ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN-POLONIA

VOL. LI.1, 18 SECTIO A 1997

JERZY ZABCZYK (Warszawa)

Stopping Problems on Polish Spaces[1]

Dedicated to Professor Dominik Szynal
on the occasion of his 60th birthday

Abstract. The paper is concerned with optimal stopping of Markov processes on Polish spaces. It extends results known for locally compact state spaces. Conditions are given under which the value function is continuous and the first moment of hitting the contact set is an optimal one. Our study is motivated by a recent aplication, due to D. Gatarek and M. Musiela, of an infinite dimensional theory, to finance.

1. Introduction. Let E be a metric, complete and separable space - called often *Polish* - and $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space equipped with an increasing family of σ-fields $\mathcal{F}_t \subset \mathcal{F}, t \geq 0$. Assume that $X(t), t \geq 0$ is a Markov process adapted to $(\mathcal{F}_t)$, having values in E and let α, φ and ψ be real functions on E. For arbitrary $(\mathcal{F}_t)$ stopping time τ and $s \in [0, +\infty)$ define functionals:

$$(1) \qquad \mathcal{J}_s(\tau, X(0)) = \mathbb{E}\Big(e^{\int_0^\tau \alpha(X(\sigma))d\sigma} \Big[\varphi(X(\tau))\chi_{\tau<s} + \psi(X(\tau))\chi_{\tau=s} \Big] \Big).$$

[1] Research supported by KBN Grant No. 2PO3A 082 08.

The paper is concerned with conditions implying existence of stopping times $\hat{\tau}$, such that,

$$\mathcal{J}_s(\tau, X(0)) \leq \mathcal{J}_s(\hat{\tau}, X(0)) = V(s, X(0)),$$

for all stopping times τ. The main emphasis is placed on various characterizations of the corresponding value function V and on showing that the hitting time, of the so called *contact* set, is an optimal one. The paper is devoted to the finite interval case $s < +\infty$, and the infinite horizon problems with the functional:

$$\mathcal{J}_\infty(\tau, X(0)) = \mathbb{E}\left(e^{\int_0^\tau \alpha(X(\sigma))d\sigma}\varphi(X(\tau))\chi_{\tau<+\infty}\right), \tag{2}$$

will be treated in a subsequent paper.

Questions discussed in the present work paper have an enormous literature see e.g. [11], [17], [18], [23], when the state space E is compact or locally compact. In the majority of papers one assumes that the transition semigroup (P_t) is strongly continuous on the space $C_0(E)$, of all bounded, continuous functions vanishing at the compactification point ∂. Thus one requires that

i) $P_t\Big(C_0(E)\Big) \subset C_0(E), t \geq 0,$

ii) for all $\psi \in C_0(E)$, $P_t(\psi) \to \psi$ as $t \to 0$ uniformly.

(Neither of the conditions i) - ii) hold for examples of interest when E is not locally compact. There exist only few papers dealing with optimal stopping of Markov processes on more general spaces. Optimal stopping of a filtering process, with values in the space of probability measures, has been discussed in [19] and [14]. The case of an infinite dimensional Ornstein-Uhlenbeck process is treated in a recent paper [8]. Stopping of general Markov processes on Polish spaces has been studied in [1, Chapter VII, § 5.2.], under the condition that the corresponding transition semigroup is strongly continuous on the space of bounded, uniformly continuous functions on E. This condition is rarely satisfied even for locally compact spaces E.

In the present paper we replace the requirement i) - ii) by a *continuity condition* which holds in all examples of interest. We show that under the continuity condition the classical techniques of the time discretization due to Bensoussan and Lions [2], El Karoui [11], Nisio [16], Mackevičius [13], Robin [17], Shiryayev [18] and Stettner and Zabczyk [19], can be applied to general gain functionals (1), general state spaces and to general Markov processes. Our results contain, as a special case, those of [8].

The initial impulse for considering optimal stopping on Polish spaces was provided by the paper [8] in which an application of some infinite dimensional results to financial mathematics has been investigated. This specific application is discussed in some detail in section 2 and at the end of the paper.

The paper is based on the preprint [24]. It which also discusses connections with parabolic variational inequalities consists of the following sections:

1. Introduction, 2. Financial model, 3. Continuity condition, 4. Continuity of the value function, 5. Extremal properties of the value function, 6. Existence of optimal moments, 7. Extension to unbounded function and an application.

2. Financial model. As a motivation to our work we describe in this section a financial problem leading directly to optimal stopping of general Markov processes on Polish spaces.

Let $r^d(t), r^f(t), t \geq 0$ be respectively domestic and foreign interest rates. The exchange rate $e(t), t \geq 0$, or equivalently the cost of the foreign currency in terms of the domestic one, satisfies the following equation, see [8],

$$de(t) = e(t)\Big\{ \big(r^d(t) - r^f(t)\big)dt + \nu dW_0(t)\Big\}$$

$$e(0) = \text{ given initial exchange rate,}$$

with a *volatility* parameter ν. The value of the American put option on the foreign currency, for the period $s > 0$ and with the striking price K, is

$$\sup_{\tau \leq s} \mathbb{E}\left(e^{-\int_0^\tau r^d(\sigma)d\sigma}(K - e(\tau))^+\right). \tag{3}$$

In general the process, $y(t) = \left(r^d(t), r^f(t), e(t)\right)$, $t \geq 0$, is not Markovian and one can not evaluate the option knowing only $r^d(0)$, $r^f(0)$, $e(0)$. Following [10] and [15], one can lift the process $y(t)$, $t \geq 0$, to a Markovian one, replacing interest rate processes by the forward rate processes. Let $r^d(t,\xi)$, $t \geq 0$, $\xi \geq 0$ and $r^f(t,\xi)$, $t \geq 0$, $\xi \geq 0$ be the domestic and foreign forward rates functions at time $t \geq 0$. Then $r^d(t,0) = r^d(t)$ and $r^f(t,0) = r^f(t)$ and $r^d(t,\xi)$ and $r^f(t,\xi)$ are the domestic and foreign exchange rates at time $t+\xi$ as perceived by the market at time t. Equivalently,

$$e^{-\int_0^\xi r^d(t,\eta)d\eta},$$

is the value at time $t \geq 0$ of a zero-coupon bond maturing at time $t+\xi$. In the paper [8], Gątarek and Musiela assumed, following [15], that $r^d(t,\cdot)$

and $r^f(t,\cdot)$, $t \geq 0$ are Gaussian - Markov processes on $H = L^2(0,+\infty)$ of the form:

$$(4)\qquad \begin{aligned} dr^d(t,\xi) &= \left(\frac{\partial}{\partial \xi} r^d(t,\xi) + a^d(\xi)\right)dt + \langle \sigma^d(\xi), dW_1(t)\rangle \\ dr^f(t,\xi) &= \left(\frac{\partial}{\partial \xi} r^f(t,\xi) + a^f(\xi)\right)dt + \langle \sigma^f(\xi), dW_2(t)\rangle, \end{aligned}$$

where $W_1(t), W_2(t)$, $t \geq 0$ are finite dimensional Wiener processes and,

$$a^d(\xi) = \langle \sigma^d(\xi), \int_0^\xi \sigma^d(\eta)d\eta\rangle,$$
$$a^f(\xi) = \langle \sigma^f(\xi), \int_0^\xi \sigma^f(\eta)d\eta\rangle,$$

$\langle\cdot,\cdot\rangle$ denoting the scalar product.

Let $W(t) = \Big(W_0(t), W_1(t), W_2(t)\Big)$ be a new Wiener process. Then the process,

$$X(t) = \Big(r^d(t,\cdot), r^f(t,\cdot), e(t)\Big), \quad t \geq 0,$$

satisfies a stochastic Ito equation on $E = H \times H \times R$, with respect to the Wiener process $W(t), t \geq 0$. Denote the components of $x \in E$ by x_1, x_2, x_3 then, the quantity (3) is equal to,

$$\sup_{\tau \leq s} \mathbb{E}\left(e^{\int_0^\tau \alpha(X(\sigma,x))d\sigma}\varphi(X(\tau,x))\right),$$

where

$$\alpha(x_1,x_2,x_3) = -x_1(0),$$
$$\varphi(x_1,x_2,x_3) = (K - x_3)^+.$$

The unpleasent feature of the discount function α is that it is not continuous, or even well defined on E and therefore the general theory is not directly applicable. This difficulty has been approached in [8] by an ad hoc method. If one replaces however the Hilbert space $L^2(0,+\infty)$ by the space $C_c[0,+\infty)$ of continuous functions on the interval $[0,+\infty]$, or by the space $W^{1,2}[0,+\infty)$ of absolutely continuous functions with square summable first desivative [var], the theory developed here is applicable. It is not difficult to formulate rather unrestrictive conditions on (4) implying existence of continuous forward rates processes in $C_c[0,+\infty)$ or in $W^{1,2}[0,+\infty)$.

Modelling the forward functions by a Gaussian process leads to a possibility of negative interest rates.Therefore it is desirable to consider non-Gaussian models. In fact the equation for the forward functions $r(\cdot,\cdot)$ derived in [10] was nonlinear:

$$
\begin{aligned}
dr(t,\xi) &= \left(\frac{\partial}{\partial\xi}r(t,\xi) + a\Big(\xi, r(t,\cdot)\Big)\right)dt \\
&\quad + \Big\langle \sigma\Big(\xi, r(t,\xi)\Big), dW(t)\Big\rangle, \quad t \geq 0,
\end{aligned} \tag{5}
$$

where

$$
a\Big(\xi, z(\cdot)\Big) = \Big\langle \sigma\Big(\xi, z(\xi)\Big), \int_0^\xi \sigma\Big(\xi, z(\eta)\Big)d\eta\Big\rangle,
$$

and σ could be, in general, a nonlinear functional of $\xi \geq 0$ and of function $z(\eta)$, $\eta \geq 0$. The solution to such an equation is a Markov processes to which the general theory of stopping is applicable.

In some papers jumps component are added to the equation (5), see [4]. As a result one obtains solutions which are discontinuous processes.

Summing up, there are good reasons to develope stopping time theory for general Markov processes on Polish spaces.

3. Continuity condition. Let (E,ρ) be a Polish space with the family $\mathcal{E} = \mathcal{B}(E)$ of its Borel subsets. By $\mathbb{D}(0,T)$ and $\mathbb{D}$ we denote the spaces of all E-valued, right-continuous functions having left-hand limits, defined on the intervals $[0,T], T > 0$ and $[0,+\infty)$ respectively, equipped with the Skorokhod topologies, see [3, Chap. 3]. For any Polish space E we denote by $\mathcal{M}(E)$ the space of all probability measures on $(E,\mathcal{E})$ equipped with the topology of weak convergence. The spaces $\mathbb{D}(0,T)$, $\mathbb{D}$ and $\mathcal{M}(E)$ are Polish. The Banach spaces of Borel, bounded and of bounded, continuous functions on E, with the supremum norm, will be denoted respectively by $B_b(E)$ and $C_b(E)$.

We assume that given is a transition function $P_t(x,\Gamma), t \geq 0, x \in E, \Gamma \in \mathcal{E}$ which defines a transition semigroup (P_t) on $B_b(E)$:

$$
P_t\psi(x) = \int_E P_t(x,dy)\psi(y), \quad \psi \in B_b(E), \quad t \geq 0, x \in E. \tag{6}
$$

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and $(\mathcal{F}_t)$ an increasing family of σ-fields. The σ-fields $\mathcal{F}_t, t \geq o$ are assumed to be complete with respect to the measure $\mathbb{P}$. An E-valued stochastic process $X(t)$, $t \geq 0$ is said to be Markov with respect to $\{(P_t),(F_t)\}$ if

$$
\mathbb{E}\left(\psi(X(t+h))|\mathcal{F}_t\right) = P_h\psi(X(t)), \quad \mathbb{P}-a.s. \tag{7}
$$

for arbitrary $\psi \in B_b(E)$ and $t, h \geq 0$.

For the presentation it is convenient to introduce a family of $\{(P_t), (\mathcal{F}_t)\}$ Markov processes $X(t,x)$, $t \geq 0$, parametrized by $x \in E$, and such that

$$(8) \qquad \lim_{t \downarrow 0} X(t,x) = x\,, \quad \mathbb{P} - a.s.$$

Although the σ- fields $(\mathcal{F}_t)$ and the probability spaces $(\Omega, \mathcal{F}, \mathbb{P})$ may be different for different versions of X the semigroup (P_t) is fixed once for all.

We will often use the following assumptions (A.1) and (A.2):

(A.1) *For arbitrary $x \in E$,*

$$\mathbb{P}(X(\cdot, x) \in \mathbb{D}) = 1\,.$$

We denote by P^x the distribution of the process $X(\cdot, x)$ on $\mathbb{D}(0,T)$.

(A.2) *For arbitrary $T > 0$ the mapping $x \to P^x$ from E to $\mathcal{M}(\mathbb{D}(0,T))$ is continuous.*

Assumption (A.2) will be called the *continuity condition*. It is equivalent to the continuous dependence of the laws of the Markov processes $X(\cdot, x)$ on the initial condition $x \in E$. Both assumptions (A.1), (A.2) are satisfied in many interesting cases.

Remark 1. We restrict our considrations to the time homogeneous Markov processes and functions α, φ, ψ, independent of time. Extensions to the time dependent case can be easily obtained by passing to the time-space Markov process $\hat{X}$ on $\hat{E} = R^1 \times E$ given by the formula:

$$\hat{X}(t, \hat{x}) = (s + t,\ X(t,x)),\ t \geq 0,\ \hat{x} = (s,x) \in \hat{E}.$$

The rest of the section is devoted to some consequences, of the introduced assumptions, needed in the sequel.

Together with the semigroup (P_t) we will work with a semigroup $(\hat{P}_t)$:

$$\hat{P}_t \psi(x) = \mathbb{E}\left(e^{\int_0^t \alpha(X(\sigma,x))d\sigma}\, \psi(X(t,x))\right),\ \psi \in B_b(E),\ t \geq 0,\ x \in E,$$

determined by a function $\alpha : E \to R^1$.

Proposition 1. *Assume that assumptions* (A.1) *and* (A.2) *hold. If $\alpha \in C_b(E)$ and $\psi \in C_b\Big([0,+\infty) \times E\Big)$ then function v:*

$$v(s,x) = \hat{P}_s \psi(s,\cdot)(x), \quad s \geq 0, \quad x \in E,$$

belongs to $C_b\Big([0,T] \times E\Big)$, for all $T < \infty$.

Before proving the proposition we recall some results from [3].

Lemma 1. *If $f \in \mathbb{D}(0,T)$ then for arbitrary $\varepsilon > 0$ there exists a finite number of points $0 = t_0 < t_1 < \ldots < t_r = T$ such that for arbitrary $k = 1,2,\ldots,r$:*

$$\sup\left\{g\big(f(t), f(s)\big); t,s \in [t_{k-1}, t_k)\right\} < \varepsilon \tag{9}$$

Proof. Define r as the supremum of all those $t \in [0,T]$ such that the interval $[0,t)$ has a decomposition satisfying (9). Since $f(0) = f(0+), r > 0$. But the limit $f(r-)$ exists and therefore also the interval $[0,r)$ has the required decomposition. Thus r cannot be smaller than T. ■

It follows from the lemma that, in particular, the image of any function $f \in \mathbb{D}(0,T)$ is contained in a compact subset of E.

Lemma 2.
i) *Arbitrary function $f \in \mathbb{D}(0,T)$ has at must a countable number of discontinuities,*
ii) *If function $f \in \mathbb{D}(0,T)$ is continuous in $t \in [0,T]$ and $f_n \to f$ in $\mathbb{D}(0,T), t_n \to t, (t_n \in [0,T])$, then $f_n(t_n) \to f(t)$.*

Proof. The proof of i) follows from Lemma 1. To show ii) notice that by the definition of the topology on $\mathbb{D}(0,T)$, there exists a sequence (λ_n) of strictly increasing functions from $[0,T]$ into $[0,T]$ such that

$$\sup_{s\in[0,T]} |\lambda_n(s) - s| \to 0, \quad \sup_{s\in[0,T]} |\lambda_n^{-1}(s) - s| \to 0.$$

Let $\tilde{t}_n = \lambda_n^{-1}(t_n), n = 1,2,\ldots.$ Then

$$|f_n(t_n) - f(t)| \leq |f_n(\lambda_n(\tilde{t}_n)) - f(\tilde{t}_n)| + |f(\tilde{t}_n) - f(t)| \to 0$$

as required. ■

Proof of Proposition 1. Fix $T > 0$ and let $x_n \to x, t_n \to t$. By the Skorokhod embedding theorem, there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and $\mathbb{D}(0,T)$ - valued random variables $\tilde{X}_n, n = 1, 2, \ldots, \tilde{X}_\infty$ such that:

$$\mathcal{L}(\tilde{X}_n) = P^{x_n}, \mathcal{L}(\tilde{X}_\infty) = P^x,$$

and $\tilde{\mathbb{P}}$ - a.s., $\tilde{X}_n \to \tilde{X}_\infty$ in $\mathbb{D}(0,T)$ as $n \to +\infty$. Assume first that $t = 0$ and $\alpha \equiv 0$. Since $\tilde{X}_\infty(0) = x$ and by Lemma 1, $\tilde{\mathbb{P}}$ - a.s. $\lim_{n\to 0} \tilde{X}_n(t_n) = \tilde{X}_\infty(0)$ therefore,

$$\tilde{\mathbb{E}}\Big(\psi(t_n, \tilde{X}_n(t_n))\Big) \to \tilde{\mathbb{E}}\Big(\psi(0, \tilde{X}_\infty(0))\Big) = \psi(0, x).$$

In particular $P_{t_n}(x_n, \cdot) \to \delta_{\{x\}}(\cdot)$ weakly as $t_n \to 0$ and $x_n \to x$.

We show next that for arbitrary $t \in [0,T], x \in E$,

$$\mathbb{P}\Big(\lim_{s\uparrow t} X(s,x) = X(t,x)\Big) = 1. \tag{10}$$

Define $\varphi(x,y) = \rho(x,y) \wedge 1,\ x, y \in E$ and let $t_n \uparrow t$. Then

$$\mathbb{E}\Big(\varphi\big(X(t_n,x), X(t,x)\big)\Big) = \mathbb{E}\Big(\int_E P_{t-t_n}\Big(X(t_n,x), dy\Big)\varphi\Big(X(t_n,x), y\Big)\Big).$$

Since $\mathbb{P}\Big(X(t_n,x) \to X(t-,x) \text{ as } n \to +\infty\Big) = 1$ and $P_{t-t_n}(z_n, \cdot) \to \delta_{\{z\}}(\cdot)$ for arbitrary sequence $z_n \to z$,

$$\mathbb{E}\Big(\varphi(X(t_n,x), X(t,x))\Big) \to \mathbb{E}\Big(\varphi(X(t-,x), X(t-,x)\Big) = 0$$

But

$$\lim_{n\to\infty} \mathbb{E}\left(\varphi(X(t_n,x), X(t,x))\right) = \mathbb{E}\left(\varphi(X(t-,x), X(t,x))\right).$$

Consequently (10) holds.

Note that

$$v(t_n, x_n) = \hat{P}_{t_n}\psi(t_n, \cdot)(x_n) = \tilde{\mathbb{E}}\left(e^{\int_0^{t_n} \alpha\left(\tilde{X}_n(\sigma)\right) d\sigma}\psi(t_n, \tilde{X}_n(t_n))\right).$$

By (10) and Lemma 2, ii), $\tilde{X}_n(t_n) \to \tilde{X}_\infty(t)$, $\tilde{\mathbb{P}}$ - a.s. as $n \to +\infty$. Moreover by Lemma 2 i), $\tilde{X}_n(s) \to \tilde{X}_\infty(s)$, for almost all $s \in [0,T]$ and by Lebesgue's theorem

$$\int_0^{t_n} \alpha\Big(\tilde{X}_n(s)\Big) ds \to \int_0^t \alpha\Big(\tilde{X}_\infty(s)\Big) ds, \text{ as } n \to +\infty,$$

$\mathbb{P}$-a.s. Therefore

$$\lim_n v(t_n, x_n) = \bar{\mathbb{E}}\left(e^{\int_0^{t_n} \alpha\left(\bar{X}_\infty(\sigma)\right)d\sigma}\psi(t, \bar{X}_\infty(t))\right) = v(t,x). \quad \blacksquare$$

Let us recall that a stochastic $(\mathcal{F}_t)$ - adapted process X is said to be *quasi-left continuous*, if for arbitrary sequence of $(\mathcal{F}_t)$ stopping times τ_n, increasing to τ:

$$X(\tau_n) \to X(\tau),\ \mathbb{P}\ a.s.\ as\ n \to \infty.$$

We will need also the following probabilistic consequences of the continuity condition (A.2).

Proposition 2. *If assumptions* (A.1), (A.2) *hold then,*

i) *Arbitrary process $X(\cdot, x)$ is quasi-left continuous,*

ii) *For arbitrary compact set $K \subset E, T > 0$ and $\varepsilon > 0$ there exists a compact set $L \subset E$ such that,*

$$\mathbb{P}\Big(X(t,x) \in L \text{ for all } t \in [0,T]\Big) \geq 1 - \varepsilon, \text{ for all } x \in K.$$

Proof. i) The proof of quasi-left continuity in the case of locally compact spaces is valid also in the present more general situation, see [9, pp. 125–126].

ii) Since the set K is compact and the transformation $x \to P^x$ is continuous, the set of measures $\mathcal{K} = \{P^x; x \in K\}$ is compact in $\mathcal{M}\big(\mathbb{D}(0,T)\big)$. Therefore there exists a compact set $\mathcal{L} \subset \mathbb{D}(0,T)$ such that,

$$P^x(\mathcal{L}) \geq 1 - \varepsilon,$$

for all $x \in K$. Define

$$L = \{y; y = f(t), \text{for some } f \in \mathcal{L} \text{ and } t \in [0,T]\}.$$

It is enough to show that the set L is compact. If $y_n \in L$ then $y_n = f_n(t_n)$ for some $f_n \in \mathcal{L}$ and $t_n \in [0,T]$. There exist subsequences $\{f_{n_k}\}, \{t_{n_k}\}$ such that $f_{n_k} \to f \in \mathcal{L}$ and $t_{n_k} \to t \in [0,T]$ as $n \to +\infty$. Moreover for some sequences of increasing functions $\lambda_k : [0,T] \to [0,T]$ and numbers $\tilde{t}_{n_k}$,

$$f_{n_k}(t_k) = f_{n_k}\left(\lambda_k(\tilde{t}_{n_k})\right) \to f(t)$$

and $y_{n_k} \to y = f(t) \in L$. The proof is complete. $\blacksquare$

4. Continuity of the value function. Bounded data. Our aim in the present section is to prove the following theorem.

Theorem 1. *Assume that (A.1) and (A.2) hold and that $\alpha, \varphi, \psi \in C_b(E)$, $\psi \geq \varphi$. Then the function V:*

$$V(s,x) = \sup_{\tau \leq s} \mathbb{E}\left(e^{\int_0^\tau \alpha\left(X(\sigma,x)\right) d\sigma} \left[\psi\left(X(\tau,x)\right)\chi_{\tau<s} + \psi\left(X(s,x)\right)\chi_{\tau=s} \right] \right)$$

defined for all $(s,x) \in [0,+\infty) \times E$ belongs to $C_b([0,T] \times E)$, for all $T < \infty$.

Proof. Let us fix $h > 0$ and $s \geq 0$ and let $k = 0, 1, \ldots$ be such that $kh < s \leq (k+1)h$; if $s = 0$ we set $k = 0$. By $\mathcal{T}$ we denote the set of all $(\mathcal{F}_t)$ stopping times. Let $\mathcal{T}^h$ be the set of all $\tau \in \mathcal{T}$ which do not take values in the interval (kh, s) and $\mathcal{T}_h$ the set of all those $\tau \in \mathcal{T}$ which take values from the set $\{0, h, \ldots, kh, s\}$ only. If $\tau \in \mathcal{T}$ then define:

$$\tau^h = \begin{cases} \tau & \text{if } \tau \leq kh \\ s & \text{if } kh < \tau \leq s, \end{cases}$$

$$\tau_h = \begin{cases} 0 & \text{if } \tau = 0 \\ (l+1)h & \text{if } lh < \tau \leq (l+1)h \leq kh \\ s & \text{if } kh < \tau \leq s. \end{cases}$$

Let $\mathcal{J}_s(\tau)$ be the functional (1) corresponding to the process $X(t) = X(t,x)$, $t \geq 0$.

We show first, in two steps, that

$$\begin{aligned} |\sup_{\tau\in\mathcal{T}} \mathcal{J}_s(\tau) - \sup_{\tau\in\mathcal{T}_h} \mathcal{J}_s(\tau)| &\leq \sup_{\tau\leq s} \mathbb{E}\left(e^{\int_0^\tau \alpha\left(X(\sigma)\right) d\sigma} r_h^\varphi(X(\tau)) \right) \\ &\quad + \sup_{\tau\leq s} \mathbb{E}\left(e^{\int_0^\tau \alpha\left(X(\sigma)\right) d\sigma} r_h^\psi(X(\tau)) \right), \end{aligned} \tag{11}$$

where

$$r_h^\varphi(x) = \sup_{t\leq h} |\varphi(x) - \hat{P}_t\varphi(x)|$$

$$r_h^\psi(x) = \sup_{t\leq h} |\psi(x) - \hat{P}_t\psi(x)|, \quad x \in E.$$

Step 1.

$$|\sup_{\tau\in\mathcal{T}} \mathcal{J}_s(\tau) - \sup_{\tau\in\mathcal{T}^h} \mathcal{J}_s(\tau)| \leq \sup_{\tau\leq s} \mathbb{E}\left(e^{\int_0^\tau \alpha\left(X(\sigma)\right) d\sigma} r_h^\psi\left(X(\tau)\right) \right).$$

Note that

$$\begin{aligned}\mathcal{J}_s(\tau) &= \mathcal{J}_s(\tau^h) + e^{\int_0^\tau \alpha\left(X(\sigma)\right)d\sigma}\varphi\Big(X(\tau)\Big)\chi_{kh<\tau<s}\\ &\quad - e^{\int_0^s \alpha\left(X(\sigma)\right)d\sigma}\psi\Big(X(s)\Big)\chi_{kh<\tau<s}\\ &\le \mathcal{J}_s(\tau^h) + e^{\int_0^\tau \alpha\left(X(\sigma)\right)d\sigma}\psi\Big(X(\tau)\Big)\chi_{kh<\tau<s}\\ &\quad - e^{\int_0^s \alpha\left(X(\sigma)\right)d\sigma}\psi\Big(X(s)\Big)\chi_{kh<\tau<s}.\end{aligned}$$

However, by the Markov property,

$$\begin{aligned}&\mathbb{E}\left(e^{\int_0^s \alpha\left(X(\sigma)\right)d\sigma}\psi\big(X(s)\big)\chi_{kh<\tau<s}\right)\\ &=\mathbb{E}\left(e^{\int_0^\tau \alpha\left(X(\sigma)\right)d\sigma}\hat{P}_{s-\tau}\psi\big(X(\tau)\big)\chi_{kh<\tau<s}\right).\end{aligned}$$

Consequently,

$$\mathcal{J}_s(\tau) \le \mathcal{J}_s(\tau^h) + \mathbb{E}\left(e^{\int_0^\tau \alpha\left(X(\sigma)\right)d\sigma}\Big(\psi\big(X(\tau)\big) - \hat{P}_{s-\tau}\psi\big(X(\tau)\big)\Big)\chi_{kh<\tau<s}\right)$$

$$\sup_{\tau\in\mathcal{T}} \mathcal{J}_s(\tau) \le \sup_{\tau\in\mathcal{T}^h} \mathcal{J}_s(\tau) + \sup_{\tau\in\mathcal{T}} \mathbb{E}\left(e^{\int_0^\tau \alpha\left(X(\sigma)\right)d\sigma}r_h\psi\Big(X(\tau)\Big)\right),$$

and therefore

$$|\sup_{\tau\in\mathcal{T}} \mathcal{J}_s(\tau) - \sup_{\tau\in\mathcal{T}^h} \mathcal{J}_s(\tau)| \le \sup_{\tau\in\mathcal{T}} \mathbb{E}\left(e^{\int_0^\tau \alpha\left(X(\sigma)\right)d\sigma}r_h\psi\Big(X(\tau)\Big)\right),$$

as required.

Step 2.

$$|\sup_{\tau\in\mathcal{T}} \mathcal{J}_s(\tau^h) - \sup_{\tau\in\mathcal{T}} \mathcal{J}_s(\tau_h)| \le \sup_{\tau\in\mathcal{T}} \mathbb{E}\left(e^{\int_0^\tau \alpha\left(X(\sigma)\right)d\sigma}r_h\phi\Big(X(\tau)\Big)\right),$$

It is clear that

$$|\sup_{\tau\in\mathcal{T}} \mathcal{J}_s(\tau^h) - \sup_{\tau\in\mathcal{T}} \mathcal{J}_s(\tau_h)| = |\sup_{\tau\in\mathcal{T}^h} \mathcal{J}_s(\tau) - \sup_{\tau\in\mathcal{T}^h} \mathcal{J}_s(\tau_h)|$$

If now $\tau \in \mathcal{T}^h$ then,

$$\mathcal{J}_s(\tau) = \mathbb{E}\left\{e^{\int_0^\tau \alpha(X(\sigma))d\sigma}\varphi(X(\tau))\chi_{\tau\le kh} + e^{\int_0^s \alpha(X(\sigma))d\sigma}\psi(X(s))\chi_{\tau=s}\right\}$$

$$\mathcal{J}_s(\tau_h) = \mathbb{E}\left\{e^{\int_0^{\tau_h} \alpha(X(\sigma))d\sigma}\varphi(X(\tau_h))\chi_{\tau\le kh} + e^{\int_0^s \alpha(X(\sigma))d\sigma}\psi(X(s))\chi_{\tau=s}\right\}$$

and

$$\mathcal{J}_s(\tau)-\mathcal{J}_s(\tau_h) = \mathbb{E}\left\{\left[e^{\int_0^\tau \alpha(X(\sigma))d\sigma}\phi(X(\tau)) - e^{\int_0^{\tau_h} \alpha(X(\sigma))d\sigma}\phi(X_{\tau_h})\right]\chi_{\tau\le kh}\right\}.$$

By the Markov property,

$$\mathbb{E}\left\{\left[e^{\int_0^\tau \alpha(X(\sigma))d\sigma}\phi(X(\tau)) - e^{\int_0^{(l+1)h} \alpha(X(\sigma))d\sigma}\phi(X(lh))\right]\chi_{\{lh<\tau\le(l+1)h\}}\right\}$$

$$= \mathbb{E}\left\{e^{\int_0^\tau \alpha(X(\sigma))d\sigma}\Big(\phi(X(\tau)) - \hat{P}_{(l+1)h-\tau}\phi(X(\tau))\Big)\chi_{\{lh<\tau\le(l+1)h\}}\right\}$$

for $l = 0,1,\dots,k-1$. Therefore the proof of step 2 and of the required basic inequality easily follows.

Denote now,

$$V^h(s,x) = \sup_{\tau\in\mathcal{T}_h} \mathbb{E}\big(\mathcal{J}_s(\tau)\big).$$

By the discrete time dynamic programming applied to the Markov chain: $X(0,x), X(h,0), \dots, X(kh,x), X(s,x)$, and to the reward functionals $\mathbb{E}\big(\mathcal{J}_s(\tau)\big)$, the functions V^h are given by the following recursive formulae :

$$\text{(12)}\qquad \begin{cases} V^h(0,x) &= \psi(x),\\ V^h(s,x) &= \max\big(\varphi(x), \hat{P}_s V^h(0,\cdot)(x)\big),\ s\in[0,h],\\ V^h(s,x) &= \max\big(\varphi(x), \hat{P}_h V^h(s-h,\cdot)(x)\big),\ s\ge h.\end{cases}$$

Therefore, by Proposition 1, $V^h(\cdot,\cdot)\in C_b([0,+\infty)\times E)$.
The first part of the proof implies that,

$$|V(s,x) - V^h(s,x)| \le \sup_{\tau\le s}\mathbb{E}\left(e^{\int_0^\tau \alpha\left(X(\sigma,x)\right)d\sigma} r_h^\varphi\Big(X(\tau,x)\Big)\right) + \sup_{\tau\le s}\mathbb{E}\left(e^{\int_0^\tau \alpha\left(X(\sigma,x)\right)d\sigma} r_h^\psi\Big(X(\tau,x)\Big)\right).$$

By Proposition 1, $r_h^\varphi \to 0$ and $r_h^\psi \to 0$ uniformly on compact sets as $h\to 0$. Therefere, by Proposition 2, for arbitrary $T>0$, arbitrary compact set K, and arbitrary $\varepsilon > 0$ there exists a compact set L and number $h_0 > 0$ such that

$$r_h^{\varphi}(y) \leq \varepsilon,\ r_h^{\psi}(y) \leq \varepsilon \text{ for all } y \in L, h \in [0, h_0]$$

and

$$\mathbb{P}\Big(X(t,x) \in L,\ t \in [0,T]\Big) \geq 1 - \varepsilon, \text{ for } x \in K.$$

Thus $|V(\cdot,\cdot) - V^h(\cdot,\cdot)| \to 0$ uniformly on $[0,T] \times K$ as $h \to 0$. This proves the continuity of the value function V. ■

5. Extremal properties of the value function. The following two characterizations of the value function are of independent interest.

Theorem 2. *Under the assumptions of* Theorem 1 *the value function* V *is the smallest bounded, measurable function with the properties:*
i) $V(0,x) \geq \psi(x), V(s,x) \geq \varphi(x)$, *for all* $s \geq 0, x \in E$
ii) $\hat{P}_s V(t,\cdot)(x) \leq V(s+t,x),\ s,t \geq 0$

Proof. Let $h_n = 1/2^n$,

$$V_n(s,x) = V^{h_n}(s,x),\ n = 1,2,\dots,\ s \geq 0,\ x \in E.$$

Since $V_n(0,x) = \psi(x)$ and $V_n(s,x) \geq \psi(x)$, $n = 1,2\dots$ and $V = \lim V_n$, therefore V satisfies i). Since for $s \geq h$, $V^h(s,x) \geq \hat{P}_h V^h(s-h,\cdot)(x)$, therefore if $t = k/2^n \leq s$, by the semigroup property,

$$V_n(s,x) \geq \hat{P}_t V_n(s-t,\cdot)(x).$$

Consequently ii) holds for all $t \geq 0$ and dyadic s and thus, by continuity of V and Proposition 1, for all $s \geq 0$. Assume now that $\tilde{V}$ is a bounded, measurable function for which i) and ii) hold. We will show that:

$$\tilde{V}(s,x) \geq V^h(s,x), \text{ for all } f > 0,\ s \geq 0, x \in E. \tag{13}$$

Fix $h > 0$ and $s \in [0,h]$. Then

$$V^h(s,x) \leq \hat{P}_s \tilde{V}(0,\cdot)(x) \leq \tilde{V}(s,x).$$

If $V^h(t,x) \leq \tilde{V}(t,x)$ for $t \leq kh$ and $kh \leq s \leq (k+1)h$, then

$$V^h(s,x) \leq \hat{P}_h \tilde{V}(s-h,\cdot)(x) \leq \tilde{V}(s,x)$$

and the result follows. ■

Let us fix $\varphi \in C_b(E)$ and $s \geq 0$, then the value function V can be regarded as an operator acting on the final function ψ:

$$V(s,x) = \mathcal{V}_s\psi(x).$$

For an $h > 0$ define a mapping $\mathcal{G}_h : C_b(E) \to C_b(E)$ by the formula:

$$\mathcal{G}_h(\psi) = \max(\varphi, \hat{P}_h\psi).$$

It follows from (12) that for $n = 0, 1, \dots$

$$V^h(nh, x) = \mathcal{G}_h\Big(V^h\big((n-1)h, \cdot\big)\Big)(x) = \mathcal{G}_h^n\psi(x).$$

This easily implies that the operators $\mathcal{V}_s, s \geq 0$ form a semigroup of nonlinear operators, compare [16], on the set

$$\mathcal{K}_c = \{\psi \in C_b(E) : \psi \geq \varphi\}.$$

Theorem 3.
i) *Under assumptions of* Theorem 2, $(\mathcal{V}_s)$ *is a semigroup on* $\mathcal{K}_c$ *such that*

$$(14) \qquad \hat{P}_s\psi \leq \mathcal{V}_s\psi, \ \textit{for all } s \geq 0, \quad \psi \in \mathcal{K}_c.$$

ii) *If* $(\tilde{\mathcal{V}}_s)$ *is a semigroup on* $\mathcal{K}_c$ *satisfying (14) then*

$$\mathcal{V}_s\psi \leq \tilde{\mathcal{V}}_s\psi, \ \textit{for all } s \geq 0, \quad \psi \in \mathcal{K}_c.$$

Proof. It is enough to show ii). Set $\tilde{V}(s,x) = \tilde{\mathcal{V}}_s\psi(x)$, $s \geq 0, x \in E$. Then $\tilde{V}(0,x) = \psi(x)$, and

$$\tilde{V}(s,x) \geq \varphi(x), \quad s \geq 0, \quad x \in E.$$

From the properties of $(\tilde{\mathcal{V}}_s)$:

$$\tilde{V}(s+t,x) = \tilde{\mathcal{V}}_{s+t}\psi(x) = \tilde{\mathcal{V}}_s(\tilde{\mathcal{V}}_t\psi)(x) \geq \hat{P}_s(\tilde{\mathcal{V}}_t\psi)(x) \geq \hat{P}_s\Big(\tilde{V}(t,\cdot)\Big)(x)$$

and by Theorem 2, $\tilde{V}(s,x) \geq V(s,x) = \mathcal{V}_s\psi(x)$. The proof is complete. ■

6. Existence of optimal moments. By $X(t)$, $t \geq 0$ we denote $\{(P_t),(\mathcal{F}_t)\}$, quasi-left continuous Markov process with trajectories in $\mathbb{D}([0,+\infty);E)$. There is no need to specify the initial condition of the process X in the present section. We use convention, $\inf\emptyset = s$.

Theorem 4. *Assume that assumptions* (A.1) *and* (A.2) *hold and that* $\alpha, \varphi, \psi \in C_b(E)$, $\psi \geq \varphi$. *For fixed* $s \geq 0$ *define*

$$\hat{\tau} = \inf\left\{t \leq s : Vb(s-t, X(t)) = \varphi(X(t))\right\}.$$

Then $\hat{\tau}$ *is an optimal stopping time and:*

$$\mathbb{E}\left(e^{\int_0^{\hat{\tau}} \alpha\left(X(\sigma)\right)d\sigma}\left[\varphi(X(\hat{\tau}))\chi_{\hat{\tau}<s} + \psi(X(\hat{\tau}))\chi_{\hat{\tau}=s}\right]\right) = \mathbb{E}(V(s, X(0))).$$

Proof. Step 1. Fix $\delta > 0$ and define

$$\sigma_\delta = \inf\left\{t \leq s;\, V\big(s-t, X(t)\big) \leq \varphi\big(X(t)\big) + \delta\right\},$$

then

$$\sigma_\delta \uparrow \hat{\tau}, \quad \text{as } \delta \downarrow 0, \quad \mathbb{P}-a.s.$$

It is clear that σ_δ is a decreasing function of $\delta > 0$ and that $\sigma_\delta \leq \hat{\tau}$, for all $\delta > 0$. Denote $\tilde{\tau} = \lim_{\delta\to 0}\sigma_\delta$. Then $\tilde{\tau} \leq \hat{\tau}$. If $\tilde{\tau} < s$ then

$$V\Big(s-\sigma_\delta, X(\sigma_\delta)\Big) \leq \varphi\Big(X(\sigma_\delta)\Big) + \delta$$

and by the quasi-left continuity of X,

$$V\Big(s-\tilde{\tau}, X(\tilde{\tau})\Big) \leq \varphi\Big(X(\tilde{\tau})\Big).$$

Since $V \geq \varphi$ one gets that $V\Big(s-\tilde{\tau}, X(\tilde{\tau})\Big) = \varphi\Big(X(\tilde{\tau})\Big)$ and $\hat{\tau} \leq \tilde{\tau}$. Thus if $\tilde{\tau} < s$ then $\tilde{\tau} = \hat{\tau}$. However if $\tilde{\tau} = s$ then also $\hat{\tau} = s$ and $\tilde{\tau} = \hat{\tau}$ in general.

The following result follows easily by backward induction. For the definitions of V^h and $\mathcal{T}_h$, see the proof of Theorem 1.

Lemma 3. *For each* $s > 0, 0 < h < s$ *define,*

$$\hat{\tau}_h = \inf\left\{kh < s;\, V^h(s-kh, X(kh)) = \varphi(X(kh))\right\}.$$

For arbitrary $\tau \in \mathcal{T}_h$ *such that* $\tau \leq \hat{\tau}_h$,

$$\mathbb{E}\left(V^h(s, X(0))\right) = \mathbb{E}\left(e^{\int_0^\tau \alpha\left(X(\sigma)\right)d\sigma} V^h\big(s-\tau, X(\tau)\big)\right). \tag{15}$$

Step 2. Let

$$\begin{aligned}\sigma_{\delta h} &= kh \quad if\ (k-1)h < \sigma_\delta \leq kh \leq s \\ &= s \quad if\ (k-1)h < s < kh.\end{aligned}$$

For h sufficiently small and all $t \in [0,s]$,

$$V^h\big(s-t, X(t)\big) + \delta \geq V\big(s-t, X(t)\big).$$

Therefore $\hat{\tau}_h \geq \sigma_{\delta h}$, $\mathbb{P}$ - a.s., for h sufficiently small. Applying (15) to $\tau = \hat{\tau}_h \wedge \sigma_{\delta h}$ one gets, by the right continuity of the process $X(.)$ and the uniform convergence of V^h to V on compact subsets of $[0,+\infty) \times E$,

$$\mathbb{E}\left(V\left(s, X(0)\right)\right) = \mathbb{E}\left(e^{\int_0^{\sigma_\delta} \alpha\left(X(\sigma)\right) d\sigma} V\left(s - \sigma_\delta, X(\sigma_\delta)\right)\right)$$

$$\leq \mathbb{E}\left(e^{\int_0^{\sigma_\delta} \alpha\left(X(\sigma)\right) d\sigma}\Big[\big(\varphi(X(\sigma_\delta)) + \delta\big)\chi_{\sigma_\delta < s} + \psi\big(X(s)\big)\chi_{\sigma_\delta = s}\Big]\right).$$

Note that

$$\begin{aligned}
&\varphi\Big(X(\sigma_\delta)\Big)\chi_{\sigma_\delta<s} + \psi\Big(X(s)\Big)\chi_{\sigma_\delta=s}\\
&= \Big[\varphi\Big(X(\sigma_\delta)\Big)\chi_{\sigma_\delta<s} + \psi\Big(X(s)\Big)\chi_{\sigma_\delta=s}\Big]\chi_{\hat{\tau}<s}\\
&+ \Big[\varphi\Big(X(\sigma_\delta)\Big)\chi_{\sigma_\delta<s} + \psi\Big(X(s)\Big)\chi_{\sigma_\delta=s}\Big]\chi_{\hat{\tau}=s}\\
&= \varphi\Big(X(\sigma_\delta)\Big)\chi_{\sigma_\delta<s}\chi_{\hat{\tau}<s}\\
&+ \Big[\varphi\Big(X(\sigma_\delta)\Big)\chi_{\sigma_\delta<s} + \psi\Big(X(s)\Big)\chi_{\sigma_\delta=s}\Big]\chi_{\hat{\tau}=s} = I_\delta^1 + I_\delta^2.
\end{aligned}$$

By the quasi-left continuity of X, $I_\delta^1 \to \varphi\Big(X(\hat{\tau})\Big)\chi_{\hat{\tau}<s}$. On the other hand if $\sigma_\delta \uparrow s$ then $\varphi\Big(X(\sigma_\delta)\Big) \to \varphi\Big(X(s)\Big) \leq \psi\Big(X(s)\Big)$ as $\delta \downarrow 0$. Consequently

$$\mathbb{E}\Big(V\big(s, X(0)\big)\Big) \leq \mathbb{E}\left(e^{\int_0^{\hat{\tau}} \alpha\left(X(\sigma)\right) d\sigma}\Big[\varphi(X(\hat{\tau}))\chi_{\hat{\tau}<s}\chi_{\hat{\tau}<s} + \psi\big(X(s)\big)\chi_{\hat{\tau}=s}\Big]\right)$$

as required. ■

7. Extension to unbounded functions and an application. We will now dispence with the restriction that functions α, φ and ψ are bounded and impose a weaker condition:

(A.3) *Functions α, φ, ψ are continuous and bounded on bounded sets and $\psi \geq \varphi$. Moreover for arbitrary compact set $K \subset E$ and arbitrary $T > 0$:*

$$\mathbb{E}\left(\sup_{\substack{x \in K\\ t\in[0,T]}} e^{\int_0^T \alpha^+\left(X(\sigma,x)\right) d\sigma}\Big|\psi\big(X(t,x)\big)\Big|\right) < +\infty. \tag{16}$$

We have the following main result.

Theorem 5. *Assume that conditions* (A.1), (A.2) *and* (A.3) *hold.*

i) *The function:* $V : [0, +\infty) \times E \to R$,

$$V(s,x) = \sup_{\tau \le s} \mathbb{E}\left(e^{\int_0^\tau \alpha^+\left(X(\sigma,x)\right)d\sigma}\left[\varphi(X(\tau,x))\chi_{\tau<s} + \psi\big(X(s,x)\big)\chi_{\tau=s}\right]\right),$$

is continuous.

ii) *For arbitrary* $\left\{(P_t), (\mathcal{F}_t)\right\}$, *quasi let-continuous Markov process* X *the moment,*

$$\hat{\tau} = \inf\left\{t \le s; V\left(s-t, X(t)\right) = \varphi\left(X(t)\right)\right\}$$

is optimal:

$$\mathbb{E}\left(V\left(s, X(0)\right)\right) = \mathcal{J}_s(\hat{\tau}, X(0)),$$

Proof. i) Let T be a fixed positive number and K a fixed compact subset of E. It follows from Proposition 2 that there exists an increasing sequence of compact sets (L_n) such that for all $x \in K$,

$$\mathbb{P}\Big(X(t,x) \in L_n \text{ for all } t \in [0,T]\Big) \ge 1 - \frac{1}{n},\ n = 1, 2 \ldots$$

For each natural n denote by α_n, φ_n and ψ_n bounded continuous functions, identical to α, φ and ψ on sets L_n and such that

$$\alpha_n^+ \le \alpha^+,\ |\varphi_n| \le |\varphi|,\ |\psi_n| \le |\psi| \text{ on } E.$$

Then functions $W_n : [0, +\infty) \times E \to R$,

$$W_n(s,x) = \sup_{\tau \le s} \mathbb{E}\left(e^{\int_0^\tau \alpha_n\left(X(\sigma,x)\right)d\sigma}\left[\varphi_n(X(\tau,x))\chi_{\tau<s} + \psi_n(X(s,x))\chi_{\tau=s}\right]\right)$$

are continuous by our previons results.Define:

$$\xi = \sup_{\substack{x \in K, \\ t \in [0,T]}} \left[e^{\int_0^\tau \alpha^+\left(X(\sigma,x)\right)d\sigma}\left|\psi\Big(X(t,x)\Big)\right|\right]$$

and

$$A_n = \left\{X(\sigma,x) \in L_n^c \text{ for some } \sigma \in [0,T]\right\}.$$

Note that for arbitrary $s \in [0,T]$, $x \in K$:

$$\begin{aligned} |W_n(s,x) - V(s,x)| &\le 2 \sup_{\tau \le s} \mathbb{E}\left(e^{\int_0^\tau \alpha^+\left(X(\sigma,x)\right)d\sigma}\left|\psi\Big(X(\tau,x)\Big)\right|\chi_{A_n}\right) \\ &\le 2\mathbb{E}(\xi\ \chi_{A_n}). \end{aligned}$$

By (A.5), ξ is an integrable random variable and by Proposition 2, $\mathbb{P}(A_n) \to 0$ as $n \to \infty$. Consequently $\mathbb{E}(\xi\,\chi_{A_n}) \to 0$ as $n \to \infty$ and

$$\sup_{\substack{s\in[0,T]\\ x\in K}} \left| W_n(s,x) - V(s,x) \right| \to 0 \text{ as} \to +\infty.$$

This proves the first part of the theorem. ii) Let V^h be the functions (12) defined in the the proof of Theorem 1. As in the bounded case functions V^h are continuous. The sequence $V^{1/2^n}$, $n = 1, 2 \dots$ increases to the continuous function V and by Dini's theorem the convergence is uniform on compact sets. Now the proof of Theorem 10 can be repeated with functions V^h replaced by $V^{1/2^n}$. ■

Application to the financial model. We conclude the paper by some comments on the applicability of the developed theory to the financial model from section 2. Let us consider equation,

$$dr(t,\xi) = \left(\frac{\partial}{\partial \xi} r(t,\xi) + a(\xi) \right) dt + \langle \sigma(\xi), dW(t) \rangle \tag{17}$$

for the forwared curve, see section 2. Denote by $E = C_0(0, \infty)$ the space of all continuous functions defined on $[0, \infty)$ vanishing at ∞. If for instance functions a and σ are in E and the first derivative of σ is in E then the equation (17) has a unique, continuous solution in E. Taking this into account and applying Fernique's result from [6] one can easily check that the assumptions of Theorem 5 are satisfied for the optimal stopping problem related to the American put option discussed in 2, see also [8].

References

[1] Bensoussan, A., *Stochastic Control by Functional Analysis Methods*, North - Holland, Amsterdam, New York, Oxford,, 1982.

[2] Bensoussan, A. and J. L. Lions, *Applications des Inéquations Variationnelles en Contrôle Stochastique*, Dunod, Paris, 1978.

[3] Billingsley, P., *Convergence of Probability Measures*, Wiley, New York, 1968.

[4] Björk, T., G. Di Masi, Yu. Kabanov and W. Runggaldier, *Towards a general theory of bond markets (preprint)*.

[5] Brace, A. and M. Musiela, *A multifactor Gauss Markov implementation of Heath, Tarrow and Morton*, Math. Finance **4** (1994), 259–283.

[6] Fernique, X., *Regularité des trajectoires des fonctions aleatoires gaussiennes*, LNiM **No.480** (1975), 2- 96.

[7] Gatarek, D., *Some remarks on the market model of interest rates*, Control Cybernet. **25** (1996), 1233-1244.

[8] Gątarek, D. and M. Musiela *Pricing of American interest rate derivatives as optimal stopping of Ornstein-Uhlenbeck processes*, 1996, manuscript.
[9] Gikhman, I. I. and A. V., *Theory of stochastic processes* vol. 2, Nauka, Moscow 1973.
[10] Heath, D. R. Jarrow and A. Morton, *Bond pricing and the term structure of interest rates: a new methodology*, Econometrica **60** (1992), 77-101.
[11] El Karoui, N., *Les aspects probabilistes du contrôle stochastique*, École d'Été de Probabilités de Saint-Flour IX-1979, Springer-Verlag, Berlin, 1981, LNiM 876.
[12] Krylov, N. V., *Controlled Difusion Processes*, Springer Verlag, Berlin, 1980.
[13] Mackevičius, V., *Passing to the limit in the optimal stopping of the Markov processes*, Liet. Mat. Rink. **XIII** (1973), 115–127.
[14] Mazziotto, G. L. Stettner, T. Szpirglas and J. Zabczyk, *On impulse control with partial observation*, SIAM J. Control Opt. **26** (1988), 964–984.
[15] Musiela, M., *Stochastic PDEs and term structure models*, International Conference in Finance, La Baule.
[16] Nisio, M., *On nonlinear semigroups for Markov processes associated with optimal stopping*, App. Math. Optim. **4** (1978), 146–169.
[17] Robin, M., *Contrôle impulsionnel des processus de Markov*, Thèse, l'Universite Paris IX, 1978.
[18] Shiryayev, A. N., *Optimal Stopping Rules*, Springer Verlag, New York, 1978.
[19] Stettner, L. and J. Zabczyk, *On optimal stopping of Feller processes*, Preprint 284, Institute of Mathematics, Polish Academy of Sciences, 1983.
[20] Vargiolu, T.; *Invariant measures for Langevin equation describing forward rates in an arbitrage free market*, Preprint 20, 1996, Scuola Normale Suyperiore, Pisa, Italy.
[21] Zabczyk, J., *Semigroup methods in stochastic contrôle*, Repost 821 of CRM, Université de Montreal, 1978.
[22] ______, *Introduction to the theory of optimal stopping*, Lecture Notes in Control and Inform. Sci. **16** (1979), 227–250.
[23] ______, *Stopping problems in stochastic control*, Proceedings of ICM, Warsaw, 1983.
[24] ______, *Optimal stopping on Polish spaces*, Preprint IM PAN, 1997.

Institute of Mathematics
Polish Academy of Siences
Warsaw, Poland

received February 12, 1997