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Multiplicative Decomposability of Probability Measures

Dedicated to Professor Dominik Szynal
on the occasion of his 60th birthday

Abstract. The paper deals with the problem of decomposition of symmetric infinitely divisible random variables into a product of independent symmetric infinitely divisible factors.

We denote by P the set of all probability measures defined on Borel subsets of the real line $\mathbb{R}$. By S denote the subset of P consisting of symmetric probability measures. By P_∞ we denote the subset of P consisting of infinitely divisible probability measures. Put $S_\infty = S \cap P_\infty$. By δ_c we denote the probability, measure concetrated at the point c. We shall also use the notation $\vartheta = \frac{1}{2}(\delta_1 + \delta_{-1})$. Given μ, $\nu \in P$ we denote by $\mu * \nu$ the convolution of μ and ν. We denote by $\widehat{\mu}$ the characteristic function of μ, i.e.

$$\widehat{\mu}(t) = \int_{-\infty}^{\infty} e^{itx}\mu(dx) \quad (t \in \mathbb{R}).$$

By $\mu \circ \nu$ we denote the multiplicative convolution of μ and ν, i.e. the probability distribution of the product XY of independent random variables

X and Y with the probability distribution μ and ν respectively. Observe that $\vartheta \circ \mu = \mu$ for $\mu \in S$ which shows that ϑ is a unit element in S for the operation $\circ$. It is clear that

$$(\mu \circ \nu)\hat{}(t) = \int_{-\infty}^{\infty} \hat{\mu}(tx)\nu(dx). \tag{1}$$

A probability measure λ from P is said to be a P_∞ - multiplicator if $\lambda \circ \mu \in P_\infty$ for every $\mu \in P$. The set of all P_∞ - multiplicators will be denoted by P_∞^0. Setting $S_\infty^0 = S \cap P_\infty^0$ and using the notation :

$$A \circ B = \{\mu \circ \nu : \mu \in A, \nu \in B\}$$

we have the inclusion

$$P \cap S_\infty^0 = S_\infty^0 \subset S_\infty . \tag{2}$$

Several authors have dealt with probability measures belonging to P_∞^0 and S_∞^0. Goldier's result on mixtures of the exponential distributions in [2] extended by F. W. Steutel in [4] showed that the set P_∞^0 is non-empty. For detailed information we refer to [5].

A probability measure λ from S_∞ is said to be multiplicativelly S_∞ - decomposable if there exist probability measures μ and ν belonging to S_∞ such that $\lambda = \mu \circ \nu$. In the opposite case λ is called multiplicatively S_∞ - indecomposable. The problem of multiplicative S_∞ - decomposability was discussed in [6]. In particular the following result was obtained.

Each probability measure μ from S_∞ other than δ_0 has a representation

$$\mu = \mu_1 \circ \mu_2 \circ \ldots \circ \mu_k , \tag{3}$$

where k is a positive integer, the factors $\mu_1, \mu_2, \ldots, \mu_k$ belong to S_∞ and are multiplicatively S_∞ - indecomposable. Moreover for every probability measure μ the set of all possible integers k in representation (3) is finite.

Let $\mu \in S_\infty$ and $\mu \neq \delta_0$. Two representations

$$\mu = \mu_1 \circ \mu_2 \circ \ldots \circ \mu_k \quad \text{and} \quad \mu = \nu_1 \circ \nu_2 \circ \ldots \circ \nu_r$$

with multiplicatively S_∞ - indecomposable factors $\mu_1, \mu_2, \ldots, \mu_k$ and $\nu_1, \nu_2, \ldots, \nu_r$ from S_∞ are said to be equivalent if $k = r$ and the k-tuple $\nu_1, \nu_2, \ldots, \nu_k$ is a permutation of the k-tuple $\delta_{a_1} \circ \mu_1, \delta_{a_2} \circ \mu_2, \ldots, \delta_{a_k} \circ \mu_k$ for some $a_1, a_2, \ldots, a_k \in \mathbb{R}$.

Lemma. *If* $\mu \in P$ *and*

$$\hat{\mu}(t) \geq 0 \quad \textit{for} \quad t \in \mathbb{R}, \tag{4}$$

then $\int_{-\infty}^{\infty} |x|^{-1}\mu(dx) = \infty$.

Proof. If μ has a positive mass at the origin, then our statement is obvious. Consequently we may assume that $\mu(\{0\}) = 0$. Since, by (4), the characteristic function $\widehat{\mu}$ is real-valued, we conclude that $\mu \in S$. Thus

$$(5) \qquad \int_{-T}^{T} \widehat{\mu}(t)dt = 2\int_{-T}^{T}\int_{0}^{\infty} \cos tx\mu(dx)dt = 4\int_{0}^{\infty} x^{-1}\sin Tx\mu(dx)$$

for every $T > 0$. Contrary to the assertion of the Lemma let us suppose that

$$(6) \qquad \int_{-\infty}^{\infty} |x|^{-1}\mu(dx) < \infty .$$

Then, by (5), $\int_{-\infty}^{\infty} \widehat{\mu}(t)dt < \infty$ which together with inequality (4) shows that the measure μ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$. Setting $\mu(dx) = g(x)dx$ we have, by (6), the inequality $\int_{0}^{\infty} x^{-1}g(x)dx < \infty$. Consequently, bv Riemann- Lebesgue Theorem

$$\lim_{T\to\infty}\int_{0}^{\infty} x^{-1}g(x)\sin Txdx = 0 .$$

Comparying this with (5) we get the equality $\int_{-\infty}^{\infty} \widehat{\mu}(t)dt = 0$ which, by (4) and the continuity of $\widehat{\mu}$, yields $\widehat{\mu}(t) = 0$ for $t \in \mathbb{R}$. But this contradicts the formula $\widehat{\mu}(0) = 1$. The Lemma is thus proved.

Theorem. *Suppose that* $\lambda \in S_{\infty}$ *and* $\int_{-\infty}^{\infty} \widehat{\lambda}(t)dt < \infty$. *Then* λ *is multiplicatively* S_{∞}*- indecomposable.*

Proof. Suppose the contrary and put $\lambda = \mu \circ \nu$ where $\mu, \nu \in S_{\infty}$. Since $\widehat{\mu}(t) > 0$ for $t \in \mathbb{R}$, we have, by (1), the formula

$$\int_{-\infty}^{\infty} \widehat{\lambda}(t)dt = \int_{-\infty}^{\infty} \widehat{\mu}(t)dt \int_{-\infty}^{\infty} |x|^{-1}\nu(dx) .$$

Consequently, by Lemma, $\int_{-\infty}^{\infty} \widehat{\lambda}(t)dt = \infty$ which contradicts the assumption. The Theorem is thus proved.

The above Theorem may serve for determining of multiplicatively S_{∞} - indecomposable factors. We shall illustrate this by some examples.

Example 1. Stable probability measure. Given $0 < p \leq 2$ we denote by σ_p the symmetric stable probability measure with the characteristicr function

$\widehat{\sigma}_p(t) = \exp(-|t|^p)$ $(t \in \mathbb{R})$. Of course $\sigma_p \in S_\infty$ and $\int_{-\infty}^{\infty} \widehat{\sigma}_p(t)dt < \infty$ which, by the Theorem, shows that the stable measures $\sigma_p\,(0 < p \leq 2)$ are multiplicatively S_∞ - indecomposable. This gives an affirmative answer to the problem raised in [6]

Example 2. The measures $\sigma_p \circ \sigma_q\,(0 < p\,, q \leq 2)$. The well-known result on variance mixtures of Gaussian probability measures can be formulated as follows. Let X be a random variable with the probability distribution λ. Suppose that the probability distribution of X^2 is infinitely divisible. Then $\sigma_2 \circ \lambda \in S_\infty$ (see [5]). Consider two independent random variables Y and Z with the probability distribution σ_2 and μ respectively. Here μ is an arbitrary probability measure from P. Observe that the probability distribution of Y^2 is the gamma distribution on $[0, \infty)$ with the density $2^{-1}(\pi x)^{-1/2}\exp(-x/4)$ and, consequently, by Steutel's Theorem from [4] belongs to P^0_∞. Hence it follows that the probability distribution the random variable Y^2Z^2 belongs to P_∞ which shows that $\sigma_2 \circ \sigma_2 \circ \mu \in S_\infty$ for every $\mu \in P$. In other words

$$\sigma_2 \circ \sigma_2 \in S^0_\infty\,. \tag{7}$$

It is well-known that for every pair $0 < p \leq r \leq 2$ there exists a probability measure $\nu_{p,r}$ concentrated on the positive half-line and fulfilling the condition

$$\sigma_p = \sigma_r \circ \nu_{p,r} \tag{8}$$

(see [1, Chapter XVII. Exercise 9]. Applying formula (1) we get the equality

$$\int_{-\infty}^{\infty} \widehat{\sigma}_p(t)dt = \int_{-\infty}^{\infty} \widehat{\sigma}_r(t)dt \int_0^{\infty} x^{-1}\nu_{p,r}(dx)$$

which yields the inequality

$$\int_0^{\infty} x^{-1}\nu_{p,r}(dx) < \infty \quad (0 < p \leq r \leq 2)\,. \tag{9}$$

Setting $r = 2$ into (8) we get the formula

$$\sigma_p \circ \sigma_q = \sigma_2 \circ \sigma_2 \circ \nu_{p,2} \circ \nu_{q,2}$$

which, by (2) and (7), yields

$$\sigma_p \circ \sigma_q \in S^0_\infty \;\; (0 < p, q \leq 2)\,.$$

Consider the case $0 < p \leq 1$, $0 < q < 2$. Setting $\lambda_r = \sigma_p \circ \nu_{q,r}$ for $q < r \leq 2$ we have, by (8),

$$\sigma_p \circ \sigma_q = \sigma_r \circ \lambda_r. \tag{10}$$

By Keilson-Steutel Theorem from [3]

$$\sigma_p \in S_\infty^0 \quad \text{for} \quad 0 < p \leq 1 \tag{11}$$

which shows that $\lambda_r \in S_\infty$. Further, by (1) and (9),

$$\int_{-\infty}^{\infty} \widehat{\lambda}_r(t)dt = \int_{-\infty}^{\infty} \widehat{\sigma}_p(t)dt \int_0^{\infty} x^{-1}\nu_{q,r}(dx).$$

Applying Theorem we conclude that the probability measures λ_r is multiplicatively S_∞ - indecomposable. Thus formula (10) defines a non-denumerable family of non-equivalent decompositions of the probability measure $\sigma_p \circ \sigma_q$ with $0 < p \leq 1$ and $0 < q < 2$.

For the probability measure $\sigma_2 \circ \sigma_2$ the situation is quite different. In fact suppose that $\sigma_2 \circ \sigma_2 = \mu \circ \nu$ with $\mu, \nu \in S_\infty$. By a simple calculation we have the formula

$$(\sigma_2 \circ \sigma_2)\widehat{\ }(t) = (1 + 4t^2)^{-1/2} \quad (t \in \mathbb{R})$$

which shows, by Theorem 2 in [6], that $\mu = \delta_a \circ \sigma_2$ and $\nu = \delta_b \circ \sigma_2$ for some $a, b \in \mathbb{R}$.

Consequently, the probability measure $\sigma_2 \circ \sigma_2$ has exactly one decomposition up to the equivalence relation.

Example 3. The multiplicative semigroup generated by σ_p. Suppose that $0 < p \leq 1$. By Zolotariev Theorem ([7, Theorem 3.5.2]) there exists a one-parameter multiplicative semigroup ρ_r $(r > 0)$ fulfilling the conditions

$$\rho_q \circ \rho_r = \rho_{q+r} \quad (q, r > 0) \tag{12}$$

and

$$\rho_1 = \sigma_p. \tag{13}$$

It is clear that

$$\rho_r \neq \delta_0 \quad \text{for} \quad r > 0. \tag{14}$$

Given $r > 1$ we have, by (12) and (13), $\rho_r = \sigma_p \circ \rho_{r-1}$ which, by (11), yields

$$\rho_r \in S_\infty \quad \text{for } r \geq 1 . \tag{15}$$

A measure λ from S is called cancellable if the equality $\lambda \circ \mu = \lambda \circ \nu$ with $\mu, \nu \in S$ yields $\mu = \nu$. Suppose that $\sigma_p \circ \mu = \sigma_p \circ \nu$ and $\mu, \nu \in S$. By formula (1) and the symmetry of μ and ν we have

$$\int_0^\infty \exp(-|t|^p x^p)\mu(dx) = \int_0^\infty \exp(-|t|^p x^p)\nu(dx) \quad (t \in \mathbb{R})$$

which, by the uniqueness of the Laplace transform, yields $\mu = \nu$. Thus the stable probability measures σ_p are cancellable. Moreover, by (12) and (13), for every positive integer m the multiplicative convolution power $\rho_m = \sigma_p^{0m}$ is cancellable.

Now we shall prove that the equality

$$\rho_q = \delta_c \circ \rho_r \tag{16}$$

for some $c \in \mathbb{R}$ and $q, r > 0$ yields $q = r$. Contrary to this assume that $q \neq r$. Since, by (14), $c \neq 0$, we may assume without loss of generality that $q = r + a$ for some $a > 0$.

Taking a positive number b such that $m = r + b$ is a positive integer we get, by (12) and (16),

$$\rho_m \circ \rho_a = \rho_m \circ \delta_c . \tag{17}$$

Since, by (15), $\rho_m \in S_\infty$, we have $\vartheta \circ \rho_m = \rho_m$. Consequenltly equality (17) can be written in the form

$$\rho_m \circ (\vartheta \circ \rho_a) = \rho_m \circ (\vartheta \circ \delta_c) .$$

Of course both probability measures $\vartheta \circ \rho_a$ and $\vartheta \circ \delta_c$ are symmetric. Since the probability measure ρ_m is cancellable, the above equality yields

$$\vartheta \circ \rho_a = \vartheta \circ \delta_c . \tag{18}$$

Observe that for every positive integer n fulfilling the condition $na > 1$ we have, by (15), $\rho_{na} \in S_\infty$ which implies $\rho_{na} = \vartheta \circ \rho_{na}$. Thus, by (12) and (18),

$$\rho_{na} = (\vartheta \circ \rho_a)^{\circ n} = \vartheta \circ \delta_c n .$$

Hence it follows that the probability measure ρ_{na} is concentrated at the two points $-c^n$ and c^n. On the other hand the support of the infinitely divisible probability measure ρ_{na} different, by (14), from δ_0 is unbounded ([1, Chapter VI. 3]) which yields the contradiction. This shows that equality (16) implies $q = r$.

From (13) it follows that the probability measure ρ_1 is multiplicatively S_∞ - indecomposable. By Proposition 1 in [6] the subset of S_∞ consisting of all multiplicatively S_∞ - indecomposable probability measures is open in the topology of weak convergence. Consequently, we can choose an integer $s \geq 2$ such that ρ_r are multiplicatively S_∞ - indecomposable whenever $1 \leq r \leq 1 + 1/s$. Given an arbitrary positive integer n and a real number q fulfilling the inequality $0 \leq q < (2s)^{-1}$ we put for the sake of notation $a(q) = 1 + q, b(q) = 1 - q + 1/s, w = (n+1)(s+1)$ and

$$u(k) = k(s+1) - 1\,, \quad v(k) = (n+1-k)s - 1 \ \ (k = 1, 2, \ldots, n)\,.$$

It is clear that $1 \leq a(q) \leq 1 + 1/s\,,\ 1 \leq b(q) \leq 1 + 1/s\,,\ u(k) \leq 1$ and $v(k) \geq 1$ for $k = 1, 2, \ldots, n$. Moreover for every $k = 1, 2, \ldots, n$ we get, by (12), a non-denumerable family of non-equivalent decompositions of the probability measure ρ_w into $(n+1)s + k$ multiplicatively S_∞ - indecomposable factors

$$\rho_w = \rho_{a(q)} \circ \rho_{b(q)} \circ \rho_{a(0)}^{\circ u(k)} \circ \rho_{b(0)}^{\circ v(k)}\,.$$

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