conditions and the assumption (A1) there exists necessarily a fixed point $\overline{x} \in X : T\overline{x} = \overline{x}$.* J. Kincses and V. Totik have conjectured in [2] *negative* answer to this question.

In this paper we provide an example of a map satisfying ⟨13⟩ and the assumption (A1), which does not have any fixed points. In the earlier publication [1] by the first of the authors and J. Carvalho e Silva the complete comparison of all these contractive conditions has been made and we will make here use of the fact that ⟨13⟩ implies ⟨38⟩, which implies ⟨88⟩ and also ⟨13⟩ implies ⟨17⟩, which implies ⟨42⟩, which implies ⟨92⟩. Therefore the example we construct provides negative answer to the above formulated question for any of the above mentioned contractive conditions and hence the conjecture of J. Kincses and V. Totik is confirmed.

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2. Main result. We are going to construct here a complete metric space X and a continuous map $T : X \longrightarrow X$ which satisfies the contractive condition ⟨13⟩ and the assumption (A1) and does not possess any fixed point.

Our X is a countable set of points $X = \left(\cup_{k=2}^{+\infty} Y_k\right) \cup Y_\infty \cup Z \cup W$, where

$$Y_k = \left\{y_k^i : k \geq 2, 1 \leq i \leq k\right\}, \quad Y_\infty = \left\{y_\infty^i : i \geq 1\right\},$$

$$W = \{w_k : k \geq 2\}, \quad Z = \{z_k : k \geq 2\}.$$

The set $(\bigcup_{k=2}^{\infty} Y_k) \cup Z \cup W$ is the sequence of iterates of the point y_2^1, while Y_∞ is the sequence of iterates of the point y_∞^1.

The metric relations we will define in a moment imply that

$$\lim_{k \to \infty} y_k^j = y_\infty^j, \quad \forall j = 1, 2, \ldots.$$

Therefore each of y_∞^j $(j = 1, 2, \ldots)$ is a cluster point of the sequence $\left\{T^n y_2^1\right\}_{n=1}^{\infty}$. All other points of X are isolated.

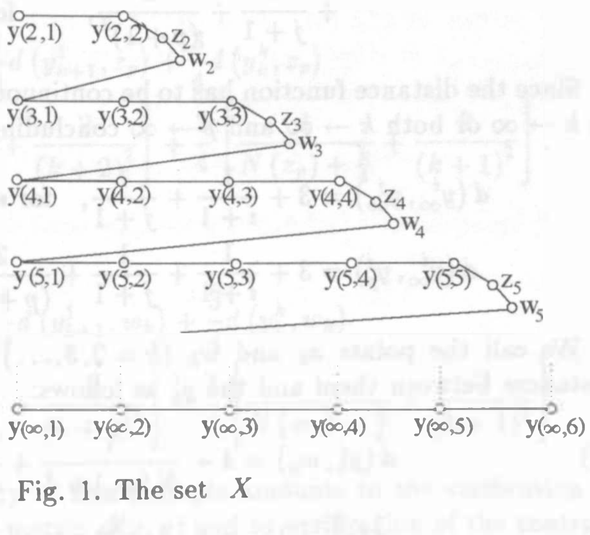

Fig. 1. The set X

Remark. On the Fig. 1 $y(k, i)$ stands for y_k^i.

We define the map $T : X \to X$ as follows:

$$T\left(y_k^i\right) = y_k^{i+1}, \text{ if } i < k; \quad T\left(y_k^k\right) = z_k; \quad T\left(z_k\right) = w_k;$$
$$T\left(w_k\right) = y_{k+1}^1 \quad \text{and} \quad T\left(y_\infty^i\right) = y_\infty^{i+1}, \quad i = 1, \ldots .$$

The set X and the map $T : X \to X$ are completely determined and now we have to endow X with a structure of complete metric space. To define the distance function $d(x, y)$ let us denote, for any x in $X \setminus Y_\infty = \left\{T^m\left(y_2^1\right) : m = 0, 1, \ldots\right\}$, $N(x) := n + 4$, where n is the order of the iterate $T^n\left(y_2^1\right)$ which coincides with x.

It is easy to compute that: $N(z_k) = (k^2 + 5k - 2)/2$ and $N(w_k) = N(z_k) + 1 = (k^2 + 5k)/2$.

We put $d\left(y_\infty^i, y_p^i\right) = 2/(p+1)^2$, which implies that $\lim_{k\to\infty} y_k^j = y_\infty^j$, $\forall j = 1, 2, \ldots$.

Also

$$d\left(y_k^i, y_p^i\right) = \frac{2}{(k+1)^2} + \frac{2}{(p+1)^2}, \quad \text{for } k \neq p.$$

We define

$$d\left(y_k^i, y_k^j\right) = 3 + \frac{1}{i+1} + \frac{1}{j+1}, \quad \text{for } i \neq j,$$

and

$$\text{(1)} \qquad d\left(y_k^i, y_p^j\right) = 3 + \frac{1}{i+1} + \frac{2}{(k+1)^2} + \frac{1}{j+1} + \frac{2}{(p+1)^2}, \quad \text{for } k \neq p \text{ and } i \neq j.$$

Since the distance function has to be continuous, we take in (1) the limits as $k \to \infty$ or both $k \to \infty$ and $p \to \infty$ concluding

$$d\left(y_\infty^i, y_\infty^j\right) = 3 + \frac{1}{i+1} + \frac{1}{j+1}, \quad \text{for } i \neq j,$$
$$d\left(y_\infty^i, y_p^j\right) = 3 + \frac{1}{i+1} + \frac{1}{j+1} + \frac{2}{(p+1)^2}, \quad \text{for } i \neq j.$$

We call the points z_k and w_k $(k = 2, 3, \ldots)$ the *knots* and define the distances between them and the y_k^i as follows:

$$\text{(2)} \qquad d\left(y_k^i, w_p\right) = 4 + \frac{1}{N(w_p) + \frac{i}{4}} + \frac{2}{(k+1)^2},$$

$$\text{(3)} \qquad d\left(y_k^i, z_p\right) = 4 + \frac{1}{N(z_p) + \frac{i}{4}} + \frac{2}{(k+1)^2}.$$

Again taking the limit in (2) and (3), as k tends to infinity, we conclude

$$d(y^i_\infty, w_p) = 4 + \frac{1}{N(w_p) + \frac{i}{4}},$$

$$d(y^i_\infty, z_p) = 4 + \frac{1}{N(z_p) + \frac{i}{4}}.$$

Finally we define the distances between the knots. For $k < p$ we put:

(4)

$$d(z_k, w_p) = \frac{1}{4}d(y^1_{k+1}, w_p) + \frac{3}{4}d(y^k_k, w_p)$$

$$= 4 + \frac{1}{4}\left[\frac{1}{N(w_p) + \frac{1}{4}} + \frac{2}{(k+2)^2}\right] + \frac{3}{4}\left[\frac{1}{N(w_p) + \frac{k}{4}} + \frac{2}{(k+1)^2}\right],$$

(5)

$$d(w_k, z_p) = \frac{1}{2}d(y^1_{k+1}, z_p) + \frac{1}{2}d(y^k_k, z_p)$$

$$= 4 + \frac{1}{2}\left[\frac{1}{N(z_p) + \frac{1}{4}} + \frac{2}{(k+2)^2}\right] + \frac{1}{2}\left[\frac{1}{N(z_p) + \frac{k}{4}} + \frac{2}{(k+1)^2}\right],$$

(6)

$$d(w_k, w_p) = \frac{1}{2}d(y^1_{k+1}, w_p) + \frac{1}{2}d(y^k_k, w_p)$$

$$= 4 + \frac{1}{2}\left[\frac{1}{N(w_p) + \frac{1}{4}} + \frac{2}{(k+2)^2}\right] + \frac{1}{2}\left[\frac{1}{N(w_p) + \frac{k}{4}} + \frac{2}{(k+1)^2}\right],$$

(7)

$$d(z_k, z_p) = \frac{1}{4}d(y^1_{k+1}, z_p) + \frac{3}{4}d(y^k_k, z_p)$$

$$= 4 + \frac{1}{4}\left[\frac{1}{N(z_p) + \frac{1}{4}} + \frac{2}{(k+2)^2}\right] + \frac{3}{4}\left[\frac{1}{N(z_p) + \frac{k}{4}} + \frac{2}{(k+1)^2}\right].$$

For $k = p$ we put:

(8)

$$d(z_k, w_k) = \frac{1}{4}d(y^1_{k+1}, w_k) + \frac{3}{4}d(y^k_k, w_k)$$

$$= 4 + \frac{1}{4}\left[\frac{1}{N(w_k) + \frac{1}{4}} + \frac{2}{(k+2)^2}\right] + \frac{3}{4}\left[\frac{1}{N(w_k) + \frac{k}{4}} + \frac{2}{(k+1)^2}\right].$$

The proof of the validity of this example amounts to the verification of triangle inequality for the metric $d(x, y)$ and to verification of the contractive condition (13). The first part is an elementary computation, which is only nontrivial when it involves one of the distances $d\left(y_k^i, y_p^i\right)$, $i = 1, 2, \ldots$, $k, p = 2, 3, \ldots$. What for the other part of the proof, then we only mark its key points, since the intermediate computations are cumbersome.

To verify ⟨13⟩ it is sufficient to prove the following inequalities:

1) for $1 \le i \le j$ and $k, p = 2, 3, \ldots$,

$$d\left(y_k^{i+1}, y_p^{j+1}\right) = d\left(Ty_k^i, Ty_p^j\right) < d\left(y_k^i, y_p^{j+1}\right) = d\left(y_k^i, Ty_p^j\right);$$

2) for $j = 1, 2, \ldots$ and $k, p = 2, 3, \ldots$,

$$d\left(y_{k+1}^1, y_p^{j+1}\right) = d\left(Tw_k, Ty_p^j\right) < d\left(w_k, y_p^{j+1}\right) = d\left(w_k, Ty_p^j\right);$$

3) for $i = 1, 2, \ldots$ and $k, p = 2, 3, \ldots$,

$$d\left(y_k^{i+1}, z_p\right) = d\left(Ty_k^i, Ty_p^p\right) < d\left(y_k^i, z_p\right) = d\left(y_k^i, Ty_p^p\right);$$

4) for $2 \le k < p$,

$$d\left(z_k, z_p\right) = d\left(Ty_k^k, Ty_p^p\right) < d\left(y_k^k, z_p\right) = d\left(y_k^k, Ty_p^p\right)$$

(by virtue of (7) it is enough to prove that $d\left(y_{k+1}^1, z_p\right) < d\left(y_k^k, z_p\right)$);

5) for $k, p = 2, 3, \ldots$,

$$d\left(y_{k+1}^1, z_p\right) = d\left(Tw_k, Ty_p^p\right) < d\left(w_k, z_p\right) = d\left(w_k, Ty_p^p\right)$$

(by virtue of (5) for $k < p$ it is enough to prove the inequality $d\left(y_{k+1}^1, z_p\right) < d\left(y_k^k, z_p\right)$, by virtue of (8) for $k = p$ we only need to prove that

$$d\left(y_{k+1}^1, z_k\right) < d\left(y_{k+1}^1, w_k\right)/4 + 3d\left(y_k^k, w_k\right)/4$$

and by virtue of (4) for $k > p$ we only need to prove that

$$d\left(y_{k+1}^1, z_p\right) < d\left(y_{p+1}^1, w_k\right)/4 + 3d\left(y_p^p, w_k\right)/4;$$

6) for $k, p = 2, 3, \ldots$ and $i = 1, 2, \ldots$,

$$d\left(y_k^{i+1}, w_p\right) = d\left(Ty_k^i, Tz_p\right) < d\left(y_k^i, w_p\right) = d\left(y_k^i, Tz_p\right);$$

7) for $k, p = 2, 3, \ldots$,

$$d\left(y_{k+1}^1, w_p\right) = d\left(Tw_k, Tz_p\right) < d\left(y_{k+1}^1, z_p\right) = d\left(Tw_k, z_p\right);$$

8) for $2 \leq k < p$,

$$d(w_k, w_p) = d(Tz_k, Tz_p) < d(z_k, w_p) = d(z_k, Tz_p)$$

(by virtue of (6) and (4) this is equivalent to the inequality $d(y^1_{k+1}, w_p) < d(y^k_k, w_p)$);

9) for $2 \leq k < p$,

$$d(w_k, z_p) = d(Tz_k, Ty^p_p) < d(z_k, z_p) = d(z_k, Ty^p_p)$$

(by virtue of (5) and (7) it is enough to prove that

$$d(y^1_{k+1}, z_p) < d(y^k_k, z_p));$$

10) for $k \geq p$,

$$d(w_k, z_p) = d(Tz_k, Ty^p_p) < d(w_k, y^p_p) = d(Tz_k, y^p_p)$$

(for $k > p$ by virtue of (4) it suffices to prove that $d(y^1_{p+1}, w_k) < d(y^p_p, w_k)$and if $k = p$, then according to (8) it is enough to prove that $d(y^1_{k+1}, w_k) < d(y^k_k, w_k)$);

11) for $1 \leq i \leq j$,

$$d(y^{i+1}_k, y^{j+1}_\infty) = d(Ty^i_k, Ty^j_\infty) < d(y^i_k, y^{j+1}_\infty) = d(y^i_k, Ty^j_\infty)$$

12) for $1 \leq j \leq i$,

$$d(y^{i+1}_k, y^{j+1}_\infty) = d(Ty^i_k, Ty^j_\infty) < d(y^{i+1}_k, y^j_\infty) = d(Ty^i_k, y^j_\infty);$$

13) for $k = 2, 3, \ldots$ and $j = 1, 2, \ldots$,

$$d(y^1_{k+1}, y^{j+1}_\infty) = d(Tw_k, Ty^j_\infty) < d(w_k, y^{j+1}_\infty) = d(w_k, Ty^j_\infty);$$

14) for $k = 2, 3, \ldots$ and $j = 1, 2, \ldots$,

$$d(w_k, y^{j+1}_\infty) = d(Tz_k, Ty^j_\infty) < d(w_k, y^j_\infty) = d(Tz_k, y^j_\infty);$$

15) for $k = 2, 3, \ldots$ and $j = 1, 2, \ldots$,

$$d(z_k, y^{j+1}_\infty) = d(Ty^k_k, Ty^j_\infty) < d(z_k, y^j_\infty) = d(Ty^k_k, y^j_\infty);$$

16) for $1 \leq i < j$,

$$d(y^{i+1}_\infty, y^{j+1}_\infty) = d(Ty^i_\infty, Ty^j_\infty) < d(y^i_\infty, y^{j+1}_\infty) < d(y^i_\infty, Ty^j_\infty).$$

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