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Construction of a Lipschitzian Retraction in the Space c_0

ABSTRACT. In this paper we give a constructive example of a lipschitzian retraction in the space c_0.

1. Introduction. Let c_0 denote the space of all sequences of real numbers convergent to 0, with maximum norm. In this paper we construct a lipschitzian retraction of the unit ball onto the unit sphere. Let us formulate this more precisely. Let $B = \{x \in c_0 : \|x\| \le 1\}$ be the closed unit ball and $S = \{x \in c_0 : \|x\| = 1\}$ be the unit sphere.

The mapping $R : B \to S$ is said to be a lipschitzian retraction of B onto S if:

(i) R satisfies the Lipschitz condition i.e. $\|Rx - Ry\| \le k\|x - y\|$ for all $x, y \in B$;

(ii) $Rx = x$ for all $x \in S$.

The problem of existence of such a retraction in any infinitely dimensional Banach space was considered in [1] and [8] a suitable construction was given. However, it is fairly complicated. Below we present a much simpler construction of a lipschitzian retraction with relatively small Lipschitz constant in the space c_0 which can be used as an estimate from above of the retraction constant $k_0(X)$ in the space c_0. The retraction constant $k_0(X)$ is the infimum of the set of all real numbers $k > 1$ for which there exists a retraction $R : B \to S$ with Lipschitz constant k in the space X.

It is known that $k_0(X) \geq 3$ for any infinitely dimensional Banach space X. For a detailed discussion of the topics mentioned above we refer the reader to [5].

For any $k \geq 0$, let $L(k)$ denote the class of Lipschitz mappings $T : B \to B$ with constant k.

2. Construction. For $k > 1$ define a mapping $T : B \to S$ by

$$Tx = T(x_1, x_2, x_3, ...) = (1, \min\{1, k|x_1|\}, \min\{1, k|x_2|\}, ...).$$

It is easy to check that $T \in L(k)$. Observe that $\|x - Tx\| > 1 - 1/k$ because the reverse inequality implies $x_i \geq 1/k$, for $i = 1, 2, 3, ...$ which is a contradiction. Define the homotopy $H : [0,1] \times S \to S$ by

$$H(c, x) = \frac{x + (c/2)Tx}{\|x + (c/2)Tx\|}.$$

The homotopy H is well defined because of

$$\|x + (c/2)Tx\| \geq 1 - c/2 \geq 1/2$$

and joins the point $H(0, x) = x$ to the point $H(1, x) = \frac{x+(1/2)Tx}{\|x+(1/2)Tx\|}$ along a path on the sphere S. The homotopy H is lipschitzian for we have

$$\begin{aligned}
\|H(c,x) - H(d,y)\| &\leq \|H(c,x) - H(d,x)\| + \|H(d,x) - H(d,y)\| \\
&\leq 4\left\|x + \frac{c}{2}Tx - x - \frac{d}{2}Tx\right\| + 4\left\|x + \frac{d}{2}Tx - y - \frac{d}{2}Ty\right\| \\
&\leq 2|c-d| + 4\left(1 + \frac{dk}{2}\right)\|x - y\| \\
&\leq 2|c-d| + 2(2+k)\|x-y\| \\
&= 2|c-d| + B(k)\|x-y\|.
\end{aligned}$$

Let us estimate

$$\left\|\frac{x}{r} + \frac{1}{2}T\left(\frac{x}{r}\right)\right\| = \max\left\{\left|\frac{x_1}{r} + \frac{1}{2}\right|, \left|\frac{x_2}{r} + \frac{1}{2}\min\left\{1, k\left|\frac{x_1}{r}\right|\right\}\right|, ...\right\}.$$

Take $x/r = (x_1/r, x_2/r, x_3/r, ...)$. Then

(i) for every $i = 1, 2, 3, ...$ $|x_i/r| < 1/k$ which implies

$$\left\|\frac{x}{r} + \frac{1}{2}T\left(\frac{x}{r}\right)\right\| \geq \left|\frac{x_1}{r} + \frac{1}{2}\right| > \frac{1}{2} - \frac{1}{k};$$

(ii) there exists a maximal index i_0 for which $|x_{i_0}/r| \geq 1/k$.
Hence

$$\left\|\frac{x}{r}+\frac{1}{2}T\left(\frac{x}{r}\right)\right\| \geq \left|\frac{x_{i_0+1}}{r}+\frac{1}{2}\min\left\{1,k\left|\frac{x_{i_0}}{r}\right|\right\}\right| \geq \frac{1}{2}-\frac{1}{k}.$$

We showed that

$$\left\|\frac{x}{r}+\frac{1}{2}T\left(\frac{x}{r}\right)\right\| \geq \frac{1}{2}-\frac{1}{k}>0 \quad \text{for } k>2.$$

Now for $k>2$ we define a retraction

$$Rx=\begin{cases} \frac{x/r+(1/2)T(x/r)}{\|x/r+(1/2)T(x/r)\|} & \text{for } \|x\| \leq r \\ H\left(1-f\left(\|x\|\right), x/\|x\|\right) & \text{for } \|x\|>r, \end{cases}$$

where f is any convex, increasing function defined on $[0,r]$ such that $f(r)=0, f(1)=1$. The retraction R is lipschitzian. For $\|x\| \leq r$, $\|y\| \leq r$ we have

$$\begin{aligned} \|Rx-Ry\| &= \left\|\frac{x/r+(1/2)T(x/r)}{\|x/r+(1/2)T\left(\frac{x}{r}\right)\|}-\frac{y/r+(1/2)T(y/r)}{\|y/r+(1/2)T(y/r)\|}\right\| \\ &\leq \frac{2}{(1/2)-(1/k)}\left\|\frac{x}{r}+\frac{1}{2}T\left(\frac{x}{r}\right)-\left(\frac{y}{r}+\frac{1}{2}T\left(\frac{y}{r}\right)\right)\right\| \\ &\leq \frac{2}{r}\,\frac{k(k+2)}{k-2}\|x-y\| = \frac{C(k)}{r}\|x-y\|. \end{aligned}$$

For $\|x\|>r$, $\|y\|>r$, after minimizing with respect to the function f and radius r we have

$$\|Rx-Ry\| \leq \frac{2B(k)}{r}\|x-y\|,$$

where r is the solution of the equation

$$\frac{2B(k)}{r}=\frac{2-2B(k)\ln r}{1-r},$$

cf. [5] Lemma 21.1. Finally we have

$$R \in L\left(\max\left\{\frac{C(k)}{r}, \frac{2B(k)}{r}\right\}\right).$$

For $k=4$ we have $B(k)=12, C(k)=24/r$. Then $R \in L(24/r)$. Numerical experiments show that $r=0.6823$ is almost optimal and hence we have a retraction with Lipschitz constant less than 35.18 which shows that $k_0(c_0)<35.18$. The problem of exact evaluation of $k_0(X)$ for at least one space remains open.

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