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On the Inversion Formula for Probability Densities

Dedicated to Professor Dominik Szynal on the occasion of his 60th birthday

Abstract. Assuming that the characteristic function φ of the distribution function F in $\mathbb{R}^d$ is square integrable, we derive from Lévy's inversion theorem a formula for probability density of F in terms of φ.

1. Introduction. Let $\varphi : \mathbb{R}^d \to \mathbb{C}$ be a characteristic function determined by certain d-dimensional distribution function $F : \mathbb{R}^d \to \langle 0,1 \rangle \subset \mathbb{R}$. The well-known Lévy's inversion theorem enables us then to reproduce F in a unique manner, but the density of F with respect to the Lebesgue measure λ_d in $\mathbb{R}^d$ can be evaluated by means of φ only in some special cases. More precisely, no satisfactory necessary and sufficient conditions for the existence of probability density expressed explicitly in terms of the characteristic function φ are known. As a matter of fact, some necessary and sufficient condition exists, namely, a function $\varphi : \mathbb{R} \to \mathbb{C}$ is the characteristic function of absolutely continuous distribution function F iff $\varphi(t) = \int_{-\infty}^{+\infty} h(t+\theta)\overline{h(\theta)}d\theta$, where $h : \mathbb{R} \to \mathbb{C}$ satisfies the condition $\int_{-\infty}^{+\infty} |h(\theta)|^2 \, d\theta = 1$ – see [8, Th. 4.2.4, Ch. IV, §4.2, p. 100], but in spite of this, in many practical situations it cannot be easily applicable. Therefore many efforts have been undertaken to find other convenient criterions. The two main and by now classical results concerning this problem are as follows:

1^0. If $|\varphi(t)|^2$ is integrable with respect to the Lebesgue measure on $\mathbb{R}$, then F possessess a density $f \in L^2(\mathbb{R})$;

2^0. If $|\varphi(t)|$ is integrable with respect to the Lebesgue measure λ_1 in $\mathbb{R}$, then F is absolutely continuous with respect to λ_1 and has a bounded, continuous density f.

The first statement was obtained perhaps by Berman [2, Lemma 2.1], and a version of it can be found in [6, Ch. XV, §3], but in most of the monographs and textbooks devoted to probability theory merely the second result is presented and 1^0 is even not mentioned, cf. [7, Part II, Ch. IV, §12.1, Corollary p. 188], [4, Ch. 8, §10, Th. 8.39, p. 178], [1], [5, Ch. 8, § 8.3, Corollary 2, p. 270], [3, Ch. V, §26, Th. 26.2 and corollaries, p. 342-343], [8, Ch. III, §3.2, Th. 3.2.2, p. 51].

The reason of such a situation is quite prosaic – to prove 1^0 far more advanced tools are needed, while for the proof of 2^0 Lévy's inversion formula is quite sufficient.

However, it should be pointed out that 1^0 can be derived also from Lévy's theorem. The aim of this note is to provide the proof of 1^0 based only on Lévy's inversion formula. The presented method may be interested from the theoretical point of view, but it seems to be more important didactically, because it may serve as an easy approach to deeper results.

It is worth mentioning that a sufficient condition of another kind for the existence of probability density, ensuring at the same time that a given mapping is the characteristic function, was also given by Pólya, but only for real valued maps on $\mathbb{R}$, cf. [8, Ch. IV, §4.3, Th. 4.3.1, p. 108].

A simple necessary condition, i.e. $\varphi(t)$ vanishes as $|t| = (t_1^2 + \ldots + t_d^2)^{1/2}$ tends to infinity, $t \in \mathbb{R}^d$, follows from Riemann-Lebesgue theorem, see [1, Ch. V, §23, Th. 23.2, p. 191], or [8, Ch. II, §2.2, (A), p. 35].

2. Notation and preliminaries. In this section we introduce the basic terminology and recall some useful facts for the future reference.

The points of d-dimensional Euclidean space $\mathbb{R}^d$ are denoted by single letters x, y, a, b, t etc., and their coordinates by the same letters with subscripts, so that the generic element $(x_1, \ldots, x_d) \in \mathbb{R}^d$ is written as x. The inequality $a < b$ for $a, b \in \mathbb{R}^d$ designates the relation $a_i < b_i$ for $1 \le i \le d$, and in such a case $\langle a, b)$ is the rectangle $\{x \in \mathbb{R}^d : a_i \le x_i < b_i$ for $1 \le i \le d\}$. Rectangles closed or opened from other sides are defined in an analogous manner. In the sequel the sets of such a kind are said to be d-dimensional intervals, or simply d-intervals. If $T \in \mathbb{R}, 0 < T < \infty$, then $\langle -T, T\rangle^d = \{x \in \mathbb{R}^d : -T \le x_i \le T$ for $1 \le i \le d\}$. More generally,

d-dimensional intervals having edges of equal length, like $\langle -T,T\rangle^d$, but not necesserily centered at zero, are called for short cubes. The boundary of the d-interval $\langle a,b\rangle$ is denoted by $\partial\langle a,b\rangle$. The usual scalar product in $\mathbb{R}^d$ is written by means of parentheses $[\cdot,\cdot]$, and the notation $\int(\cdot)dx$ signifies integration with respect to the Lebesgue measure λ_d in $\mathbb{R}^d$.

Suppose now that $\psi : X \to \mathbb{C}$ is a complex-valued measurable function on a measure space $(X,\mathcal{A},\mu)$ integrable with respect to μ on the set $A\in\mathcal{A}$. It can be easily seen that

$$(1) \qquad \left|\int_A \psi d\mu\right| \le \int_A |\psi|\, d\mu\ ,$$

– see e.g. [1, Ch. V, Th. 21.1, p. 179]. We shall use (1) mainly for Borel sets $A\subset\mathbb{R}^d$ and the Lebesgue measure. Also the inequalities of Schwarz and Hölder will be applied later on for various integrals, but we do not quote them here.

To simplify the writing, throughout the paper the symbol F denotes certain distribution function on $\mathbb{R}^d$ as well as the probability measure generated by F on $(\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d))$, where $\mathcal{B}(\cdot)$ stands for the Borel σ-field.

3. The results.

Our first goal is to prove a criterion for continuity of distribution functions expressed in terms of characteristic functions.

Theorem 1. *Let $\varphi:\mathbb{R}^d\to\mathbb{C}$ be a characteristic function corresponding to d-dimensional distribution function $F:\mathbb{R}^d\to\langle 0,1\rangle\subset\mathbb{R}$. If $|\varphi|\in L^p(\mathbb{R}^d)$ for some $1\le p<\infty$, i.e.*

$$\|\varphi\|^p_{L^p(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |\varphi(t)|^p\, dt < \infty\ ,$$

then F is continuous. Moreover, F satisfies the following condition of Lipschitz - Hölder type:

$$F[\langle a,b)] \le M_p\cdot\prod_{r=1}^{d}(b_r-a_r)^{1/p}\ , \quad \textit{for all}\ \ a,b\in\mathbb{R}^d,\ a<b\ ,$$

where $0<M_p<\infty$ is a constant.

Proof. Suppose first that $1<p<\infty$ and $|\varphi|\in L^p(\mathbb{R}^d)$. According to Lévy's theorem, for an arbitrary d-interval $\langle a,b)\subset\mathbb{R}^d$ with $F[\partial\langle a,b)]=0$, we have

$$F[\langle a,b)] = \lim_{T\to\infty}(2\pi)^{-d}\int_{<-T,T>^d}\left[\prod_{r=1}^{d}\frac{\exp\{-it_ra_r\}-\exp\{-it_rb_r\}}{it_r}\right]\varphi(t)dt\ .$$

We will show that $F[\langle a,b\rangle]$ can be arbitrarily small as $\lambda_d(\langle a,b\rangle)$ is sufficiently close to zero. Choose a real number $1<q<\infty$ in such a way that $1/p+1/q=1$. By (1) and Hölder's inequality we have

(2)
$$\begin{aligned} &F[\langle a,b\rangle]\\ &\le \lim_{T\to\infty}(2\pi)^{-d}\int_{<-T,T>^d}\left|\prod_{r=1}^{d}\frac{\exp\{-it_ra_r\}-\exp\{-it_rb_r\}}{it_r}\right||\varphi(t)|\,dt\\ &\le \lim_{T\to\infty}(2\pi)^{-d}\left\{\int_{<-T,T>^d}\left|\prod_{r=1}^{d}\frac{\exp\{-it_ra_r\}-\exp\{-it_rb_r\}}{it_r}\right|^q dt\right\}^{1/q}\\ &\qquad\times\lim_{T\to\infty}\left\{\int_{<-T,T>^d}|\varphi(t)|^p\,dt\right\}^{1/p}. \end{aligned}$$

Observe next that

(3)
$$\begin{aligned} &\left|\frac{\exp\{-it_ra_r\}-\exp\{-it_rb_r\}}{it_r}\right|\\ &=|\exp\{-it_r(a_r+b_r)/2\}|\cdot\left|\frac{\exp\{-it_r(a_r-b_r)/2\}-\exp\{-it_r(b_r-a_r)/2\}}{it_r}\right|\\ &=\left|\frac{2\sin[t_r(b_r-a_r)/2]}{t_r}\right|\le\begin{cases}|b_r-a_r| & \text{for } |t_r|\le 2/|b_r-a_r|\,,\\ 2/|t_r| & \text{for } |t_r|>2/|b_r-a_r|\,,\end{cases} \end{aligned}$$

and therefore

(4)
$$\begin{aligned} &\int_{-\infty}^{\infty}\left|\frac{\exp\{-it_ra_r\}-\exp\{-it_rb_r\}}{it_r}\right|^q dt_r\\ &\le 2\int_0^{2/|b_r-a_r|}|b_r-a_r|^q\,dt_r+2\int_{2/|b_r-a_r|}^{\infty}2^q/t_r^q\,dt_r\\ &=4\,|b_r-a_r|^{q-1}+4\,|b_r-a_r|^{q-1}/(q-1)=4p\,|b_r-a_r|^{q/p}. \end{aligned}$$

Applying now the well-known Fubini theorem and (4) we conclude that the right-hand side of (2) is bounded by

$$(2\pi)^{-d}\left\{\prod_{r=1}^{d}\int_{\mathbb{R}}\left|\frac{\exp\{-it_ra_r\}-\exp\{-it_rb_r\}}{it_r}\right|^q dt_r\right\}^{1/q}\left\{\int_{\mathbb{R}^d}|\varphi(t)|^p\,dt\right\}^{1/p}$$

$$\leq (2\pi)^{-d} \left\{ \prod_{r=1}^{d} 4p\, |b_r - a_r|^{q/p} \right\}^{1/q} \|\varphi\|_{L^p(\mathbb{R}^d)}$$

$$= (2\pi)^{-d} \{4p\}^{d/q} \|\varphi\|_{L^p(\mathbb{R}^d)} \prod_{r=1}^{d} (b_r - a_r)^{1/p} .$$

Hence it follows that $F[\langle a, b)] \to 0$ as $\lambda_d[\langle a, b)] = \prod_{r=1}^{d} (b_r - a_r) \to 0$. If $p = 1$, then on the basis of (3) and the inequality

$$\left| \frac{2 \sin[t_r(b_r - a_r)/2]}{t_r} \right| \leq |b_r - a_r|$$

we obtain

$$\text{(5)} \qquad F[\langle a,b)] \leq (2\pi)^{-d} \prod_{r=1}^{d} |b_r - a_r| \lim_{T\to\infty} \int_{<-T,T>^d} |\varphi(t)|\, dt$$

$$= (2\pi)^{-d} \|\varphi\|_{L^1(\mathbb{R}^d)} \prod_{r=1}^{d} (b_r - a_r) .$$

In view of the lower-left continuity of each distribution function F in $\mathbb{R}^d$, the above inequalities are valid for all $a, b \in \mathbb{R}^d$, $a < b$.

The continuity of F is a straightforward consequence of the obtained estimates when $d = 1$. To prove the continuity of F in $\mathbb{R}^d$ for $d > 1$, observe that the F-measure of each hyperplane $H \subset \mathbb{R}^d$ parallel to some axes of the system of coordinates is equal to zero. Indeed, let $H_r = \{x \in \mathbb{R}^d : x_r = a_r\}$. Consider the sequence of d-intervals $\langle a^{(n)}, b^{(n)}\rangle$, $n \geq 1$, such that $a_r^{(n)} = a_r$, $a_r < b_r^{(n)} \searrow a_r$, $a_j^{(n)} \searrow -\infty$ and $b_j^{(n)} \nearrow \infty$ for $1 \leq j \leq d$, $j \neq r$, in such a way that $\lambda_d\left(\langle a^{(n)}, b^{(n)}\rangle\right) \to 0$ as $n \to \infty$. Then $H_r \cap \langle a^{(n)}, b^{(n)}\rangle$, $n \geq 1$, forms an increasing sequence of sets, and so

$$F[H_r] = F[\bigcup_n (H_r \cap \left\langle a^{(n)}, b^{(n)}\right\rangle)] = \lim_n F[H_r \cap \left\langle a^{(n)}, b^{(n)}\right\rangle] .$$

However, on account of the above considerations

$$F[H_r \cap \left\langle a^{(n)}, b^{(n)}\right\rangle] \leq M_p \cdot \lambda_d[\left\langle a^{(n+m)}, b^{(n+m)}\right\rangle]^{1/p}$$

for an arbitrary $m \geq 1$, and thus $F[H_r \cap \langle a^{(n)}, b^{(n)}\rangle] = 0$ for all $n \geq 1$. Consequently, $F[H_r] = 0$ for $r = 1, 2, ..., d$.

Let now $a, y^{(n)} \in \mathbb{R}^d$ and $y^{(n)} \to a$. Then $\left|F(a) - F(y^{(n)})\right|$ does not exceed the F-measure of the closure of symmetric difference $(-\infty, a) \div (-\infty, y^{(n)})$ formed by means of infinite d-dimensional intervals $(-\infty, a) = \{x \in \mathbb{R}^d : x < a\}$, and similarly $(-\infty, y^{(n)}) = \{x \in \mathbb{R}^d : x < y^{(n)}\}$. Clearly, $\operatorname{cl}[(-\infty, a) \div (-\infty, y^{(n)})]$ is contained in the finite sum $\bigcup_{r=1}^d A_r^{(n)}$ of sets

$$A_r^{(n)} = \{x \in \mathbb{R}^d : \min(a_r, y_r^{(n)}, y_r^{(n+1)}, \ldots) \le x_r \le \max(a_r, y_r^{(n)}, y_r^{(n+1)}, \ldots)\} .$$

If $y^{(n)} \to a$, then $A_r^{(n)} \searrow H_r$, and so $F[A_r^{(n)}]$ can be arbitrarily close to zero for sufficiently large n. Therefore

$$\left|F(a) - F(y^{(n)})\right| \le \sum_{r=1}^{d} F[A_r^{(n)}] \to 0 \quad \text{as} \quad y^{(n)} \to a . \qquad \square$$

Remark. If $d = 1$, then

$$\lim_{T\to\infty} \frac{1}{2T} \int_{\langle -T,T\rangle} |\varphi(t)|^2 \, dt$$

is the sum of all the jumps of F, thus the condition $|\varphi(t)| \to 0$ as $|t| \to \infty$ implies that F is continuous, cf. [8, Ch. III, §3.3, Th. 3.3.4, p. 60].

Basing on this observation we can obtain a weaker sufficient condition for continuity of distribution functions F in $\mathbb{R}^d$ expressed in terms of their characteristic functions φ. Denote $\varphi(0; t_j) = \varphi(0, \ldots, 0, t_j, 0, \ldots, 0)$. Since $\varphi(0; t_j)$, $1 \le j \le d$, are characteristic functions of 1-dimensional marginal distributions, the condition

$$\forall_{1\le j\le d} \quad |\varphi(0; t_j)| \to 0 \quad \text{as} \quad |t_j| \to \infty$$

implies that the distribution function F in $\mathbb{R}^d$ corresponding to φ has no $(d-1)$-dimensional hyperplanes of discontinuity. Therefore F is continuous.

Using the above condition, one can derive another proof of continuity of F under the assumptions of Theorem 1. *A contrario*, suppose that there exist $1 \le j \le d$, a number $\delta > 0$ and a sequence $\{t_j^{(n)}\}$, $\left|t_j^{(n)}\right| \nearrow \infty$, such that $\left|t_j^{(n)} - t_j^{(n+k)}\right| \ge \theta > 0$, $n, k \ge 1$, and

$$\left|\varphi(0; t_j^{(n)})\right| \ge \delta > 0 \quad \text{for all} \quad n \ge 1 .$$

Since φ is uniformly continuous, a neighbourhood of zero can be found in $\mathbb{R}^d$, say $K(0,\tau) = \{t \in \mathbb{R}^d : |t| = (t_1^2 + ... + t_d^2)^{1/2} < \tau\}$, $\tau \leq \theta/2$, such that

$$|\varphi(t)| \geq \delta/2 > 0 \quad \text{for all} \quad t \in K\left((0; t_j^{(n)}), \tau\right) .$$

Then $K\left((0; t_j^{(n)}), \tau\right)$, $n \geq 1$, are disjoint and

$$\int_{\mathbb{R}^d} |\varphi(t)|^p \, dt \geq (\delta/2)^p \sum_n \lambda_d \left[K\left((0; t_j^{(n)}), \tau\right)\right] = \infty ,$$

which leads to a contradiction. Hence it follows that F is continuous. It is interesting that the behaviour of φ on the axes of the system of coordinates in $\mathbb{R}^d$ decides in advance about the continuity of F.

Theorem 2. *Let φ and F be as in* Theorem 1. *If $|\varphi| \in L^p(\mathbb{R}^d)$ for some $1 \leq p \leq 2$, then F is absolutely continuous with respect to the Lebesgue measure λ_d in $\mathbb{R}^d$, and has density $f \in L^2(\mathbb{R}^d)$. Moreover, the density f of F is given by the formula*

$$f(x) = \lim_{c_\epsilon \to 0} \left\{ (2\pi)^{-d} \prod_{r=1}^{d} (b_{r,x}^{(\epsilon)} - a_{r,x}^{(\epsilon)})^{-1} \right. \tag{6}$$

$$\left. \times \lim_{T \to \infty} \int_{\langle -T,T\rangle^d} \prod_{r=1}^{d} \left[\frac{\exp\{-it_r a_{r,x}^{(\epsilon)}\} - \exp\{-it_r b_{r,x}^{(\epsilon)}\}}{it_r} \right] \cdot \varphi(t) dt \right\}$$

for a.a. $x \in \mathbb{R}^d$, where $x \in \left\langle a_{1,x}^{(\epsilon)}, b_{1,x}^{(\epsilon)} \right\rangle \times ... \times \left\langle a_{d,x}^{(\epsilon)}, b_{d,x}^{(\epsilon)} \right\rangle$ and $0 < b_{r,x}^{(\epsilon)} - a_{r,x}^{(\epsilon)} = c_\epsilon \to 0$ uniformly in $1 \leq r \leq d$.

Proof. Obviously, if $1 \leq p < 2$, then we have $|\varphi(t)|^2 \leq |\varphi(t)|^p$, $t \in \mathbb{R}^d$. Therefore there is no loss of generality if we assume that $|\varphi| \in L^2(\mathbb{R}^d)$. It is clear that the boundary $\partial \langle a, b\rangle$ of any d-interval $\langle a, b\rangle \subset \mathbb{R}^d$ is contained in the sum of hyperplanes defined by its edges,

$$\partial \langle a, b\rangle \subset \{x \in \mathbb{R}^d : (x_1 = a_1) \vee (x_1 = b_1) \vee ... \vee (x_d = a_d) \vee (x_d = b_d)\} ,$$

thus $F[x \in \mathbb{R}^d : x_i = a_i] = 0 = F[x \in \mathbb{R}^d : x_i = b_i]$ for $1 \leq i \leq d$ implies that $F[\partial \langle a, b\rangle] = 0$.

Choose maximal sets $D_i \subset \mathbb{R}$ in such a way that $s \in D_i$ iff $-s \in D_i$ and $F[x \in \mathbb{R}^d : x_i \in D_i] = 0$, $1 \leq i \leq d$, and next put $\mathcal{D} = D_1 \times ... \times D_d$. Since each distribution function has at most countably many parallel hyperplanes of discontinuity, the sets D_i are dense in $\mathbb{R}$ and in consequence $\mathcal{D}$ is dense in $\mathbb{R}^d$. Furthermore, for all $a, b \in \mathcal{D}$, $a < b$, we have $F[\partial \langle a, b\rangle] = 0$.

Denote by $\mathcal{S}$ the class of functions

$$v = \sum_{j=1}^{n} c_j \mathcal{J}_{<a^{(j)}, b^{(j)})} ,$$

where c_j are arbitrary real numbers and $\langle a^{(j)}, b^{(j)}) \subset \mathbb{R}^d$ are disjoint d-intervals with endpoints $a^{(j)}, b^{(j)} \in \mathcal{D}$, $a^{(j)} < b^{(j)}$, $1 \le j \le n$. Notice that $\mathcal{S}$ is a linear space. Using the Fourier transform Ψ_v of v,

$$\Psi_v(t) = \int_{\mathbb{R}^d} \exp\{i\,[t,x]\} v(x) dx ,$$

define the linear functional L on $\mathcal{S}$ by the formula

$$Lv = (2\pi)^{-d} \lim_{T\to\infty} \int_{<-T,T>^d} \Psi_v(t)\varphi(t) dt .$$

It can be easily seen that

$$\int_{\mathbb{R}^d} \exp\{i\,[t,x]\} \mathcal{J}_{<a^{(j)},b^{(j)})}(x) dx = \prod_{r=1}^{d} \int_{a_r^{(j)}}^{b_r^{(j)}} \exp\{it_r x_r\} dx_r$$

$$= \prod_{r=1}^{d} \frac{\exp\{it_r b_r^{(j)}\} - \exp\{it_r a_r^{(j)}\}}{it_r} ,$$

and thus

$$\Psi_v(t) = \sum_{j=1}^{n} c_j \prod_{r=1}^{d} \frac{\exp\{it_r b_r^{(j)}\} - \exp\{it_r a_r^{(j)}\}}{it_r} .$$

Note next that, by analogy to (3),

$$\left| \frac{\exp\{it_r b_r^{(j)}\} - \exp\{it_r a_r^{(j)}\}}{it_r} \right|^2$$

$$= \left| \frac{1}{t_r} \left[2 \sin \left(t_r \frac{b_r^{(j)} - a_r^{(j)}}{2} \right) \right] \right|^2 \le \begin{cases} (b_r^{(j)} - a_r^{(j)})^2 & \text{for } |t_r| \le 1 , \\ 4/t_r^2 & \text{for } |t_r| > 1 , \end{cases}$$

and in addition,

$$|\Psi_v(t)|^2 \le \sum_{j=1}^{n} c_j^2 \prod_{r=1}^{d} \left| \frac{\exp\{it_r b_r^{(j)}\} - \exp\{it_r a_r^{(j)}\}}{it_r} \right|^2$$

$$+\sum_{j\neq k}|c_jc_k|\prod_{r=1}^{d}\left|\frac{\exp\{it_rb_r^{(j)}\}-\exp\{it_ra_r^{(j)}\}}{it_r}\right|\left|\frac{\exp\{it_rb_r^{(k)}\}-\exp\{it_ra_r^{(k)}\}}{it_r}\right|$$

$$\leq\sum_{j=1}^{n}c_j^2\prod_{r=1}^{d}\left[\frac{2}{t_r}\sin\left(t_r\frac{b_r^{(j)}-a_r^{(j)}}{2}\right)\right]^2$$

$$+\frac{1}{2}\sum_{j\neq k}\left\{c_j^2\prod_{r=1}^{d}\left[\frac{2}{t_r}\sin\left(t_r\frac{b_r^{(j)}-a_r^{(j)}}{2}\right)\right]^2+c_k^2\prod_{r=1}^{d}\left[\frac{2}{t_r}\sin\left(t_r\frac{b_r^{(k)}-a_r^{(k)}}{2}\right)\right]^2\right\}$$

$$=n\sum_{j=1}^{n}c_j^2\prod_{r=1}^{d}\left[\frac{2}{t_r}\sin\left(t_r\frac{b_r^{(j)}-a_r^{(j)}}{2}\right)\right]^2,$$

therefore $|\Psi_v(t)|^2$ is integrable over the whole space $\mathbb{R}^d$ with respect to the Lebesgue measure. Since $\int_{\mathbb{R}^d}|\varphi(t)|^2\,dt<\infty$, from Hölder's inequality we infer that

$$|Lv|\leq(2\pi)^{-d}\left\{\int_{\mathbb{R}^d}|\Psi_v(t)|^2\,dt\right\}^{1/2}\|\varphi\|_{L^2(R^d)}<\infty,$$

and so the functional L is well-defined for all $v\in\mathcal{S}$. Moreover, if $v=\sum_{j=1}^{n}c_j\mathcal{J}_{<a^{(j)},b^{(j)})}$, where the d-intervals $\langle a^{(j)},b^{(j)}\rangle\subset\mathbb{R}^d$ are disjoint and their endpoints $a^{(j)},b^{(j)}\in\mathcal{D}$, then

$$(2\pi)^{2d}|Lv|^2=\left|\lim_{T\to\infty}\int_{<-T,T>^d}\sum_{j=1}^{n}c_j\prod_{r=1}^{d}\frac{\exp\{it_rb_r^{(j)}\}-\exp\{it_ra_r^{(j)}\}}{it_r}\varphi(t)dt\right|^2$$

$$\leq\lim_{T\to\infty}\int_{<-T,T>^d}\left|\sum_{j=1}^{n}c_j\prod_{r=1}^{d}\frac{\exp\{it_rb_r^{(j)}\}-\exp\{it_ra_r^{(j)}\}}{it_r}\right|^2dt\cdot\int_{R^d}|\varphi(t)|^2\,dt$$

$$=\lim_{T\to\infty}\int_{<-T,T>^d}\sum_{j=1}^{n}\sum_{k=1}^{n}c_jc_k\prod_{r=1}^{d}\frac{\exp\{it_rb_r^{(j)}\}-\exp\{it_ra_r^{(j)}\}}{it_r}$$

$$\times\prod_{r=1}^{d}\frac{\exp\{-it_rb_r^{(k)}\}-\exp\{-it_ra_r^{(k)}\}}{-it_r}dt\cdot\|\varphi\|^2_{L^2(\mathbb{R}^d)}.$$

To evaluate the above limiting expression we can use Lévy's formula: for any continuity points $a<b$ of distribution function G in $\mathbb{R}^d$ (such that $G[\partial\langle a,b\rangle]=0$) with characteristic function $\gamma(t)$, we have

$$G[\langle a,b\rangle]=\lim_{T\to\infty}(2\pi)^{-d}\int_{<-T,T>^d}\prod_{r=1}^{d}\left[\frac{\exp\{-it_ra_r\}-\exp\{-it_rb_r\}}{it_r}\right]\cdot\gamma(t)dt.$$

Suppose

$$\gamma(t)=\prod_{r=1}^{d}\frac{\exp\{it_rb_r^{(j)}\}-\exp\{it_ra_r^{(j)}\}}{it_r(b_r^{(j)}-a_r^{(j)})}$$

is the characteristic function of the uniform distribution G concentrated on $\langle a^{(j)},b^{(j)})$ in $\mathbb{R}^d$ with density $g=\mathcal{J}_{<a^{(j)},b^{(j)})}\prod_{r=1}^{d}(b_r^{(j)}-a_r^{(j)})^{-1}$. Since $G(x)=\int_{\{y\in\mathbb{R}^d:y<x\}}g(y)dy$ is then a continuous function of x in the whole domain $\mathbb{R}^d$, we conclude that

$$\lim_{T\to\infty}(2\pi)^{-d}\Big\{\int_{<-T,T>^d}\prod_{r=1}^{d}\frac{\exp\{-it_ra_r^{(k)}\}-\exp\{-it_rb_r^{(k)}\}}{it_r}$$

$$\times\prod_{r=1}^{d}\frac{\exp\{it_rb_r^{(j)}\}-\exp\{it_ra_r^{(j)}\}}{it_r(b_r^{(j)}-a_r^{(j)})}dt\cdot\prod_{r=1}^{d}(b_r^{(j)}-a_r^{(j)})\Big\}$$

$$=\int_{<a^{(k)},b^{(k)})}\mathcal{J}_{<a^{(j)},b^{(j)})}(x)dx=\begin{cases}0 & \text{if } k\neq j,\\ \prod_{r=1}^{d}(b_r^{(j)}-a_r^{(j)}) & \text{if } k=j,\end{cases}$$

(recall that $\langle a^{(j)},b^{(j)})$, $1\le j\le n$, are here disjoint). Therefore

$$\lim_{T\to\infty}\int_{<-T,T>^d}\left|\sum_{j=1}^{n}c_j\prod_{r=1}^{d}\frac{\exp\{-it_ra_r^{(j)}\}-\exp\{-it_rb_r^{(j)}\}}{it_r}\right|^2dt \tag{7}$$

$$=\Big\{\lim_{T\to\infty}\int_{<-T,T>^d}\sum_{k=1}^{n}\sum_{j=1}^{n}c_kc_j\Big[\prod_{r=1}^{d}\frac{\exp\{-it_ra_r^{(k)}\}-\exp\{-it_rb_r^{(k)}\}}{it_r}$$

$$\times\prod_{r=1}^{d}\frac{\exp\{it_ra_r^{(j)}\}-\exp\{it_rb_r^{(j)}\}}{-it_r}\Big]dt\Big\}=(2\pi)^d\sum_{j=1}^{n}c_j^2\prod_{r=1}^{d}(b_r^{(j)}-a_r^{(j)})\,,$$

and so

$$|Lv|^2\le(2\pi)^{-d}\left\{\sum_{j=1}^{n}c_j^2\prod_{r=1}^{d}(b_r^{(j)}-a_r^{(j)})\right\}\|\varphi\|^2_{L^2(\mathbb{R}^d)}\,.$$

On the other hand,

$$\int_{\mathbb{R}^d}(v(x))^2dx=\int_{\mathbb{R}^d}\sum_{j=1}^{n}\sum_{k=1}^{n}c_jc_k\mathcal{J}_{<a^{(j)},b^{(j)})}(x)\mathcal{J}_{<a^{(k)},b^{(k)})}(x)dx$$

$$=\sum_{j=1}^{n}c_j^2\prod_{r=1}^{d}(b_r^{(j)}-a_r^{(j)})=\|v\|^2_{L^2(\mathbb{R}^d)}\,.$$

Consequently,

$$|Lv|^2 \le (2\pi)^{-d} \|v\|^2_{L^2(\mathbb{R}^d)} \|\varphi\|^2_{L^2(\mathbb{R}^d)} = C_{\varphi,d} \|v\|^2_{L^2(\mathbb{R}^d)} ,$$

where $0 < C_{\varphi,d} < \infty$ is a constant. Hence it follows that $L : \mathcal{S} \to \mathbb{R}$ is a continuous linear functional on $\mathcal{S} \subset L^2(\mathbb{R}^d)$ (taking only real values in view of Lévy's formula). Since $\mathcal{D}$ is a dense set in $\mathbb{R}^d$, the linear space $\mathcal{S}$ is dense in $L^2(\mathbb{R}^d)$, and thus by continuity L can be extended uniquely into the whole Hilbert space $L^2(\mathbb{R}^d)$. Basing on Riesz's representation theorem for continuous linear functionals in a Hilbert space, we conclude that there exists a real function $f \in L^2(\mathbb{R}^d)$ such that

$$Lv = \int_{R^d} v(x) f(-x) dx \quad \text{for all} \quad v \in L^2(\mathbb{R}^d) . \tag{8}$$

According to definition of L, if $v = \mathcal{J}_{<-b,-a)}$ for some points $a, b \in \mathcal{D}$, $a < b$, then on account of Lévy's theorem

$$\begin{aligned} Lv &= (2\pi)^{-d} \lim_{T\to\infty} \int_{<-T,T>^d} \prod_{r=1}^{d} \frac{\exp\{-it_r a_r\} - \exp\{-it_r b_r\}}{it_r} \cdot \varphi(t) dt \\ &= F[\langle a, b)] , \end{aligned}$$

while in view of (8),

$$\begin{aligned} Lv &= \int_{\mathbb{R}^d} v(x) f(-x) dx = \int_{\mathbb{R}^d} \mathcal{J}_{<-b,-a)}(x) f(-x) dx \\ &= \int_{<-b,-a)} f(-x) dx = \int_{(a,b>} f(u) du , \end{aligned}$$

i.e.

$$F[\langle a, b)] = \int_{(a,b>} f(u) du , \quad a, b \in \mathcal{D} , \ a < b .$$

However, $\mathcal{D} \subset \mathbb{R}^d$ is dense in $\mathbb{R}^d$, thus by continuity of the distribution function F on $\mathbb{R}^d$ (cf. Theorem 1) we conclude that the equality

$$F[\langle a, b\rangle] = \int_{<a,b>} f(u) du$$

is valid for all $a, b \in \mathbb{R}^d$, $a < b$. Hence it follows easily that F is absolutely continuous with respect to the Lebesgue measure λ_d in $\mathbb{R}^d$. Furthermore, basing on the properties of absolutely continuous additive set functions of d-intervals in $\mathbb{R}^d$, see e.g. [9, Th. 4.7, Ch. X, 4, p. 399], we infer that

$$F[\langle a, b)] = \int_{<a,b>} F'(x) dx ,$$

where

$$(9) \qquad F'(x) = \lim_{\lambda_d(K_{x,\epsilon})\to 0} \frac{F(K_{x,\epsilon})}{\lambda_d(K_{x,\epsilon})} = f(x) \text{ for } \lambda_d - a.a.\ x \in \mathbb{R}^d,$$

and the limit is determined by a sequence $\{K_{x,\epsilon}\}$ of d-dimensional closed cubes containing x. In other words, the density f of F is given by (6) and satisfies the condition $f \in L^2(\mathbb{R}^d)$. □

The dichotomy between the case $p = 1$ and $1 < p \le 2$ becomes more apparent in view of the below result. Although the mentioned statement is rather known, we include it here for convenience of the reader.

Corollary. *If $|\varphi| \in L^1(\mathbb{R}^d)$, i.e. $\int_{\mathbb{R}^d} |\varphi(t)|\,dt < \infty$, then F is absolutely continuous with respect to the Lebesgue measure λ_d in $\mathbb{R}^d$ and possesses bounded, uniformly continuous density $f_1 \in L^2(\mathbb{R}^d)$ given by*

$$f_1(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \exp\{-i\,[t,x]\} \cdot \varphi(t)dt \quad \text{for all} \quad x \in \mathbb{R}^d.$$

Furthermore, the derivative in (6) or (9) is equal to $f_1(x)$ for every $x \in \mathbb{R}^d$, and we have

$$(10) \qquad \|f_1\|_\infty = \sup_{x\in\mathbb{R}^d} |f_1(x)| \le (2\pi)^{-d} \cdot \|\varphi\|_{L^1(\mathbb{R}^d)} \ .$$

Proof. The existence of density follows directly from Theorem 1, because for an arbitrary elementary figure $E = \bigcup_{k=1}^{s} \langle a^{(k)}, b^{(k)} \rangle$ in $\mathbb{R}^d$ consisting of a finite number of non-overlapping d-intervals we have

$$F[E] \le M_p \cdot \sum_{k=1}^{s} \prod_{r=1}^{d} \left(b_r^{(k)} - a_r^{(k)} \right) = M_p \cdot \lambda_d(E)\,,$$

and so F is absolutely continuous with respect to the Lebesgue measure – cf. [9, Ch. X, §4, Th. 4.7, p. 399]. The same conclusion is also an easy consequence of Theorem 2. In fact, $|\varphi(t)|^2 \le |\varphi(t)|$ for all $t \in \mathbb{R}^d$, whence on account of our Theorem 2 we infer that F has density $f \in L^2(\mathbb{R}^d)$ defined by (6) with respect to the Lebesgue measure λ_d in $\mathbb{R}^d$.

Let $K_{x,n} = \langle x - 1/n, x + 1/n \rangle \subset \mathbb{R}^d$, $x \pm 1/n = (x_1 \pm 1/n, ..., x_d \pm 1/n)$. Then

$$F(K_{x,n})$$

$$= \lim_{T\to\infty} (2\pi)^{-d} \int_{<-T,T>^d} \prod_{r=1}^{d} \frac{\exp\{-it_r(x_r - \frac{1}{n})\} - \exp\{-it_r(x_r + \frac{1}{n})\}}{it_r} \cdot \varphi(t)dt$$

$$= \lim_{T\to\infty} (2\pi)^{-d} \int_{<-T,T>^d} \prod_{r=1}^{d} \left[\exp\{-it_r x_r\} \frac{2\sin(t_r/n)}{t_r}\right] \cdot \varphi(t) dt \, .$$

Since

$$\left| \prod_{r=1}^{d} \exp\{-it_r x_r\} \frac{2\sin(t_r/n)}{t_r} \right| \le \left(\frac{2}{n}\right)^d$$

and $|\varphi(t)|$ is integrable on $\mathbb{R}^d$, we conclude that the last limit is equal to

$$(2\pi)^{-d} \int_{R^d} \prod_{r=1}^{d} \left[\exp\{-it_r x_r\} \frac{2\sin(t_r/n)}{t_r}\right] \cdot \varphi(t) dt \, .$$

Moreover, $\lambda_d(K_{x,n}) = (2/n)^d$, thus the same estimate as above and the Lebesgue theorem on dominated convergence imply that

$$f(x) = \lim_{n\to 0} \frac{F(K_{x,n})}{\lambda_d(K_{x,n})}$$

$$= \lim_{n\to 0} (2\pi)^{-d}(2/n)^{-d} \int_{R^d} \prod_{r=1}^{d} \left[\exp\{-it_r x_r\} \frac{2\sin(t_r/n)}{t_r}\right] \cdot \varphi(t) dt$$

$$= (2\pi)^{-d} \int_{R^d} \prod_{r=1}^{d} \left[\exp\{-it_r x_r\} \lim_{n\to 0} \frac{2\sin(t_r/n)}{2t_r/n}\right] \cdot \varphi(t) dt$$

$$= (2\pi)^{-d} \int_{R^d} \prod_{r=1}^{d} [\exp\{-it_r x_r\}] \cdot \varphi(t) dt \quad \text{for} \quad \lambda_d - a.a.\ x \in \mathbb{R}^d .$$

Denote the last formula by $f_1(x)$ and observe that it is a continuous function of $x \in \mathbb{R}^d$. Indeed, taking $h = (h_1, ..., h_d) \in \mathbb{R}^d$ we obtain

$$|f_1(x+h) - f_1(x)|$$

$$\le (2\pi)^{-d} \int_{\langle -T,T\rangle^d} \left| \prod_{r=1}^{d} \exp\{-it_r(x_r + h_r)\} - \prod_{r=1}^{d} \exp\{-it_r x_r\} \right| \cdot |\varphi(t)| \, dt$$

$$+(2\pi)^{-d} \int_{\mathbb{R}^d \setminus \langle -T,T\rangle^d} \left| \prod_{r=1}^{d} \exp\{-it_r(x_r + h_r)\} - \prod_{r=1}^{d} \exp\{-it_r x_r\} \right| \cdot |\varphi(t)| \, dt \, .$$

Given any $\varepsilon > 0$, one can choose $0 < T < \infty$ so large that the second term is less than $\varepsilon/2$, and next select h sufficiently close to the origin, $|h| = (h_1^2 + ... + h_d^2)^{1/2} < \delta = \delta(\varepsilon)$, in such a way that

$$\left|\prod_{r=1}^{d}\exp\{-it_r(x_r+h_r)\}-\prod_{r=1}^{d}\exp\{-it_rx_r\}\right|<\frac{\varepsilon}{2}\cdot\frac{(2\pi)^d}{\|\varphi\|_{L^1(\mathbb{R}^d)}}$$

uniformly in $t \in \langle -T,T\rangle^d$ and $x \in \mathbb{R}^d$. The above argument then shows that

$$|f_1(x+h)-f_1(x)|<\varepsilon \quad \text{whenever} \quad |h|<\delta\,,$$

i.e. f_1 is uniformly continuous on $\mathbb{R}^d$. Furthermore, $f(x)=f_1(x)$ $\lambda_d-a.e.$ on $\mathbb{R}^d$, thus $f_1 \in L^2(\mathbb{R}^d)$ and

$$F(A)=\int_A f_1(x)dx\,,\quad A\in\mathcal{B}(\mathbb{R}^d)\,.$$

Hence, on the basis of the well-known theorem concerning differentiability of integrals of continuous functions (see e.g. [9, Th. 4.10, Ch. X, § 4, p. 400]) it follows that

$$\lim_{n\to 0}\frac{F(K_{x,n})}{\lambda_d(K_{x,n})}=f_1(x)=\lim_{\lambda_d(K_{x,\epsilon})\to 0}\frac{F(K_{x,\epsilon})}{\lambda_d(K_{x,\epsilon})}\quad\text{for all}\quad x\in R^d.$$

The boundedness of f_1 and (10) follows from (5). □

Remark. The upper bound in (10) cannot be improved, because it is attained for the standard normal distribution function in $\mathbb{R}^d$.

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