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Approximating Common Fixed Points of Nonexpansive Semigroups by the Mann Iteration Process

ABSTRACT. In this paper we introduce a new iteration procedure of Mann's type for approximating common fixed points for a family of nonexpansive mappings in a Hilbert space. Then, using some ideas in the nonlinear ergodic theory, we prove that the iterates converge weakly to a common fixed point for a family of mappings. Further, we prove the strong convergence theorems for a noncommutative family of nonexpansive mappings in a Hilbert space.

1. Introduction. Let C be a nonempty closed convex subset of a real Hilbert space H. Then a mapping $T : C \to C$ is called nonexpansive, if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of T.

Mann [11] introduced an iteration procedure for approximating fixed points of a mapping T in a Hilbert space as follows:

$$(1) \quad x_1 = x \in C, \quad x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n \quad \text{for every} \quad n \ge 1,$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$.

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Later, Reich [13] discussed this iteration procedure in a uniformly convex Banach space whose norm is Fréchet differentiable and obtained that the iterates $\{x_n\}$ converge weakly to a fixed point of T if $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$ and $F(T) \neq \emptyset$.

On the other hand, Baillon [1] proved the first nonlinear ergodic theorem for nonexpansive mappings in a Hilbert space: Let C be a nonempty closed convex subset of a Hilbert space and let T be a nonexpansive mapping of C into itself. If the set $F(T)$ is nonempty, then for each $x \in C$, the Cesàro means

$$S_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converge weakly to some $y \in F(T)$. This result has been extended to nonlinear ergodic theorems for families of nonexpansive mappings by several authors (see, e.g. [2], [6], [7], [14], [15], 17]).

2. Preliminaries. Throughout this paper we assume that H is a real Hilbert space. In a real Hilbert space H, we have

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$$

for all $x, y \in H$ and $\lambda \in \mathbb{R}$ with $0 \leq \lambda \leq 1$. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ of vectors converges weakly to x. Similarly $x_n \to x$ (or $\lim_{n\to\infty} x_n = x$) will symbolize strong convergence. We denote by $\mathbb{R}$ and $\mathbb{R}^+$ the set of all real numbers and the set of all nonnegative real numbers, respectively. For a subset A of H, $\operatorname{co} A$ and $\overline{\operatorname{co}}A$ mean the convex hull of A and the closure of the convex hull of A, respectively.

Let S be a semigroup and let $B(S)$ be the Banach space of all bounded real valued functions on S with supremum norm. Then, for each $s \in S$ and $f \in B(S)$, we can define elements $r_s f \in B(S)$ and $l_s f \in B(S)$ by $(r_s f)(t) = f(ts)$ and $(l_s f)(t) = f(st)$ for all $t \in S$, respectively. We also denote by r_s^* and l_s^* the conjugate operators of r_s and l_s, respectively. Let D be a subspace of $B(S)$ and let μ be an element of D^*. Then, we denote by $\mu(f)$ the value of μ at $f \in D$. Sometimes, $\mu(f)$ will be also denoted by $\mu_t(f(t))$ or $\int f(t)d\mu(t)$. When D contains constants, a linear functional μ on D is called a mean on D: if $\|\mu\| = \mu(1) = 1$. We also know that μ is a mean on D if and only if

$$\inf_{s\in S} f(s) \leq \mu(f) \leq \sup_{s\in S} f(s)$$

for each $f \in D$. For $s \in S$, we can define a point evaluation δ_s by $\delta_s(f) = f(s)$ for every $f \in B(S)$. A convex combination of point evaluations is called a finite mean on S. A finite mean on S is also a mean on any subspace D of $B(S)$ containing constants. Further, let D be a subspace of $B(S)$ containing constants which is r_s-invariant i.e., $r_s D \subset D$ for each $s \in S$. Then, a mean μ on D is called right invariant if $\mu(r_s f) = \mu(f)$ for all $s \in S$ and $f \in D$. Similarly, we can define a left invariant mean on a l_s-invariant subspace of $B(S)$ containing constants. A right and left invariant mean is called an invariant mean.

The following definition which was introduced by Takahashi [15] is crucial in the nonlinear ergodic theory for abstract semigroups. Let u be a function of S into H such that the weak closure of $\{u(t) : t \in S\}$ is weakly compact and $\langle u(\cdot), y\rangle \in D$ for every $y \in H$. And let μ be an element of D^*. Then, by the Riesz theorem, there exists a unique element $u_\mu \in H$ such that $\langle u_\mu, y\rangle = \mu_s\langle u(s), y\rangle$ for all $y \in H$. If μ is a mean on D, then u_μ is contained in $\overline{\text{co}}\{u(t) : t \in S\}$ (for example, see [8], [9], 15]). Sometimes, u_μ will be denoted by $\int u(t)d\mu(t)$.

Let C be a subset of a Hilbert space H. Then, a mapping T of C into itself is said to be nonexpansive on C if $\|Tx - Ty\| \le \|x - y\|$ for every $x, y \in C$. Let T be a mapping of C into itself. Then we denote by $F(T)$ the set of fixed points of T. On the other hand, a family $\mathcal{S} = \{T(s) : s \in S\}$ of mappings of C into itself is called a nonexpansive semigroup on C if it satisfies the following conditions:

(i) $T(st) = T(s)T(t)$ for all $s, t \in S$;

(ii) $\|T(s)x - T(s)y\| \le \|x - y\|$ for all $x, y \in C$ and $s \in S$.

We denote by $F(\mathcal{S})$ the set of common fixed points of $T(t), t \in S$, that is, $F(\mathcal{S}) = \bigcap_{t\in S} F(T(t))$.

We know that a Hilbert space H satisfies Opial's condition [12], that is, for any sequence $\{x_n\} \subset E$ with $x_n \rightharpoonup x \in E$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \tag{2}$$

holds for every $y \in E$ with $y \ne x$.

3. Weak convergence theorems for nonexpansive semigroups. Let S be a semigroup, let C be a nonempty closed convex subset of a Hilbert space H and let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \ne \emptyset$. Let D be a subspace of $B(S)$ such that D contains constants and for any $x \in C$ and $y \in H$, $\langle T(\cdot)x, y\rangle \in D$. For any mean μ on D and $x \in C$, there exists a unique element $T_\mu x$ in C such that $\langle T_\mu x, z\rangle = \mu_s\langle T(s)x, z\rangle$ for all $z \in H$; see [7], [15].

Now consider the following iteration scheme :

(3) $x_1 = x \in C$ and $x_{n+1} = \alpha_n x_n + (1-\alpha_n) T_{\mu_n} x_n$ for every $n \geq 1$,

where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $[0,1]$ and $\{\mu_n\}$ is a sequence of means on D. Putting $T_n x = \alpha_n x + (1-\alpha_n) T_{\mu_n} x$ for every $x \in C$, the mapping T_n of C into itself is also nonexpansive. In fact, let $x, y \in C$. Then, for any $z \in C$, we have

$$\begin{aligned}\|T_{\mu_n} x - T_{\mu_n} y\| &= \sup_{\|z\| \leq 1} \left| \int \langle T(s)x - T(s)y, z\rangle d\mu_n(s) \right| \\ &\leq \sup_{\|z\|\leq 1} \int \|T(s)x - T(s)y\| \|z\| d\mu_n(s) \\ &\leq \int \|T(s)x - T(s)y\| d\mu_n(s) \leq \|x-y\|\end{aligned}$$

and hence

$$\begin{aligned}\|T_n x - T_n y\| &= \|\{\alpha_n x + (1-\alpha_n) T_{\mu_n} x\} - \{\alpha_n y + (1-\alpha_n) T_{\mu_n} y\}\| \\ &\leq \alpha_n \|x-y\| + (1-\alpha_n)\|T_{\mu_n} x - T_{\mu_n} y\| \\ &\leq \alpha_n \|x-y\| + (1-\alpha_n)\|x-y\| = \|x-y\|.\end{aligned}$$

Further, we have $F(\mathcal{S}) \subset F(T_{\mu_n}) \subset F(T_n)$ for every $n \geq 1$ and hence $F(\mathcal{S}) \subset \bigcap_{n=1}^{\infty} F(T_n)$.

Using ideas of [2], [6], we can prove the following lemma.

Lemma 3.1. *Let C be a nonempty bounded closed convex subset of a Hilbert space H and let S be a semigroup. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C and let D be a subspace of $B(S)$ containing constants and invariant under every $l_s, s \in S$. Suppose that for each $x \in C$ and $z \in H$, the function $t \mapsto \langle T(t)x, z\rangle$ is in D. Let $\{\mu_n\}$ be a sequence of means on D such that $\lim_{n\to\infty} \|\mu_n - l_s^* \mu_n\| = 0$ for every $s \in S$. Then,*

$$\lim_{n\to\infty} \sup_{x \in C} \|T_{\mu_n} x - T(t) T_{\mu_n} x\| = 0$$

for every $t \in S$.

Proof. Let $u \in H$. We have that

$$\begin{aligned}\|T_{\mu_n} x - u\|^2 &= \langle T_{\mu_n} x - u, T_{\mu_n} x - u\rangle = (\mu_n)_t \langle T(t)x - u, T_{\mu_n} x - u\rangle \\ &= (\mu_n)_t (\mu_n)_s \langle T(t)x - u, T(s)x - u\rangle.\end{aligned}$$

Since

$$2\langle T(t)x-u, T(s)x-u\rangle = \|T(t)x-u\|^2 + \|T(s)x-u\|^2 - \|T(t)x-T(s)x\|^2,$$

we have

$$\begin{aligned} &2(\mu_n)_t(\mu_n)_s\langle T(t)x-u, T(s)x-u\rangle \\ (4)\quad &= (\mu_n)_t(\mu_n)_s\{\|T(t)x-u\|^2 + \|T(s)x-u\|^2 - \|T(t)x-T(s)x\|^2\} \\ &= 2(\mu_n)_t\|T(t)x-u\|^2 - (\mu_n)_t(\mu_n)_s\|T(t)x-T(s)x\|^2. \end{aligned}$$

Then, putting $u = T_{\mu_n}x$ in (4), we have

$$(\mu_n)_t(\mu_n)_s\|T(t)x-T(s)x\|^2 = 2(\mu_n)_t\|T(t)x-T_{\mu_n}x\|^2.$$

So, it follows that

$$(5)\qquad \|T_{\mu_n}x-u\|^2 = (\mu_n)_t\|T(t)x-u\|^2 - (\mu_n)_t\|T(t)x-T_{\mu_n}x\|^2.$$

Let $s \in S$. Putting $u = T(s)(T_{\mu_n}x)$ in (5),

$$\|T_{\mu_n}x-T(s)T_{\mu_n}x\|^2 = (\mu_n)_t\|T(t)x-T(s)T_{\mu_n}x\|^2 - (\mu_n)_t\|T(t)x-T_{\mu_n}x\|^2.$$

Then, we have that

$$\begin{aligned} &\|T_{\mu_n}x-T(s)T_{\mu_n}x\|^2 \\ &=(\mu_n-l_s^*\mu_n)_t\|T(t)x-T(s)T_{\mu_n}x\|^2 - (\mu_n)_t\|T(t)x-T_{\mu_n}x\|^2 \\ &\qquad + (l_s^*\mu_n)_t\|T(t)x-T(s)T_{\mu_n}x\|^2 \\ &=(\mu_n-l_s^*\mu_n)_t\|T(t)x-T(s)T_{\mu_n}x\|^2 - (\mu_n)_t\|T(t)x-T_{\mu_n}x\|^2 \\ &\qquad + (\mu_n)_t\|T(s)T(t)x-T(s)T_{\mu_n}x\|^2 \\ &\le (\mu_n-l_s^*\mu_n)_t\|T(t)x-T(s)T_{\mu_n}x\|^2 - (\mu_n)_t\|T(t)x-T_{\mu_n}x\|^2 \\ &\qquad + (\mu_n)_t\|T(t)x-T_{\mu_n}x\|^2 \\ &=(\mu_n-l_s^*\mu_n)_t\|T(t)x-T(s)T_{\mu_n}x\|^2 \le \|\mu_n-l_s^*\mu_n\|\cdot M, \end{aligned}$$

where $M = 4\sup_{x\in C}\|x\|^2$. So, we have that $\lim_{n\to\infty}\sup_{x\in C}\|T_{\mu_n}x-T(s)T_{\mu_n}x\| = 0$ for every $s \in S$.

□

We have the following lemma for iterates $\{x_n\}$ defined by (3).

Lemma 3.2. *Let C be a nonempty closed convex subset of a Hilbert space H and let S be a semigroup. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Let D be a subspace of $B(S)$ containing constants and invariant under every $l_s, s \in S$. Suppose that for each $x \in C$ and $z \in H$, the function $t \mapsto \langle T(t)x, z\rangle$ is in D. Let $\{\mu_n\}$ be a sequence of means on D. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)T_{\mu_n}x_n \quad \text{for every} \quad n \geq 1,$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$. Let w be a common fixed point of $T(t), t \in S$. Then, $\lim_{n\to\infty} \|x_n - w\|$ exists.

Proof. Let w be a common fixed point of $T(t), t \in S$. Then, we have

$$\begin{aligned}\|x_{n+1} - w\| &= \|\alpha_n x_n + (1-\alpha_n)T_{\mu_n}x_n - w\| \\ &\leq \alpha_n\|x_n - w\| + (1-\alpha_n)\|T_{\mu_n}x_n - w\| \\ &\leq \alpha_n\|x_n - w\| + (1-\alpha_n)\|x_n - w\| \\ &= \|x_n - w\|\end{aligned}$$

and hence $\lim_{n\to\infty} \|x_n - w\|$ exists.

□

Using Lemma 3.1, we obtain the following lemma which is essential to prove the weak and strong convergence theorems.

Lemma 3.3. *Let C be a nonempty closed convex subset of a Hilbert space H and let S be a semigroup. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$ and let D be a subspace of $B(S)$ containing constants and invariant under every $l_s, s \in S$. Suppose that for each $x \in C$ and $z \in H$, the function $t \mapsto \langle T(t)x, z\rangle$ is in D. Let $\{\mu_n\}$ be a sequence of means on D such that $\lim_{n\to\infty} \|\mu_n - l_s^*\mu_n\| = 0$ for every $s \in S$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)T_{\mu_n}x_n \quad \text{for every} \quad n \geq 1,$$

where $0 \leq \alpha_n \leq a$ for some a with $0 < a < 1$. Then,

$$\lim_n \|T(t)x_n - x_n\| = 0 \quad \text{for every} \quad t \in S.$$

In particular, $x_{n_i} \rightharpoonup y_0$ implies $y_0 \in F(\mathcal{S})$.

Proof. For $x \in C$ and $f \in F(\mathcal{S})$, put $r = \|x - f\|$ and set

$$X = \{u \in H : \|u - f\| \leq r\} \cap C.$$

Then X is a nonempty bounded closed convex subset of C which is $T(t)$-invariant for every $t \in S$ and contains $x_1 = x$. So, without loss of generality, we may assume that C is bounded. Then, it follows from the definition of $\{x_n\}$ that $x_{n+1} - T_{\mu_n} x_n = \alpha_n(x_n - T_{\mu_n} x_n)$.

Let w be a common fixed point of $T(t), t \in S$. Then, from

$$\begin{aligned}\|x_{n+1} - w\|^2 &= \|\alpha_n(x_n - w) + (1-\alpha_n)(T_{\mu_n} x_n - w)\|^2 \\ &= \alpha_n\|x_n - w\|^2 + (1-\alpha_n)\|T_{\mu_n} x_n - w\|^2 \\ &\quad - \alpha_n(1-\alpha_n)\|T_{\mu_n} x_n - x_n\|^2\end{aligned}$$

we have

$$\begin{aligned}\alpha_n(1-a)\|T_{\mu_n} x_n - x_n\|^2 &\le \alpha_n(1-\alpha_n)\|T_{\mu_n} x_n - x_n\|^2 \\ &= \alpha_n\|x_n - w\|^2 + (1-\alpha_n)\|T_{\mu_n} x_n - w\|^2 - \|x_{n+1} - w\|^2 \\ &\le \alpha_n\|x_n - w\|^2 + (1-\alpha_n)\|x_n - w\|^2 - \|x_{n+1} - w\|^2 \\ &= \|x_n - w\|^2 - \|x_{n+1} - w\|^2.\end{aligned}$$

Then, from Lemma 3.2, we obtain

$$\lim_{n\to\infty} \alpha_n\|T_{\mu_n} x_n - x_n\| = 0. \tag{6}$$

Since, for each $t \in S$,

$$\begin{aligned}\|T(t)x_{n+1} - x_{n+1}\| &\le \|T(t)x_{n+1} - T(t)T_{\mu_n} x_n\| \\ &\quad + \|T(t)T_{\mu_n} x_n - T_{\mu_n} x_n\| + \|T_{\mu_n} x_n - x_{n+1}\| \\ &\le 2\|T_{\mu_n} x_n - x_{n+1}\| + \|T(t)T_{\mu_n} x_n - T_{\mu_n} x_n\| \\ &= 2\alpha_n\|x_n - T_{\mu_n} x_n\| + \|T(t)T_{\mu_n} x_n - T_{\mu_n} x_n\|,\end{aligned}$$

from (6) and Lemma 3.1, we have

$$\lim_{n\to\infty} \|T(t)x_n - x_n\| = 0. \tag{7}$$

Assume $x_{n_i} \rightharpoonup y_0$ and $y_0 \notin F(S)$. Then, we have $y_0 \neq T(s)y_0$ for some $s \in S$. Since H satisfies Opial's condition [12] from (7), we obtain,

$$\begin{aligned}\liminf_{i\to\infty} \|x_{n_i} - y_0\| &< \liminf_{i\to\infty} \|x_{n_i} - T(s)y_0\| \\ &= \liminf_{i\to\infty} \|x_{n_i} - T(s)x_{n_i} + T(s)x_{n_i} - T(s)y_0\| \\ &= \liminf_{i\to\infty} \|T(s)x_{n_i} - T(s)y_0\| \le \liminf_{i\to\infty} \|x_{n_i} - y_0\|.\end{aligned}$$

This is a contradiction. Hence, we obtain that y_0 is a common fixed point of $T(t), t \in S$.

□

Now we can prove a weak convergence theorem for nonexpansive semigroups in a Hilbert space.

Theorem 3.4. *Let C be a nonempty closed convex subset of a Hilbert space H and let S be a semigroup. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$ and let D be a subspace of $B(S)$ containing constants and invariant under every $l_s, s \in S$. Suppose that for each $x \in C$ and $z \in H$, the function $t \mapsto \langle T(t)x, z\rangle$ is in D. Let $\{\mu_n\}$ be a sequence of means on D such that $\lim_{n\to\infty} \|\mu_n - l_s^*\mu_n\| = 0$ for every $s \in S$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)T_{\mu_n}x_n \quad \text{for every} \quad n \geq 1,$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$. If $\{\alpha_n\}$ is chosen so that $\alpha_n \in [0,a]$ for some a with $0 < a < 1$, then $\{x_n\}$ converges weakly to a common fixed point y_0 of $T(t), t \in S$.

Proof. Let w be a common fixed point of $T(t), t \in S$. Then, from Lemma 3.2 $\lim_{n\to\infty} \|x_n - w\|$ exists. As in the proof of Lemma 3.3, we may assume that C is bounded. So, $\{x_n\}$ must contain a subsequence which converges weakly to a point in C. So, let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be two subsequences of $\{x_n\}$ such that $x_{n_i} \rightharpoonup z_1$ and $x_{n_j} \rightharpoonup z_2$. Then, from Lemma 3.3, we have that z_1 and z_2 are common fixed points of $T(t), t \in S$. Next, we show $z_1 = z_2$. If not, then since H satisfies Opial's condition [12], we have

$$\begin{aligned}\lim_{n\to\infty} \|x_n - z_1\| &= \lim_{i\to\infty} \|x_{n_i} - z_1\| \\ &< \lim_{i\to\infty} \|x_{n_i} - z_2\| = \lim_{n\to\infty} \|x_n - z_2\| = \lim_{j\to\infty} \|x_{n_j} - z_2\| \\ &< \lim_{j\to\infty} \|x_{n_j} - z_1\| = \lim_{n\to\infty} \|x_n - z_1\|.\end{aligned}$$

This is a contradiction. Hence, we obtain $x_n \rightharpoonup y_0 \in F(\mathcal{S})$. □

As direct consequences of Theorem 3.4, we have the following corollaries.

Corollary 3.5. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \frac{1}{n+1}x_n + \left(1 - \frac{1}{n+1}\right)\frac{1}{n}\sum_{i=1}^{n} T^i x_n = \frac{1}{n+1}\sum_{i=0}^{n} T^i x_n$$

for every $n \geq 1$. Then, $\{x_n\}$ converges weakly to a fixed point of T.

Proof. Let $S = \{0,1,2,\dots\}, \mathcal{S} = \{T^i : i \in S\}$, $D = B(S)$ and $\lambda_n(f) = \frac{1}{n}\sum_{i=0}^{n-1} f(i)$ for all $n = 1,2,\dots$ and $f \in D$. Then, $\{\lambda_n : n = 1,2,\dots\}$ is a sequence of means. Further, we have

$$\|\lambda_n - l_1^*\lambda_n\| = \sup_{\|f\|\le 1} |(\lambda_n - l_1^*\lambda_n)(f)| = \frac{1}{n}\sup_{\|f\|\le 1} |f(0) - f(n)| \le \frac{2}{n} \to 0,$$

as $n \to \infty$ and hence for $k \ge 2$,

$$\|\lambda_n - l_k^*\lambda_n\| \le \|l_k^*\lambda_n - l_{k-1}^*\lambda_n\| + \cdots + \|l_1^*\lambda_n - \lambda_n\| \le k\|\lambda_n - l_1^*\lambda_n\| \to 0,$$

as $n \to \infty$. Therefore, we obtain Corollary 3.5 by using Theorem 3.4. □

Let $\mathrm{N} = \{0,1,2,\dots\}$ and let $Q = \{q_{n,m}\}_{n,m\in\mathrm{N}}$ be a matrix satisfying the following conditions:

(a) $\sup_{n\ge 0} \sum_{m=0}^{\infty} |q_{n,m}| < \infty;$

(b) $\lim_{n\to\infty} \sum_{m=0}^{\infty} q_{n,m} = 1;$

(c) $\lim_{n\to\infty} \sum_{m=0}^{\infty} |q_{n,m+1} - q_{n,m}| = 0.$

Then, according to Lorentz [10], Q is called a strongly regular matrix. If Q is a strongly regular matrix, then for each $m \in \mathrm{N}$, we have that $|q_{n,m}| \to 0$, as $n \to \infty$ (see [7]).

Corollary 3.6. *Let H and C be as in* Corollary 3.5. *Let T be a nonexpansive mapping of C into itself such that $F(T) \ne \emptyset$. Let $Q = \{q_{n,m}\}_{n,m\in\mathrm{N}}$ be a strongly regular matrix. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by $x_{n+1} = \alpha_n x_n + (1-\alpha_n)\sum_{m=0}^{\infty} q_{n,m}T^m x_n$ for every $n \ge 1$, where $\{\alpha_n\}$ is a sequence in $[0,1]$. If $\{\alpha_n\}$ is chosen so that $\alpha_n \in [0,a]$ for some a with $0 < a < 1$, then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. Let $S = \{0,1,2,\dots\}, \mathcal{S} = \{T^n : n \in S\}$, $D = B(S)$ and $\lambda_n(f) = \frac{1}{n}\sum_{m=0}^{\infty} q_{n,m}f(m)$ for each $n = 1,2,\dots$ and $f \in D$. Then, $\{\lambda_n : n = 1,2,\dots\}$ is a sequence of means. Further, we have $\|\lambda_n - l_k^*\lambda_n\| \to 0$ for every $k = 0,1,2,\dots$. Indeed, we have that

$$\|\lambda_n - l_1^*\lambda_n\| = \sup_{\|f\|\le 1} |(\lambda_n - l_1^*\lambda_n)(f)| = \sup_{\|f\|\le 1} \left|\sum_{m=0}^{\infty} q_{n,m}\{f(m) - f(m+1)\}\right|$$

$$= \sup_{\|f\|\le 1} \left| q_{n,0} f(0) + \sum_{m=0}^{\infty} q_{n,m+1} f(m+1) - \sum_{m=0}^{\infty} q_{n,m} f(m+1) \right|$$

$$\le \sum_{m=0}^{\infty} |q_{n,m+1} - q_{n,m}| + |q_{n,0}| \to 0,$$

as $n \to \infty$ and hence for $k \ge 2$,

$$\|\lambda_n - l_k^* \lambda_n\| \le \|l_k^* \lambda_n - l_{k-1}^* \lambda_n\| + \cdots + \|l_1^* \lambda_n - \lambda_n\|$$
$$\le k \|\lambda_n - l_1^* \lambda_n\| \to 0,$$

as $n \to \infty$. So, using Theorem 3.4, we obtain Corollary 3.6.

□

Corollary 3.7. *Let H and C be as in* Corollary 3.5. *Let U and T be nonexpansive mappings of C into itself with $UT = TU$ and $F(T) \cap F(U) \ne \emptyset$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by $x_{n+1} = \alpha_n x_n + (1 - \alpha_n) \frac{1}{n^2} \sum_{i,j=0}^{n-1} U^i T^j x_n$ for every $n \ge 1$, where $\{\alpha_n\}$ is a sequence in $[0,1]$. If $\{\alpha_n\}$ is chosen so that $\alpha_n \in [0, a]$ for some a with $0 < a < 1$, then $\{x_n\}$ converges weakly to a common fixed point of T and U.*

Proof. Let $S = \{0,1,2,\dots\} \times \{0,1,2,\dots\}, \mathcal{S} = \{U^i T^j : (i,j) \in S\}$, $D = B(S)$ and $\lambda_n(f) = \frac{1}{n^2} \sum_{i,j=0}^{n-1} f(i,j)$ for each $n = 1,2,\dots$ and $f \in D$. Then, $\{\lambda_n : n = 1,2,\dots\}$ is a sequence of means. Further, we have that for each $(l,m) \in S$,

$$\|\lambda_n - l_{(l,m)}^* \lambda_n\| = \sup_{\|f\|\le 1} |(\lambda_n - l_{(l,m)}^* \lambda_n)(f)|$$
$$= \sup_{\|f\|\le 1} \left| \frac{1}{n^2} \sum_{i,j=0}^{n-1} f(i,j) - \frac{1}{n^2} \sum_{i,j=0}^{n-1} f(i+l, j+m) \right|$$
$$\le \frac{1}{n^2} \{l \cdot n + m(n-l) + l \cdot n + m(n-l)\}$$
$$= \frac{1}{n^2} \{2n(l+m) - 2ml\} \to 0,$$

as $n \to \infty$. Therefore, using Theorem 3.4, we obtain Corollary 3.7.

□

Let C be a bounded closed convex subset of a Hilbert space H and let $\mathcal{S}' = \{T(t) : t \in \mathbb{R}^+\}$ be a family of nonexpansive mappings of C into itself. Then, $\mathcal{S}'$ is called a one-parameter nonexpansive semigroup on C if it satisfies the following conditions:
$T(0) = I$, $T(t+s) = T(t)T(s)$ for all $t, s \in \mathbb{R}^+$ and $T(t)x$ is continuous in $t \in \mathbb{R}^+$ for each $x \in C$.

Corollary 3.8. *Let H and C be as in Corollary 3.5. Let $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ be a one-parameter nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by $x_{n+1} = \alpha_n x_n + (1-\alpha_n)\frac{1}{s_n}\int_0^{s_n} T(t)x_n dt$ for every $n \geq 1$, where $s_n \to \infty$ as $n \to \infty$ and $\{\alpha_n\}$ is a sequence in $[0,1]$. If $\{\alpha_n\}$ is chosen so that $\alpha_n \in [0,a]$ for some a with $0 < a < 1$, then $\{x_n\}$ converges weakly to a common fixed point of $T(t), t \in S$*

Proof. Let $S = \mathbb{R}^+$, $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ and let D be the Banach space $C(S)$ of all bounded continuous functions on S with the supremum norm. Define $\lambda_s(f) = \frac{1}{s}\int_0^s f(t)dt$ for every $s > 0$ and $f \in D$. Then, we obtain that for any k with $0 < k < \infty$,

$$\begin{aligned}
\|\lambda_s - l_k^*\lambda_s\| &= \sup_{\|f\|\leq 1}\left|\frac{1}{s}\int_0^s f(t)dt - \frac{1}{s}\int_0^s f(t+k)dt\right| \\
&= \frac{1}{s}\sup_{\|f\|\leq 1}\left|\int_0^s f(t)dt - \int_k^{s+k} f(t)dt\right| \\
&= \frac{1}{s}\sup_{\|f\|\leq 1}\left|\int_0^k f(t)dt - \int_s^{s+k} f(t)dt\right| \\
&\leq \frac{1}{s}\sup_{\|f\|\leq 1}\left(\int_0^k |f(t)|dt + \int_s^{s+k} |f(t)|dt\right) \\
&= \frac{2k}{s} \to 0,
\end{aligned}$$

as $s \to \infty$. Therefore, using Theorem 3.4, we obtain Corollary 3.8. □

Corollary 3.9. *Let H and C be as in Corollary 3.5. Let $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ be a one-parameter nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) r_n \int_0^\infty e^{-r_n t} T(t) x_n dt$$

for every $n \geq 1$, where $r_n \to 0$ as $n \to \infty$ and $\{\alpha_n\}$ is a sequence in $[0,1]$. If $\{\alpha_n\}$ is chosen so that $\alpha_n \in [0,a]$ for some a with $0 < a < 1$, then $\{x_n\}$ converges weakly to a common fixed point of $T(t), t \in S$.

Proof. Let $S = \mathbb{R}^+$, $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ and $D = C(S)$. Define $\lambda_r(f) = r\int_0^\infty e^{-rt} f(t)dt$ for each $r > 0$ and $f \in D$. Then, we have that for each s with $0 < s < \infty$,

$$\begin{aligned} \|\lambda_r - l_s^* \lambda_r\| &= \sup_{\|f\|\le 1} \left| r\int_0^\infty e^{-rt} f(t)dt - r\int_0^\infty e^{-rt} f(s+t)dt \right| \\ &= \sup_{\|f\|\le 1} \left| r\int_0^s e^{-rt} f(t)dt + r(1-e^{rs})\int_s^\infty e^{-rt} f(t)dt \right| \\ &\le rs + |1 - e^{rs}| \to 0, \end{aligned}$$

as $r \to 0$. Therefore, using Theorem 3.4, we obtain Corollary 3.9. □

Let $Q = \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be a function satisfying the following conditions:

(a) $\displaystyle \sup_{s\ge 0} \int_0^\infty |Q(s,t)|dt < \infty;$

(b) $\displaystyle \lim_{s\to\infty} \int_0^\infty Q(s,t)dt = 1;$

(c) $\displaystyle \lim_{s\to\infty} \int_0^\infty |Q(s,t+h) - Q(s,t)|dt = 0$ for every $h \in \mathbb{R}^+$.

Then, Q is called a strongly regular kernel.

Corollary 3.10. *Let H and C be as in Corollary 3.5. Let $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ be a one-parameter nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by $x_{n+1} = \alpha_n x_n + (1-\alpha_n)\int_0^\infty Q(s_n,t)T(t)x_n dt$ for every $n \ge 1$, where where $s_n \to \infty$ as $n \to \infty$ and $\{\alpha_n\}$ is a sequence in $[0,1]$. If $\{\alpha_n\}$ is chosen so that $\alpha_n \in [0,a]$ for some a with $0 < a < 1$, then $\{x_n\}$ converges weakly to a common fixed point of $T(t), t \in S$.*

Proof. Let $S = \mathbb{R}^+$, $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ and $D = C(S)$. Define $\lambda_s(f) = \int_0^\infty Q(s,t)f(t)dt$ for every $s > 0$ and $f \in D$. Then, we have that for each h with $0 < h < \infty$,

$$\begin{aligned} \|\lambda_s - l_h^* \lambda_s\| &= \sup_{\|f\|\le 1} |(\lambda_s - l_h^* \lambda_s)(f)| \\ &= \sup_{\|f\|\le 1} \left| \int_0^\infty Q(s,t)f(t)dt - \int_0^\infty Q(s,t)f(t+h)dt \right| \end{aligned}$$

$$= \sup_{\|f\|\leq 1} \left| \int_0^h Q(s,t)f(t)dt + \int_0^\infty Q(s,t+h)f(t+h)dt - \int_0^\infty Q(s,t)f(t+h)dt \right|$$

$$\leq \left| \int_0^h Q(s,t)dt \right| + \left| \int_0^\infty |Q(s,t+h) - Q(s,t)|\, dt \right| \to 0,$$

as $s \to \infty$. Therefore, using Theorem 3.4, we obtain Corollary 3.10. □

4. Strong convergence theorems. In this section, we shall prove strong convergence theorems for iterates defined by (3).

Theorem 4.1. *Let C be a nonempty closed convex subset of a Hilbert space H and let S be a semigroup.*

Let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C such that $\bigcup_{t\in S} T(t)(C) \subset K \subset C$ for some compact subset K of C. Let D be a subspace of $B(S)$ containing constants and invariant under every $l_s, s \in S$. Suppose that for each $x \in C$ and $z \in H$, the function $t \mapsto \langle T(t)x, z\rangle$ is in D. Let $\{\mu_n\}$ be a sequence of means on D such that $\lim_{n\to\infty} \|\mu_n - l_s^*\mu_n\| = 0$ for every $s \in S$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)T_{\mu_n} x_n \quad \text{for every} \quad n \geq 1,$$

where $\{\alpha_n\}_{n=1}^\infty$ is a sequence in $[0,1]$. If $\{\alpha_n\}$ is chosen so that $\alpha_n \in [0,a]$ for some a with $0 < a < 1$, then $\{x_n\}$ converges strongly to a common fixed point y_0 of $T(t), t \in S$.

Proof. From Mazur's theorem [5], $\overline{\text{co}}\left(\{x_1\} \cup \bigcup_{t\in S} T(t)(C)\right)$ is a compact subset of C containing $\{x_n\}$. Then, there exist a subsequence $\{x_{n_i}\}$ of the sequence $\{x_n\}$ and a point $y_0 \in C$ such that $x_{n_i} \to y_0$. So, from Lemma 3.3, we obtain $T(t)y_0 = y_0$ for every $t \in S$. Then, since $\lim_{n\to\infty} \|x_n - y_0\|$ exists, we have

$$\lim_{n\to\infty} \|x_n - y_0\| = \lim_{i\to\infty} \|x_{n_i} - y_0\| = 0.$$

Therefore, $\{x_n\}$ converges strongly to a common fixed point of $T(t), t \in S$. □

The following is a strong convergence theorem which is connected with the metric projections.

Theorem 4.2. *Let C be a nonempty closed convex subset of a Hilbert space H, let S be a semigroup and let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Let D be a subspace of $B(S)$ containing constants and invariant under every $l_s, s \in S$. Suppose that for each $x \in C$ and $z \in H$, the function $t \mapsto \langle T(t)x, z\rangle$ is in D. Let $\{\mu_n\}$ be a sequence of means on D such that $\lim_{n\to\infty} \|\mu_n - l_s^* \mu_n\| = 0$ for every $s \in S$. Let P be the metric projection of C onto $F(\mathcal{S})$. Suppose that $\{x_n\}$ is given by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T_{\mu_n} x_n \quad \text{for every} \quad n \geq 1,$$

where $\alpha_n \in [0, 1]$. Then, $\lim_{n\to\infty} Px_n$ exists. Further, if $z_0 = \lim_{n\to\infty} Px_n$, then z_0 is a unique element of $F(\mathcal{S})$ such that

$$\lim_{n\to\infty} \|x_n - z_0\| = \inf\{\lim_{n\to\infty} \|x_n - w\| : w \in F(\mathcal{S})\}.$$

Proof. Since $F(\mathcal{S})$ is nonempty, as in the proof of Lemma 3.3, we may assume that C is bounded. From Lemma 3.2, we know that $g(w) = \lim_{n\to\infty} \|x_n - w\|$ exists for all $w \in F(\mathcal{S})$. Let $R = \inf\{g(w) : w \in F(\mathcal{S})\}$ and $K = \{u \in F(\mathcal{S}) : g(u) = R\}$. Then, since g is convex and continuous on $F(\mathcal{S})$ and $g(w) \to \infty$ as $\|w\| \to \infty$, K is a nonempty closed convex subset of $F(\mathcal{S})$. Fix $z_0 \in K$ with $g(z_0) = R$. Since P is the metric projection of H onto $F(\mathcal{S})$, we have $\|x_n - Px_n\| \leq \|x_n - y\|$ for all $n \geq 1$ and $y \in F(\mathcal{S})$ and hence

$$\limsup_{n\to\infty} \|x_n - Px_n\| \leq R.$$

Suppose that $\limsup_{n\to\infty} \|x_n - Px_n\| < R$. Then, we may choose $\delta > 0$ and $n_0 \geq 1$ so that $\|x_n - Px_n\| \leq R - \delta$ for all $n \geq n_0$. From Lemma 3.2, we have that

$$\|x_{n+k} - Px_n\| \leq \|x_n - Px_n\| \leq R - \delta < R$$

for all $n \geq n_0$ and $k \geq 0$. Therefore, we obtain that

$$R \leq \lim_{k\to\infty} \|x_{n+k} - Px_n\| = \lim_{k\to\infty} \|x_k - Px_n\| \leq R - \delta < R$$

for all $n \geq n_0$. This contradicts the definition of R. So, we conclude that $\limsup_{n\to\infty} \|x_n - Px_n\| = R$.

Now, we claim that $\lim_{n\to\infty} Px_n = z_0$. If not, then there exists $\varepsilon > 0$ such that for any $k \geq 1$, $\|Px_{k'} - z_0\| \geq \varepsilon$ for some $k' \geq k$. Choose $a > 0$ so that

$a < \sqrt{R^2 + \frac{\varepsilon^2}{4}} - R$. Then, there exists k' such that $\|x_{k'} - Px_{k'}\| \le R + a$ and $\|x_{k'} - z_0\| \le R + a$. Therefore, we have, for $n \ge 1$,

$$\begin{aligned}
R^2 &\le \left\| x_{n+k'} - \frac{Px_{k'} + z_0}{2} \right\|^2 \\
&\le \left\| x_{k'} - \frac{Px_{k'} + z_0}{2} \right\|^2 \\
&= 2\left\| \frac{x_{k'} - Px_{k'}}{2} \right\|^2 + 2\left\| \frac{x_{k'} - z_0}{2} \right\|^2 - \left\| \frac{Px_{k'} - z_0}{2} \right\|^2 \\
&\le 2\cdot\left(\frac{R+a}{2}\right)^2 + 2\cdot\left(\frac{R+a}{2}\right)^2 - \frac{\varepsilon^2}{4} \\
&= (R+a)^2 - \frac{\varepsilon^2}{4} < R^2.
\end{aligned}$$

This is a contradiction. Thus, we have $\lim_{n\to\infty} Px_n = z_0$. Consequently, the element $z_0 \in F(\mathcal{S})$ with $g(z_0) = \inf\{g(w) : w \in F(\mathcal{S})\}$ is unique. □

Using Theorems 3.4 and 4.2, we have also the following theorem.

Theorem 4.3. *Let C be a nonempty closed convex subset of a Hilbert space H and let S be a semigroup. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \ne \emptyset$ and let D be a subspace of $B(S)$ containing constants and invariant under every l_s, $s \in S$. Suppose that for each $x \in C$ and $z \in H$, the function $t \mapsto \langle T(t)x, z\rangle$ is in D. Let $\{\mu_n\}$ be a sequence of means on D such that $\lim_{n\to\infty} \|\mu_n - l_s^*\mu_n\| = 0$ for every $s \in S$. Let P be the metric projection of C onto $F(\mathcal{S})$. Suppose that $\{x_n\}$ is given by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)T_{\mu_n} x_n \quad \textit{for every } n \ge 1,$$

where $\alpha_n \in [0, a]$ for some a with $0 < a < 1$. Then, $\{x_n\}$ converges weakly to an element z of $F(\mathcal{S})$, where $z = \lim_{n\to\infty} Px_n$.

Proof. From Theorem 3.4, $\{x_n\}$ converges weakly to an element y_0 of $F(\mathcal{S})$. From Theorem 4.2, $\{Px_n\}$ converges strongly to an element z_0 of $F(\mathcal{S})$. Since P is the metric projection of H onto $F(\mathcal{S})$, we also know that $\langle x_n - Px_n, Px_n - w\rangle \ge 0$ for all $w \in F(\mathcal{S})$. So, we have $\langle y_0 - z_0, z_0 - w\rangle \ge 0$ for all $w \in F(\mathcal{S})$. Putting $w = y_0$, we obtain $-\|y_0 - z_0\|^2 \ge 0$ and hence $y_0 = z_0$.

□

References

[1] Baillon, J. B., *Un théorème de type ergodique pour les contractions non linéaires dans un espace de Hilbert*, C. R. Acad. Sci. Paris Sér. A-B **280** (1975), 1511-1514.

[2] Brezis, H. and F. E. Browder, *Remarks on nonlinear ergodic theory*, Adv. Math. **25** (1977), 165-177.

[3] Bruck, R. E., *A simple proof of the mean ergodic theorem for nonlinear contractions in Banach spaces*, Israel J. Math. **32** (1979), 107-116.

[4] Day, M. M., *Amenable semigroups*, Illionois J. Math. **1** (1957), 509-544.

[5] Dunford, N. and J.T. Schwartz, *Linear Operators*, Part I, Interscience, New York, 1958.

[6] Hirano, N. and W. Takahashi, *Nonlinear ergodic theorems for nonexpansive mappings in Hilbert spaces*, Kodai Math. J. **2** (1979), 11-25.

[7] Hirano, N., K. Kido and W. Takahashi, *Nonexpansive retractions and nonlinear ergodic theorems in Banach spaces*, Nonlinear Anal. **12** (1988), 1269-1281.

[8] Kido, K. and W. Takahashi, *Mean ergodic theorems for semigroups of linear continuous operators in Banach spaces*, J. Math. Anal. Appl. **103** (1984), 387–394.

[9] ———, *Means on commutative semigroups and nonlinear ergodic theorems*, J. Math. Anal. Appl. **111** (1985), 585–605.

[10] Lorentz, G. G., *A contribution to the theory of divergent series*, Acta Math. **80** (1948), 167-190.

[11] Mann, W. R., *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506-510.

[12] Opial, Z., *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc. **73** (1967), 591-597.

[13] Reich, S., *Weak convergence theorems for nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl. **67** (1979), 274-276.

[14] Rodé, G., *An ergodic theorem for semigroups of nonexpansive mappings in a Hilbert space*, J. Math. Anal. Appl. **85** (1982), 172-178.

[15] Takahashi, W., *A nonlinear ergodic theorem for an amenable semigroup of nonexpansive mappings in a Hilbert space*, Proc. Amer. Math. Soc. **81** (1981), 253-256.

[16] ———, *Nonlinear Functional Analysis (in Japanese)*, Kindai-kagakusha, Japan, 1988.

[17] ———, *Fixed point theorem and nonlinear ergodic theorem for nonexpansive semigroup without convexity*, Canad. J. Math. **44** (1992), 880-887.

[18] Takahashi, W. and J. Y. Park , *On the asymptotic behavior of almost orbits of commutative semigroups in Banach spaces*, in Nonlinear and Convex Analysis (B. L. Lin and S. Simons, Eds.), Lecture Notes in Pure and Appl. Math., Marcel Dekker, Inc., New York, 1987, 271-293.

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